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Weak time-derivatives and no arbitrage pricing*

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Abstract

The risk-neutral pricing formula provides the valuation of random payoffs in continuous-time markets. Despite the variety of payoffs, no arbitrage price dynamics are driven by the same (possibly stochastic) interest rate. We formalize this intuition by showing that no arbitrage prices constitute the solution of a differential equation, where interest rates are prominent. To achieve this goal, we introduce the notion of weak time-derivative, which permits to differentiate adapted processes. This instrument isolates drifts of semimartingales and it is null for martingales. Finally, we reformulate the eigenvalue problem of Hansen and Scheinkman (2009) by employing weak time-derivatives.

Keywords: no arbitrage pricing; weak time-derivative; martingale component; special semimartingales; stochastic interest rates.

Mathematics Subject Classification (2010): 60G07, 91G80, 49J40.

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1 Introduction

The no arbitrage pricing formula for the valuation of random payoffs is a milestone of asset pricing theory. For instance, it states that the *proper* price of a European option at time t is given by the conditional expectation of the discounted future payoff under a risk-neutral probability Q. If the derivative has payoff h_T at maturity T, a risk-neutral price $\pi(t)$ is

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-r(T-t)} h_T \right],$$

where r is a constant interest rate. From this formulation it is clear that the instantaneous rate r plays a fundamental role in the derivative pricing. Indeed, price dynamics are determined by the same interest rate process, whatever the terminal payoff of the option. This intuition goes back to Cox and Ross (1976), who derived the Black and Scholes (1973) result by exploiting the accounting relations between bonds, stocks and options. This line of reasoning actually stems from the original approach of Modigliani and Miller (1958).

In this paper we formalize the intuition that risk-neutral valuation is driven by the process of interest rates by generalizing the ordinary differential equation satisfied by the (deterministic) risk-free bond price $B(t) = e^{\int_0^t r(s)ds}$, namely

$$\frac{dB}{dt} = rB.$$

When the price process under consideration is stochastic, the issue of properly differentiating a random process arises. We address this problem by introducing the notion of *weak time-derivative* for adapted processes - denoted by $\mathcal{D}$ - and showing the basic rules of its calculus. By using this differential instrument, we prove that the no arbitrage pricing formula is the unique solution of the equation

$$\mathcal{D}\pi = r\pi \tag{1}$$

with the terminal condition $\pi(T) = h_T$. We prove this result in Theorems 18 and 21, where we suppose interest rates to be stochastic. The relation formalized by (1) stems from the foundations of asset pricing theory since it rephrases the equality between the instantaneous return of a risky asset - namely $\mathcal{D}\pi/\pi$ - and the instantaneous interest rate r, that needs to be valid in arbitrage-free markets. Indeed, by employing weak time-derivatives we are able to characterize no arbitrage - in particular, equivalent martingale measures - for a wide class of semimartingale price processes (see Proposition 17).

The definition of weak time-derivative requires a suitable set of test functions (see Definitions 1 and 20) and involves the conditional expectation of an adapted process. This instrument provides a handy characterization of martingales with respect to a given probability measure. Indeed, as claimed in Proposition 4, the weak time-derivative of a stochastic process is null if and only if the process is a martingale. Hence, the weak time-derivative allows us to qualify martingales as the stochastic counterpart of constant sequences in deterministic settings.

The parallel between calculus of weak time-derivatives - that we develop in Subsections 2.3 and 2.4 - and deterministic differential calculus is even deeper. For instance, submartingality and supermartingality are monotonicity properties related to positive or negative signs of weak time-derivatives. In terms of interpretation, the weak time-derivative provides an indication of the upward or downward growth rate of the conditional expectation of a random process.

More generally, by Theorem 9 the set of weakly time-differentiable processes coincides with a large class of special semimartingales. Specifically, weak time-derivatives capture the drift of processes under consideration, a useful property when canonical decompositions are unknown. Furthermore, the identification of the drift and of the martingale component reminds of the logic behind Girsanov Theorem and, in fact, is crucial for the analysis of eq. (1).

A nice feature of the weak time-derivative is that it applies to any adapted processes, without requiring the Feller property (as the *infinitesimal generator*) or Markovianity (as the *extended infinitesimal generator*). In addition, under suitable assumptions, the weak time-derivative specializes to both these notions, as we illustrate in Corollaries 14 and 15. In fact, the weak time-derivative allows us to deal with generalized formulations of problems that are usually formalized by these instruments.

As anticipated, the main results of the paper are summarized by Theorems 18 and 21, which show existence and uniqueness of the solution of eq. (1) under both deterministic and stochastic interest rates. In addition, Proposition 19 provides an interesting generalization of eq. (1) to cashflows.

When the risk-free rate is constant, by rewriting eq. (1) in operator form, we obtain a reformulation of the eigenvalue-eigenvector problem, analyzed by Hansen and Scheinkman (2009), which employs the weak time-derivative in place of the extended infinitesimal generator. Following Hansen and Scheinkman, we obtain in our setting a decomposition of the stochastic discount factor into a martingale and a transient component.

Our paper combines different areas of mathematical analysis and stochastic calculus. The overall approach comes from variational calculus, it exploits the theory of Sobolev spaces and weak formulations of differential equations. See, for example, Brezis (2010), Adams and Fournier (2003) and Lions (1971) for a comprehensive introduction to variational calculus, and Revuz and Yor (1999) for stochastic calculus.

From a financial point of view, our work builds directly on the foundations of no arbitrage asset pricing theory illustrated, for instance, in Björk (2004), Hansen and Richard (1987), Föllmer and Schied (2011) and Delbaen and Schachermayer (1998). In addition, our eigenvalue formulation refers to the long-term risk literature, in particular to Hansen and Scheinkman (2009) and related works, like Alvarez and Jermann (2005).

The paper is organized as follows. Section 2 develops the mathematical formalism of the weak time-derivative, proves its main properties (Subsections 2.3 and 2.4) and relates it to infinitesimal generators (Subsections 2.5 and 2.6). In Sections 3 and 5 we solve the no arbitrage pricing equation with deterministic and stochastic interest rates, respectively. A brief discussion of the special case of Black-Scholes model is presented in Subsection 3.1, while Subsection 3.2 discusses the risk-neutral pricing of cashflows. Section 4 deals with the eigenvalue-eigenvector problem and the decomposition of the stochastic discount factor. In particular, Subsection 4.1 compares the roles of weak time-derivative and infinitesimal generator in the eigenvalue-eigenvector formulation.

2 The weak time-derivative

After describing the technical framework, we define weak time-derivatives and illustrate their peculiar features. Then, in the last subsections we contrast weak time-derivatives with infinitesimal generators of Feller processes and with extended infinitesimal generators of Markov processes.

2.1 Filtration, measurability and identification

Given a probability space $(\Omega, \mathcal{F}, P)$ with strictly positive P, we fix $T > 0$, $N \in \mathbb{N}$ and consider a vectorial process $X = \{X_t\}_{t\in[0,T]}$ such that $X : [0,T] \times \Omega \longrightarrow \mathbb{R}^N$. In particular $X_t = \left[X_t^{(1)}, \ldots, X_t^{(N)}\right]'$ for all $t \in [0,T]$. For example, X_t may be a bunch of primary asset prices at time t. By convenience we will equivalently use both the notations X_t and $X(t)$. Moreover, $\mathbb{R}^N$ is endowed with the Borel σ-algebra and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ denotes the filtration generated by X.

We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfies the usual conditions, namely $\mathbb{F}$ is complete and right-continuous. Accordingly, we write $\mathcal{F}_t = \mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$. In addition, we assume left-continuity in T, i.e. $\mathcal{F}_T = \mathcal{F}_{T^-}$.

Throughout the paper, random variables are identified almost surely. Also inequalities between random variables are meant almost surely. Moreover, we identify stochastic processes up to modifications.

On the given filtered probability space, we consider processes $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ that are progressively measurable (progressive, for brevity), namely, for all $t \in [0,T]$, the function

$[0,t] \times \Omega \longrightarrow \mathbb{R}$ that maps (τ,ω) to the valuation $u_\omega(\tau)$ is Borel$([0,t]) \otimes \mathcal{F}_t$-measurable (see Revuz and Yor (1999)). In particular, such processes are adapted. Since any measurable and adapted u admits a progressive modification - see Karatzas and Shreve (2012) - and we identify stochastic processes up to modifications, we will simply require adaptability.

Seen as function of time, our processes take values in $L^1(\mathcal{F}_T)$ and, with a small abuse of notation, we simply write $u : [0,T] \longrightarrow L^1(\mathcal{F}_T)$. We will also often require that u is L^1-continuous, meaning that, for every $t \in [0,T]$, $\mathbb{E}[|u(\tau) - u(t)|]$ tends to zero whenever τ goes to t.

Moreover, we denote as $C([0,T],\mathbb{R})$ the space of continuous functions $f : [0,T] \longrightarrow \mathbb{R}$ while $C_c^n([0,T],\mathbb{R})$ contains n-times continuously differentiable functions from $[0,T]$ to $\mathbb{R}$ with compact support.

We define the space $\mathcal{V}$ as

$$\begin{aligned}\mathcal{V} = \Big\{&\text{adapted } u : [0,T] \longrightarrow L^1(\mathcal{F}_T), \quad L^1\text{-right-continuous in } [0,T), \\ &L^1\text{-left-continuous in } T, \quad \int_0^T \mathbb{E}[|u(\tau)|]\, d\tau < +\infty\Big\}. \end{aligned} \tag{2}$$

As a result, any $u \in \mathcal{V}$ is Bochner integrable and the Bochner integral of u is the element of $L^1(\mathcal{F}_T)$ denoted by $\int_0^T u(\tau)\, d\tau$. Indeed, condition (2) is necessary and sufficient for Bochner integrability of u. In addition, progressive measurability ensures the joint measurability of u on $[0,T] \times \Omega$. This property guarantees that the Bochner integral of u coincides almost surely with the pathwise Lebesgue integral, that we will employ at some point. See Diestel and Uhl (1977) and Aliprantis and Border (2006) as general references.

Observe that the space $\mathcal{V}$ contains all martingales defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The one requirement to check is L^1-continuity. In fact, for any $t \in [0,T)$, if $\tau \to t^+$,

$$\mathbb{E}[|u(\tau) - u(t)|] = \mathbb{E}[|\mathbb{E}_\tau[u(T)] - \mathbb{E}_t[u(T)]|] = \mathbb{E}[|\mathbb{E}_\tau[u(T)] - \mathbb{E}_{t^+}[u(T)]|] \longrightarrow 0$$

by Lévy's Downward Theorem. Moreover, if $\tau \to T^-$,

$$\mathbb{E}[|u(\tau) - u(T)|] = \mathbb{E}[|\mathbb{E}_\tau[u(T)] - \mathbb{E}_{T^-}[u(T)]|] \longrightarrow 0$$

by Lévy's Upward Theorem. See, for instance, Williams (1991).

2.2 Weak time-differentiability

We have now all the instruments to introduce the concept of *weak time-differentiability* for processes in $\mathcal{V}$.

Definition 1. *Given $u \in \mathcal{V}$, we say that u is* weakly time-differentiable *when there exists $w \in \mathcal{V}$ such that for every $t \in [0,T]$*

$$\int_t^T \mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = -\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T],\mathbb{R}\right).$$

In this case we call w a weak time-derivative *of u.*

Similarly, $u \in \mathcal{V}$ is said to be *twice weakly time-differentiable* when there exists $z \in \mathcal{V}$ such that for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[z\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = \int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi''(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^2\left([t,T],\mathbb{R}\right).$$

Observe that, if $u \in \mathcal{V}$, the integrals

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \quad \text{and} \quad \int_t^T \mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau$$

are finite for any choice of $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$. As a result, Definition 1 is well-posed.

The definition of weak time-derivative combines a variational approach, employed for weak solutions of (deterministic) partial differential equations, with the information structure due to the filtered probability space. Hence, the weak time-derivative builds a bridge between the *information-free* setting of calculus of variations and the adaptability concerns of stochastic process.

The presence of indicator functions reveals that weak time-differentiability actually involves the conditional expectation of random processes. The definition is stated in terms of the physical measure P. However, weak time-differentiability may be established also with respect to other probability measures. This is the case of asset pricing applications in which risk-neutral measures equivalent to P are employed for no arbitrage pricing, as illustrated in Section 3. Moreover, measure changes associated with the choice of a convenient numéraire are common practices in option pricing and interest rate theory. See, e.g., Margrabe (1978) and Geman et al (1995), among the others. Therefore, the weak time-derivative may be a fruitful instrument for asset pricing applications also in these settings.

The weak time-derivative is unique, up to modifications.

Proposition 2. *Let $u \in \mathcal{V}$ be weakly time-differentiable. Then, the weak time-derivative of u is unique.*

Proof. Let w and $\hat{w}$ be two weak time-derivatives of u. Then, for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[\left\{w\left(\tau\right) - \hat{w}\left(\tau\right)\right\} \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau = 0$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T], \mathbb{R}\right).$$

By Lemma 1 in Appendix, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}\left[\left\{w\left(\tau\right) - \hat{w}\left(\tau\right)\right\} \mathbf{1}_{A_t}\right] = \mathbb{E}\left[0 \mathbf{1}_{A_t}\right] = 0 \qquad \forall A_t \in \mathcal{F}_t.$$

Hence, the null process fits the definition of conditional expectation of $w\left(\tau\right) - \hat{w}\left(\tau\right)$ with respect to $\mathcal{F}_t$. Therefore, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}_t\left[w\left(\tau\right)\right] = \mathbb{E}_t\left[\hat{w}\left(\tau\right)\right].$$

Now, consider a sequence $\{\tau_i\}_{i \in \mathbb{N}} \subset [t,\tau]$ such that $\tau_i \longrightarrow t^+$ and $\mathbb{E}_t\left[w\left(\tau_i\right)\right] = \mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ for all i. Since w is L^1-continuous from the right, $\mathbb{E}_t\left[w\left(\tau_i\right)\right]$ converges in L^1 to $w(t)$ as τ_i approaches t because

$$\mathbb{E}\left[\left|\mathbb{E}_t\left[w\left(\tau_i\right)\right] - w(t)\right|\right] = \mathbb{E}\left[\left|\mathbb{E}_t\left[w\left(\tau_i\right) - w(t)\right]\right|\right] \leqslant \mathbb{E}\left[\left|w\left(\tau_i\right) - w(t)\right|\right] \longrightarrow 0.$$

Simultaneously, $\mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ converges in L^1 to $\hat{w}(t)$. However, $\mathbb{E}_t\left[w\left(\tau_i\right)\right]$ and $\mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ coincide a.s. Therefore, by uniqueness of the L^1-limit, $w\left(t\right) = \hat{w}\left(t\right)$. $\square$

We denote by $\mathcal{D}u$ the weak time-derivative of u. Moreover, we introduce the space

$$\mathcal{W} = \{\text{weakly time-differentiable } u \in \mathcal{V}\}$$

with norm

$$\|u\|_{\mathcal{W}} = \int_0^T \mathbb{E}\left[\left|u\left(\tau\right)\right|\right] d\tau + \int_0^T \mathbb{E}\left[\left|\mathcal{D}u\left(\tau\right)\right|\right] d\tau.$$

When we consider deterministic processes, the weak time-derivative coincides with the classical derivative of calculus.

Proposition 3. *Let $u \in \mathcal{V}$, $g \in C\left([0,T], \mathbb{R}\right)$ and, for every $t \in [0,T]$*

$$u\left(t\right) = \int_0^t g(s) ds.$$

Then, $\mathcal{D}u = g$.

Proof. As g is deterministic and continuous, $g \in \mathcal{V}$. Taken any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1([t,T],\mathbb{R})$, for every $t \in [0,T]$ we have

$$\begin{aligned}\int_t^T \mathbb{E}\left[u(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \mathbb{E}\left[\int_0^\tau g(s)ds\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T \left(\int_0^\tau g(s)ds\right)\varphi'(\tau)d\tau \\ &= -\mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T g(\tau)\varphi(\tau)d\tau \\ &= -\int_t^T \mathbb{E}\left[g(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau\end{aligned}$$

because φ has compact support. Hence, g is the weak time-derivative of u. □

2.3 Calculus of the weak time-derivative

We start by establishing the equivalence between processes with null weak time-derivative and martingales.

Proposition 4. *A process u belongs to $\mathcal{W}$ and it has $\mathcal{D}u = 0$ if and only if it is a martingale.*

Proof. Assume that u is a martingale. As observed in Subsection 2.1, u belongs to $\mathcal{V}$. Fixed $t \in [0,T]$, for all $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$,

$$\begin{aligned}\int_t^T \mathbb{E}\left[u(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \mathbb{E}\left[\mathbb{E}_t\left[u(\tau)\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \int_t^T \mathbb{E}\left[u(t)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \mathbb{E}\left[u(t)\mathbf{1}_{A_t}\right]\int_t^T \varphi'(\tau)d\tau \\ &= 0\end{aligned}$$

because φ is a function in $C_c^1([t,T],\mathbb{R})$. As a result, $w(t) = 0$ for all $t \in [0,T]$ satisfies the definition of weak time-derivative of u, i.e. $\mathcal{D}u = 0$.

Conversely, suppose that $u \in \mathcal{W}$ has $\mathcal{D}u = 0$. We first show that, given $t \in [0,T]$, $\mathbb{E}_t[u(\tau)]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take a continuous function $\eta : [t,T] \longrightarrow \mathbb{R}$ with compact support such that $\int_t^T \eta(\tau)d\tau = 1$. Given a continuous function $\xi : [t,T] \longrightarrow \mathbb{R}$ with compact support, we define the function $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(s) = \xi(s) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(s).$$

k_ξ is continuous with compact support and $\int_t^T k_\xi(\tau)d\tau = 0$. Hence, k_ξ has a primitive K_ξ which is continuous with compact support. As $K_\xi \in C_c^1([t,T],\mathbb{R})$, we employ it as a test function in the definition of weak time-derivative of u. Since $\mathcal{D}u = 0$, for all $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}
0 &= \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\left\{\xi(s) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(s)\right\}ds \\
&= \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\xi(s)ds - \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\left(\int_t^T \xi(\tau)d\tau\right)\eta(s)ds \\
&= \int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\xi(\tau)d\tau - \int_t^T \left\{\int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds\right\}\xi(\tau)d\tau \\
&= \int_t^T \left\{\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds\right\}\xi(\tau)d\tau.
\end{aligned}$$

By the density of continuous functions ξ, Lemma 1 in Appendix ensures that for a.e. $\tau \in [t,T]$

$$\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds.$$

Since $\int_t^T \eta(s)ds = 1$, we can rewrite the left-hand side as $\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\eta(s)ds$ so that

$$\int_t^T \left\{\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] - \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\right\}\eta(s)ds = 0.$$

As the last equality holds for any continuous function η with compact support in $[t,T]$, we find that, for a.e. $s \in [t,T]$,

$$\mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right] = \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]$$

and so $\mathbb{E}_t\left[u\left(s\right)\right] = \mathbb{E}_t\left[u\left(\tau\right)\right]$. As a result, $\mathbb{E}_t\left[u\left(\tau\right)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and we can state that $\mathbb{E}_t\left[u\left(\tau\right)\right] = f_t$ for some $f_t \in L^1\left(\mathcal{F}_t\right)$.

Since u is L^1-right-continuous,

$$\mathbb{E}_t\left[u\left(\tau\right)\right] \xrightarrow{L^1} u\left(t\right) \qquad \tau \longrightarrow t^+.$$

Since for a.e. $\tau \in [t,T]$, $\mathbb{E}_t\left[u\left(\tau\right)\right]$ coincides a.s. with f_t, which is not dependent on τ, the uniqueness of the L^1-limit implies that $f_t = u\left(t\right)$. Therefore, for any $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[u\left(\tau\right)\right] = u\left(t\right).$$

This property is actually satisfied by any $\tau \in [t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,T]$ such that $\tau_i \longrightarrow \tau^+$ and $\mathbb{E}_t\left[u\left(\tau_i\right)\right] = u\left(t\right)$. Since u is L^1-right-continuous, the L^1-limit of $\mathbb{E}_t\left[u\left(\tau_i\right)\right]$ is $\mathbb{E}_t\left[u\left(\tau\right)\right]$. Nevertheless, $\mathbb{E}_t\left[u\left(\tau_i\right)\right] = u\left(t\right)$ for all i and so, by uniqueness of the L^1-limit, $\mathbb{E}_t\left[u\left(\tau\right)\right] = u\left(t\right)$. □

A simple corollary of Proposition 4 shows that, given a weak time-derivative w, all processes $u \in \mathcal{W}$ such that $\mathcal{D}u = w$ differ by a martingale.

Corollary 5. *Let $w \in \mathcal{V}$ be the weak time-derivative of $u_1 \in \mathcal{W}$. w is also the weak time-derivative of $u_2 \in \mathcal{W}$ if and only if*

$$u_2 = u_1 + m,$$

where m is a martingale

Proof. Assume that w is also the weak time-derivative of $u_2 \in \mathcal{W}$ and consider the process $m = u_2 - u_1 \in \mathcal{W}$. The weak time-derivative of m is null and so, by Proposition 4, m is a martingale.

The converse implication follows from the fact that the weak time-derivative of any martingale is null. □

Moreover, the following result holds.

Proposition 6. *Let u be defined, for all $t \in [0, T]$, by*

$$u(t) = \int_0^t g(s)\, ds + m(t),$$

where $g \in \mathcal{V}$ and m is a martingale. Then, u belongs to $\mathcal{W}$ and $\mathcal{D}u = g$.

Proof. As $g \in \mathcal{V}$, g is Bochner integrable. Thus, for all $t \in [0, T]$, the process $G(t) = \int_0^t g(s)ds$ is well-defined and adapted.

Firstly, observe that

$$\begin{aligned}\int_0^T \mathbb{E}\left[|G(\tau)|\right] d\tau &\leqslant \int_0^T \mathbb{E}\left[\int_0^\tau |g(s)|\, ds\right] d\tau \leqslant \int_0^T \mathbb{E}\left[\int_0^T |g(s)|\, ds\right] d\tau \\ &= T\mathbb{E}\left[\int_0^T |g(s)|\, ds\right] = T\int_0^T \mathbb{E}\left[|g(s)|\right] ds,\end{aligned}$$

which is finite because $g \in \mathcal{V}$. Here, the exchange between the order of expectation (which is a bounded operator) and Bochner integral is made possible by Lemma 11.45 in Aliprantis and Border (2006).

Secondly, for any $t, \tau \in [0, T]$,

$$\mathbb{E}\left[|G(\tau) - G(t)|\right] \leqslant \mathbb{E}\left[\int_{\min\{\tau,t\}}^{\max\{\tau,t\}} |g(s)|\, ds\right] = \int_{\min\{\tau,t\}}^{\max\{\tau,t\}} \mathbb{E}\left[|g(s)|\right] ds$$

and the last quantity tends to zero as τ approaches t, ensuring L^1-continuity. As a result, G belongs to $\mathcal{V}$.

We now show that $\mathcal{D}G = g$. Given $t \in [0,T]$, consider any $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$. Then

$$\int_t^T \mathbb{E}\left[u(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau = \int_t^T \mathbb{E}\left[\int_0^\tau g(s)\,ds\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau.$$

We exchange the order of expectation and Bochner integral and, later, we apply Fubini's Theorem and we exploit the compact support of φ:

$$\begin{aligned}\int_t^T \mathbb{E}\left[G(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \left(\int_0^\tau \mathbb{E}\left[g(s)\mathbf{1}_{A_t}\varphi'(\tau)\right]ds\right)d\tau \\ &= \int_0^t \left(\int_t^T \mathbb{E}\left[g(s)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\right)ds \\ &\quad + \int_t^T \left(\int_s^T \mathbb{E}\left[g(s)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\right)ds \\ &= \int_0^t \left(\mathbb{E}\left[g(s)\mathbf{1}_{A_t}\right]\int_t^T \varphi'(\tau)d\tau\right)ds \\ &\quad + \int_t^T \left(\mathbb{E}\left[g(s)\mathbf{1}_{A_t}\right]\int_s^T \varphi'(\tau)d\tau\right)ds \\ &= -\int_t^T \mathbb{E}\left[g(s)\mathbf{1}_{A_t}\right]\varphi(s)ds.\end{aligned}$$

In consequence, g is the weak time-derivative of G.

As for m, this process belongs to $\mathcal{W}$ and $\mathcal{D}m = 0$ (see Proposition 4). Therefore, by additivity, $u \in \mathcal{W}$ and $\mathcal{D}u = g$. □

Example 7. *Assume that for every $t \in [0,T]$*

$$u(t) = \alpha t + m(t) \tag{3}$$

with $\alpha \in \mathbb{R}$ and m a martingale. Then $\mathcal{D}u = \alpha$. In addition, by Corollary 5 all processes $u \in \mathcal{W}$ such that $\mathcal{D}u = \alpha$ may be written as in eq. (3). In other words, a process in $\mathcal{W}$ is the sum of a deterministic trend and a martingale if and only if it has constant weak time-derivative. In this case, the value of $\mathcal{D}u$ identifies the linear drift.

If $N = 1$ this feature may be retrieved, for instance, in the Black-Scholes model, where the stock price satisfies

$$X_t = X_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$

with $r \in \mathbb{R}, \sigma > 0$ and W denoting a Wiener process under the risk-neutral measure Q. In fact, log prices are the sum of a deterministic drift and a martingale process, namely

$$log(X_t) = \log(X_0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t.$$

Here, the weak time-derivative of log prices properly captures the drift coefficient $r - \sigma^2/2$ under Q.

Example 8. *When $N = 1$ another illustration of Proposition 6 comes from continuous Itô semimartingales (or generalized diffusions) as, for instance, the process X in $\mathcal{V}$ such that*

$$X_t = X_0 + \int_0^t g(s)\, ds + \int_0^t h(s) dW_s,$$

where $g \in \mathcal{V}$, h is adapted and $\int_0^T \mathbb{E}\left[h^2(s)\right] ds$ finite. The stochastic differential of the process is usually written as

$$dX_t = g(t)dt + h(t)dW_t.$$

The Itô integral of h with respect to the Wiener process W is a martingale and it belongs to $\mathcal{W}$. As a result, the weak time-derivative of X is the drift g:

$$\mathcal{D}X = g.$$

Moreover, any process defined by $v(t) = f(X_t)$ for all $t \in [0, T]$ with f twice continuously differentiable is a continuous Itô semimartingale, too. The stochastic differential of v can, then, be derived by Itô's formula and the weak time-derivative of v is identified as the drift in the stochastic differential of v.

Inspired by Example 8, we provide a characterization of weakly time-differentiable processes in terms of special semimartingales (see Protter (2004)). Although by Definition 1 weak time-differentiability is established through an integration by parts relation, this notion identifies a class of stochastic processes widely employed in dynamic asset pricing modelling.

Theorem 9. *$\mathcal{W}$ is the space of special semimartingales u such that*

$$u = a + m,$$

with $a(t) = \int_0^t g(s)ds$, $g \in \mathcal{V}$ and m a martingale.

Proof. Take $u \in \mathcal{W}$ and let $\mathcal{D}u$ denote its weak time-derivative. For any $t \in [0, T]$ define

$$a(t) = \int_0^t \mathcal{D}u(s)ds.$$

$a \in \mathcal{W}$ and it has the same weak time-derivative of u. Therefore, by Corollary 5, $m = u - a$ is a martingale. As a result, u decomposes into the sum $u = a + m$. Here, a has finite variation (because $\mathcal{D}u$ is integrable), it is càdlàg (because its paths are continuous) and adapted. Moreover, m is a local martingale (because it is a martingale). These features make u a semimartingale. In addition, a is also predictable (because it is left-continuous and adapted). As a result, u is a special semimartingale.

Conversely, let u be a special semimartingale such that, for all $t \in [0, T]$

$$u(t) = \int_0^t g(s)ds + m(t),$$

with $g \in \mathcal{V}$ and m a martingale. By Proposition 6, u belongs to $\mathcal{W}$ and $\mathcal{D}u = g$. □

It is now apparent that weak time-differentiability captures a class of special semimartingales that feature a (unique) absolutely continuous finite variation term and a (unique) local martingale term which is actually a martingale. The innovation of our approach relies on the fact that this class of processes is characterized via a synthetic differentiability condition that does not require the knowledge, ex-ante, of the canonical decomposition of the semimartingale into consideration. In other words, the weak time-derivative allows the identification of the drift term of semimartingales in $\mathcal{W}$ even when the canonical decomposition is not available.

In the next example we dig into the martingale term of Theorem 9. Indeed, in general contexts m combines a continuous martingale term with a pure-jump martingale.

Example 10. *Set $N = 1$ and consider in the time interval $[0, T]$ the process X driven by the dynamics*

$$\frac{dX_t}{X_{t^-}} = \mu dt + \sigma dW_t + dH_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, W is a Wiener process and H is a compound Poisson process. In particular, $H_t = \sum_{k=1}^{N_t} z_k$, i.e. $dH_t = z_{N_{t^-}+1} dN_t$, where N is a Poisson process with intensity λ and z_k are i.i.d. random variables independent of W and N such that $\mathbb{E}[z_k] = z$ and $z_k \geqslant -1$. This jump-diffusion process has been, firstly, employed by Merton (1976) for option pricing. A taxonomy of similar SDEs applied in asset valuation can be retrieved, for instance, in Platen and Bruti-Liberati (2010).

Although H is not a martingale in general, the compensated *Poisson process $\hat{H}$ defined by $\hat{H}_t = Y_t - \lambda z t$ for all $t \in [0, T]$ is a martingale. Therefore, we can rewrite the dynamics of X as*

$$\frac{dX_t}{X_{t^-}} = (\mu + \lambda z)dt + \sigma dW_t + d\hat{H}_t.$$

In the last formulation, the finite variation term - eventually captured by the weak time-derivative - is driven by the deterministic drift coefficient $\mu + \lambda z$. However, the martingale term is the sum of two process: a continuous martingale given by the Wiener process and a pure-jump martingale individuated by the compensated Poisson process.

As Example 10 suggests, the decomposition of Theorem 9 holds for a wide class of Lévy processes. Beyond Merton's model, such processes are extensively employed for modelling asset price dynamics. Relevant examples are provided by Kou (2002), Barndorff-Nielsen (1997) and Carr et al (2002), among the others.

2.4 Monotonicity and convexity

We now discuss the connection between monotonicity and sign of the weak time-derivative. In conditional terms, we can associate the positivity of $\mathcal{D}u$ with the increasing monotonicity of u. Similarly, a negative weak time-derivative reveals the decreasing monotonicity of u. Interestingly, these features can be related to *submartingality* and *supermartingality* respectively.

Proposition 11. *Let $u \in \mathcal{W}$. We have $\mathcal{D}u \geqslant 0$ if and only if, for every $t \in [0,T]$ and $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 \geqslant \tau_1$,*

$$\mathbb{E}_t \left[u \left(\tau_2 \right) \right] \geqslant \mathbb{E}_t \left[u \left(\tau_1 \right) \right].$$

In this case, u is a submartingale.

An analogous result holds with $\leqslant$ instead of $\geqslant$.

Proof. As in the proof of Proposition 4, we consider the continuous with compact support functions $\eta, \xi : [t,T] \longrightarrow \mathbb{R}$, with $\int_t^T \eta(s)ds = 1$ and we define $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(\tau) = \xi(\tau) - \left(\int_t^T \xi(s)ds \right) \eta(\tau).$$

The primitive

$$K_\xi(\tau) = \int_t^\tau \xi(s)ds - \left(\int_t^T \xi(s)d\tau \right) \int_t^\tau \eta(s)ds$$

belongs to $C_c^1 \left([t,T], \mathbb{R} \right)$ and we employ it as a test function in the definition of weak time-derivative of u. In addition, we require that $\int_t^T \xi(s)ds = 1$, so that we consider

$$K_\xi(\tau) = \int_t^\tau \left(\xi(s) - \eta(s) \right) ds, \qquad\qquad k_\xi(\tau) = \xi(\tau) - \eta(\tau).$$

As $u \in \mathcal{W}$, for all $\varphi \in C_c^1 \left([t,T], \mathbb{R} \right)$ and $A_t \in \mathcal{F}_t$, we have

$$\int_t^T \mathbb{E} \left[u \left(\tau \right) \mathbf{1}_{A_t} \right] \left(\xi(\tau) - \eta(\tau) \right) d\tau$$
$$= - \int_t^T \mathbb{E} \left[\mathcal{D}u \left(\tau \right) \mathbf{1}_{A_t} \right] \left(\int_t^\tau \left(\xi(s) - \eta(s) \right) ds \right) d\tau.$$

By Fubini's Theorem,

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\left(\xi(\tau)-\eta(\tau)\right)d\tau$$
$$= -\int_t^T \left(\int_s^T \mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]d\tau\right)\left(\xi(s)-\eta(s)\right)ds$$

so that

$$\int_t^T \left(\mathbb{E}\big[u\left(\tau\right)\mathbf{1}_{A_t}\big]+\int_\tau^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds\right)\xi(\tau)d\tau$$
$$= \int_t^T \left(\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]+\int_\tau^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds\right)\eta(\tau)d\tau.$$

By the density of continuous functions ξ, η, Lemma 1 in Appendix implies that for a.e. $\tau_1, \tau_2 \in [t, T]$

$$\mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{A_t}\right]+\int_{\tau_1}^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds$$
$$= \mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{A_t}\right]+\int_{\tau_2}^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds,$$

namely

$$\mathbb{E}\left[\left(u\left(\tau_2\right)-u\left(\tau_1\right)\right)\mathbf{1}_{A_t}\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds. \tag{4}$$

If $\mathcal{D}u \geqslant 0$, then, for a.e. $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 \geqslant \tau_1$,

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds \geqslant 0$$

for any $A_t \in \mathcal{F}_t$ and so $\mathbb{E}\left[\left(u\left(\tau_2\right)-u\left(\tau_1\right)\right)\mathbf{1}_{A_t}\right] \geqslant 0$. Since this holds for any $\mathcal{F}_t$-measurable set A_t, we infer that

$$\mathbb{E}_t\left[u\left(\tau_2\right)-u\left(\tau_1\right)\right] \geqslant 0$$

for a.e. $\tau_1, \tau_2 \in [t, T]$ with $\tau_2 \geqslant \tau_1$.

We are left to show that the last inequality is satisfied by all $\tau_1, \tau_2 \in [t, T]$ with $\tau_2 \geqslant \tau_1$. Indeed, suppose by contradiction that there exists a pair $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 > \tau_1$ and

$$\mathbb{E}_t\left[u\left(\tau_2\right)\right]\mathbf{1}_{E_t} < \mathbb{E}_t\left[u\left(\tau_1\right)\right]\mathbf{1}_{E_t},$$

where E_t is a $\mathcal{F}_t$-measurable set with positive measure. Then,

$$\mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{E_t}\right] < \mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{E_t}\right].$$

Now take into consideration a sequence $\{t_i\}_{i\in\mathbb{N}} \subset [\tau_1, \tau_2]$ such that $t_i \longrightarrow \tau_1^+$ and

$$\mathbb{E}_t \left[u\left(\tau_2\right)\right] \geqslant \mathbb{E}_t \left[u\left(t_i\right)\right].$$

This ensures that

$$\mathbb{E}\left[u\left(\tau_2\right) \mathbf{1}_{E_t}\right] \geqslant \mathbb{E}\left[u\left(t_i\right) \mathbf{1}_{E_t}\right].$$

Since u is L^1-right-continuous,

$$\mathbb{E}_t\left[u\left(t_i\right)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u\left(\tau_1\right)\right] \qquad t_i \longrightarrow \tau_1^+$$

and so

$$\mathbb{E}_t\left[u\left(t_i\right)\right] \mathbf{1}_{E_t} \xrightarrow{L^1} \mathbb{E}_t\left[u\left(\tau_1\right)\right] \mathbf{1}_{E_t} \qquad t_i \longrightarrow \tau_1^+.$$

As a result,

$$\mathbb{E}\left[u\left(t_i\right) \mathbf{1}_{E_t}\right] \longrightarrow \mathbb{E}\left[u\left(\tau_1\right) \mathbf{1}_{E_t}\right] \qquad t_i \longrightarrow \tau_1^+,$$

where we deal with a sequence of real numbers. Since $\mathbb{E}\left[u\left(\tau_1\right) \mathbf{1}_{E_t}\right] > \mathbb{E}\left[u\left(\tau_2\right) \mathbf{1}_{E_t}\right]$, by permanence of sings we can find an index $\hat{i}$ such that

$$\mathbb{E}\left[u\left(t_i\right) \mathbf{1}_{E_t}\right] > \mathbb{E}\left[u\left(\tau_2\right) \mathbf{1}_{E_t}\right]$$

for all $i > \hat{i}$. Therefore, we obtain a contradiction.

Conversely, if $\mathbb{E}_t\left[u\left(\tau_2\right) - u\left(\tau_1\right)\right] \geqslant 0$ for every $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 \geqslant \tau_1$, then

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right) \mathbf{1}_{A_t}\right] ds \geqslant 0.$$

Since this holds for any $\tau_1, \tau_2 \in [t, T]$, it follows that, for a.e. $\tau \in [t, T]$

$$\mathbb{E}\left[\mathcal{D}u\left(\tau\right) \mathbf{1}_{A_t}\right] \geqslant 0.$$

As A_t is any $\mathcal{F}_t$-measurable set, we have

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \geqslant 0.$$

By L^1-right-continuity of du/dt, as $\tau \longrightarrow t^+$,

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \xrightarrow{L^1} \mathcal{D}u\left(t\right).$$

As a result, $\mathcal{D}u \geqslant 0$ for all $t \in [0, T]$.

We are left to show that, in this case, u is a submartingale: for every $t \in [0, T]$ and $\tau \in [t, T]$

$$\mathbb{E}_t\left[u(\tau)\right] \geqslant u(t).$$

By contradiction, suppose that there exists $\tau \in (t, T]$ and a set $F_t \in \mathcal{F}_t$ with $P(F_t) > 0$ such that

$$\mathbb{E}_t [u(\tau)] \mathbf{1}_{F_t} < u(t) \mathbf{1}_{F_t}$$

This implies that

$$\mathbb{E} [u(\tau) \mathbf{1}_{F_t}] < \mathbb{E} [u(t) \mathbf{1}_{F_t}] .$$

Now, consider a sequence $\{t_i\}_{i \in \mathbb{N}} \subset [t, \tau]$ such that $t_i \longrightarrow t^+$. For all i we have

$$\mathbb{E}_t [u(\tau)] \geqslant \mathbb{E}_t [u(t_i)] .$$

It follows that

$$\mathbb{E} [u(\tau) \mathbf{1}_{F_t}] \geqslant \mathbb{E} [u(t_i) \mathbf{1}_{F_t}] .$$

Since u is L^1-continuous from the right,

$$\mathbb{E}_t [u(t_i)] \xrightarrow{L^1} u(t) \qquad t_i \longrightarrow t^+$$

and so

$$\mathbb{E}_t [u(t_i)] \mathbf{1}_{F_t} \xrightarrow{L^1} u(t) \mathbf{1}_{F_t} \qquad t_i \longrightarrow t^+.$$

In consequence,

$$\mathbb{E} [u(t_i) \mathbf{1}_{F_t}] \longrightarrow \mathbb{E} [u(t) \mathbf{1}_{F_t}] \qquad t_i \longrightarrow t^+,$$

which is the limit of a sequence of real numbers. As observed at the beginning,

$$\mathbb{E} [u(t) \mathbf{1}_{F_t}] > \mathbb{E} [u(\tau) \mathbf{1}_{F_t}]$$

and so, by permanence of signs, it is possible to find an index $\hat{i}$ such that, for all $i > \hat{i}$,

$$\mathbb{E} [u(t_i) \mathbf{1}_{F_t}] > \mathbb{E} [u(\tau) \mathbf{1}_{F_t}] .$$

Hence, we meet a contradiction. □

Next we focus on the increments of weak time-derivatives, i.e. we deal with convexity (or concavity). The following result shows that a process $u \in \mathcal{W}$ satisfies a convexity property when $\mathcal{D}u$ is increasing, after taking the conditional expectation.

Proposition 12. *Let $u \in \mathcal{W}$. For every $t \in [0, T]$ and $\tau_1, \tau_2 \in [t, T]$ such that $\tau_1 \leqslant \tau_2$*

$$\mathbb{E}_t [\mathcal{D}u(\tau_1)] \leqslant \mathbb{E}_t [\mathcal{D}u(\tau_2)]$$

if and only if, for every $t \in [0, T]$ and $\tau_1, \tau_2 \in [t, T]$ such that $\tau_1 \leqslant \tau_2$

$$\mathbb{E}_t [\mathcal{D}u(\tau_1)] \leqslant \frac{\mathbb{E}_t [u(\tau_2)] - \mathbb{E}_t [u(\tau_1)]}{\tau_2 - \tau_1} \leqslant \mathbb{E}_t [\mathcal{D}u(\tau_2)] .$$

An analogous result holds with $\geqslant$ instead of $\leqslant$.

Proof. Following the proof of Proposition 11, in particular eq. (4), given $t \in [0,T]$, for a.e. $\tau_1, \tau_2 \in [t,T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[u\left(\tau_2\right)-u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right]=\int_{\tau_1}^{\tau_2}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(s\right)\right]\mathbf{1}_{A_t}\right]ds.$$

Let $\tau_1 \leqslant \tau_2$. If

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\leqslant\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right],$$

we have

$$\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right]\leqslant\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right]$$

and this monotonicity ensures that

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] &\leqslant \frac{\int_{\tau_1}^{\tau_2}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(s\right)\right]\mathbf{1}_{A_t}\right]ds}{\tau_2-\tau_1}\\ &\leqslant \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

By the initial equality,

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] &\leqslant \frac{\mathbb{E}\left[\mathbb{E}_t\left[u\left(\tau_2\right)-u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right]}{\tau_2-\tau_1}\\ &\leqslant \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

As this holds for any $A_t \in \mathcal{F}_t$, we deduce that

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\leqslant\frac{\mathbb{E}_t\left[u\left(\tau_2\right)\right]-\mathbb{E}_t\left[u\left(\tau_1\right)\right]}{\tau_2-\tau_1}\leqslant\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right].$$

Conversely, if the last inequality holds, it is clear that

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\leqslant\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right].$$

Although the results written so far hold for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$, by exploiting L^1-continuity as in the proof of Proposition 11, we obtain that all conclusions are satisfied for every $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$. □

Summing up, the calculus of weak time-derivatives generalizes in a natural way some key results of standard differential calculus.

2.5 Comparison with the infinitesimal generator

We relate the notion of weak time-derivative with the one of *infinitesimal generator*, widely employed in stochastic calculus. A further comparison of the applications of both instruments in option pricing is discussed in Subsection 4.1.

We begin by considering the infinitesimal incremental ratios of conditional expectations. These quantities converge to the weak time-derivative.

Proposition 13. *Let $u \in \mathcal{W}$ and $t \in [0,T]$. If for any $\tau \in [t,T]$*

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}$$

is convergent in L^1 when $h \longrightarrow 0^+$, then

$$\frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h} \xrightarrow{L^1} \mathcal{D}u\left(t\right) \qquad h \longrightarrow 0^+.$$

Proof. We first show that, for a.e. $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \qquad h \longrightarrow 0^+. \tag{5}$$

By following the same steps of the proof of Proposition 11 we find that, for a.e. $\tau, \hat{\tau} \in [t,T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[u\left(\hat{\tau}\right)-u\left(\tau\right)\right]\mathbf{1}_{A_t}\right] = \int_\tau^{\hat{\tau}} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds.$$

By setting $\hat{\tau} = \tau + h$ for some $h > 0$, we have

$$\mathbb{E}\left[\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}\mathbf{1}_{A_t}\right] = \frac{1}{h}\int_\tau^{\tau+h} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds.$$

Now we take the limit as $h \longrightarrow 0^+$. By Lebesgue Differentiation Theorem, the right-hand side converges to $\mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]$. Moreover, if $w\left(\tau\right)$ denotes the $\mathcal{F}_t$-measurable L^1-limit of $\frac{\mathbb{E}_t[u(\tau+h)-u(\tau)]}{h}$, the left-hand side converges to $\mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]$. Consequently,

$$\mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right] = \mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]$$

for every $\mathcal{F}_t$-measurable set A_t. Hence, by definition of conditional expectation,

$$w\left(\tau\right) = \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right].$$

As a result, the convergence in eq. (5) is proved.

Now recall that, since u and $\mathcal{D}u$ are L^1-right-continuous, as $\tau \longrightarrow t^+$,

$$\mathbb{E}_t\left[u\left(\tau\right)\right] \xrightarrow{L^1} u\left(t\right), \qquad \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \xrightarrow{L^1} \mathcal{D}u\left(t\right).$$

Also, the fact that, for any $\tau \in [t,T]$,

$$\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h} \xrightarrow{L^1} w(\tau) \qquad h \longrightarrow 0^+$$

ensures that, for any $\tau \in [t,T]$,

$$\mathbb{E}_t\left[u(\tau+h)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u(\tau)\right] \qquad h \longrightarrow 0^+.$$

Indeed, for every $h > 0$,

$$\begin{aligned}\mathbb{E}\big[\big|\mathbb{E}_t[u(\tau+h)-u(\tau)]\big|\big] &= h\mathbb{E}\left[\frac{\left|\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]\right|}{h}\right] \\ &\leqslant h\left\{\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h}-w(\tau)\right|\right]+\mathbb{E}\left[\left|w(\tau)\right|\right]\right\}\end{aligned}$$

and this quantity converges to 0 as $h \longrightarrow 0^+$ since $w(\tau)$ is in L^1. In particular, for any fixed $h > 0$, we have

$$\mathbb{E}_t\left[u(\tau+h)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u(t+h)\right] \qquad \tau \longrightarrow t^+$$

and so

$$\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h} \xrightarrow{L^1} \frac{\mathbb{E}_t\left[u(t+h)\right]-u(t)}{h} \qquad \tau \longrightarrow t^+.$$

Putting things together, for any $\tau \in [t,T], h > 0$ we have

$$\begin{aligned}&\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u(t+h)\right]-u(t)}{h}-\mathcal{D}u(t)\right|\right] \\ &\qquad \leqslant \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u(t+h)\right]-u(t)}{h}-\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h}\right|\right] \\ &\qquad + \mathbb{E}\left[\left|-\mathcal{D}u(t)+\mathbb{E}_t\left[\mathcal{D}u(\tau)\right]\right|\right] \\ &\qquad + \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h}-\mathbb{E}_t\left[\mathcal{D}u(\tau)\right]\right|\right].\end{aligned}$$

The previous convergences allow us to choose $\tau \in [t,T]$ in a way that the first two terms in the right-hand side are arbitrarily small and the convergence in eq. (5) allows us to choose h so that the last term is arbitrarily little. Hence, when $h \longrightarrow 0^+$,

$$\frac{\mathbb{E}_t\left[u(t+h)\right]-u(t)}{h} \xrightarrow{L^1} \mathcal{D}u(t).$$

□

Fixing $t \in [0,T]$, the outcome of Proposition 13 can be restated as

$$\mathbb{E}_t\left[u(\tau)\right]-u(t)-\mathcal{D}u(t)(\tau-t) \xrightarrow{L^1} 0 \qquad \tau \longrightarrow t^+,$$

which is a first-order expansion of $\mathbb{E}_t [u(\tau)]$ in a right neighbourhood of t, with the limit taken in L^1.

As described in Revuz and Yor (1999), the infinitesimal generator of a Feller process X is the operator $\mathcal{A}$ that maps any continuous bounded function f belonging to the domain of $\mathcal{A}$ into the function $\mathcal{A}f$ such that, for any $t \in [0,T]$,

$$\mathcal{A}f(X_t) = \lim_{h \to 0^+} \frac{\mathbb{E}_t [f(X_{t+h})] - f(X_t)}{h}.$$

The limit, here, is in the uniform topology over all states $\omega \in \Omega$ and $\mathcal{A}f$ is continuous and bounded.

Proposition 13 shows that the weak time-derivative in $\mathcal{W}$ generally works as the infinitesimal generator with the limit $h \longrightarrow 0^+$ taken in the L^1-norm without requiring the Feller property of the underlying process. We now show that the weak time-derivative and the infinitesimal generator coincide when they are both well-defined.

Corollary 14. *Let X be a Feller process. Let $u \in \mathcal{W}$ be such that, for every $t \in [0,T]$, $u(t) = f(X_t)$ with f continuous and bounded in the domain of $\mathcal{A}$. Then, for every $t \in [0,T]$,*

$$\mathcal{D}u(t) = \mathcal{A}f(X_t).$$

Proof. The function f is continuous and bounded, $\mathcal{A}f$ is continuous and bounded and $\mathcal{A}f(X_t)$ belongs to L^1. Since, for every $\tau \in [t,T]$,

$$\frac{\mathbb{E}_\tau [f(X_{\tau+h})] - f(X_\tau)}{h}$$

converges to $\mathcal{A}f(X_\tau)$ as $h \longrightarrow 0^+$ in the uniform topology,

$$\frac{\mathbb{E}_t [f(X_{\tau+h}) - f(X_\tau)]}{h} \xrightarrow{L^1} \mathbb{E}_t [\mathcal{A}f(X_\tau)] \qquad\qquad h \longrightarrow 0^+.$$

Indeed, since f is in the domain of the infinitesimal generator $\mathcal{A}$, we can find an arbitrary small $\varepsilon > 0$ such that

$$\begin{aligned}
\left| \frac{\mathbb{E}_\tau [f(X_{\tau+h})] - f(X_\tau)}{h} \right| &\leqslant \left| \frac{\mathbb{E}_t [f(X_{\tau+h})] - f(X_\tau)}{h} - \mathcal{A}f(X_\tau) \right| + |\mathcal{A}f(X_\tau)| \\
&\leqslant \sup_{\omega \in \Omega} \left| \frac{\mathbb{E}_t [f(X_{\tau+h})] - f(X_\tau)}{h} - \mathcal{A}f(X_\tau) \right| + |\mathcal{A}f(X_\tau)| \\
&\leqslant \varepsilon + |\mathcal{A}f(X_\tau)|.
\end{aligned}$$

By the Conditional Dominated Convergence Theorem, when $h \longrightarrow 0^+$

$$\mathbb{E}_t \left[\frac{\mathbb{E}_\tau [f(X_{\tau+h})] - f(X_\tau)}{h} \right] \xrightarrow{a.s.} \mathbb{E}_t [\mathcal{A}f(X_\tau)],$$

that is

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{a.s.} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right].$$

Moreover,

$$\begin{aligned}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\right| &= \left|\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right]\right| \\ &\leqslant \mathbb{E}_t\left[\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right|\right] \\ &\leqslant \mathbb{E}_t\left[\varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|\right] \\ &= \varepsilon+\mathbb{E}_t\left[\left|\mathcal{A}f\left(X_\tau\right)\right|\right].\end{aligned}$$

Therefore, by the Dominated Convergence Theorem, for every $t \in [0,T]$, for every $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right] \qquad h \longrightarrow 0^+.$$

In particular,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \mathcal{A}f\left(X_t\right) \qquad h \longrightarrow 0^+.$$

Since $\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}$ is convergent in L^1 as $h \longrightarrow 0^+$ for every $t \in [0,T]$ and every $\tau \in [t,T]$, Proposition 13 applies. In consequence,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \mathcal{D}u\left(t\right) \qquad h \longrightarrow 0^+.$$

By uniqueness of the L^1-limit, we infer that

$$\mathcal{D}u\left(t\right)=\mathcal{A}f\left(X_t\right).$$

□

As we will see in Subsection 4.1, weak time-derivatives provide more general formulations of operator equations that are usually expressed through infinitesimal generators, such as the eigenvalue-eigenvector problem $\mathcal{A}f = rf$. This kind of generalization is made possible by the fact that both instruments provide a similar characterization of martingales. Indeed, the process $\left\{f\left(X_t\right)\right\}_{t\in[0,T]}$ is a martingale when the infinitesimal generator of f is null, as ensured by Proposition 1.6 in Chapter VII of Revuz and Yor (1999). This result actually shares the same insight of Proposition 4 that relates the martingale property to the nullity of weak time-derivatives.

2.6 Comparison with the extended infinitesimal generator

To illustrate the relation between the weak time-derivative and the extended infinitesimal generator, we begin with a corollary of Proposition 6.

Corollary 15. *Let $u \in \mathcal{W}$. Then, the process v defined, for all $t \in [0,T]$, by*

$$v(t) = u(t) - u(0) - \int_0^t \mathcal{D}u(\tau)d\tau$$

is a martingale.

Proof. v belongs to $\mathcal{V}$ as discussed in the proof of Proposition 6. Still by Proposition 6, the weak time-derivative of the process $U(t) = \int_0^t \mathcal{D}u(s)ds$ is $\mathcal{D}u$. Since the weak time-derivative of $u(0)$ is null, by additivity we conclude that

$$\mathcal{D}v = \mathcal{D}u - 0 - \mathcal{D}u = 0.$$

In consequence, by Proposition 4, v is a martingale. $\square$

This result rephrases Dynkin's formula for Markov processes. See, e.g., Protter (2004). In particular, it ensures that, for all $t \in [0,T]$ and $\tau \in [t,T]$,

$$\mathbb{E}_t\left[u(\tau)\right] = u(t) + \mathbb{E}_t\left[\int_t^\tau \mathcal{D}u(s)ds\right].$$

In case X is a Markov process and $u(t) = f\left(X_t\right)$ for all $t \in [0,T]$, the martingality of v implied by Corollary 15 guarantees that the weak time-derivative of u coincides with the *extended infinitesimal generator* of f. In fact, the extended infinitesimal generator of a measurable function f of X_t is a measurable function g such that $g\left(X_t\right)$ is integrable over time and the process v defined, for all $t \in [0,T]$, by

$$v(t) = f\left(X_t\right) - f\left(X_0\right) - \int_0^t g\left(X_\tau\right)d\tau$$

is a martingale (see Revuz and Yor (1999)). We summarize this observation in a proper statement.

Corollary 16. *Let X_t be a Markov process. Let $u \in \mathcal{W}$ be such that, for every $t \in [0,T]$, $u(t) = f\left(X_t\right)$. Then, $\mathcal{D}u$ is the extended infinitesimal generator of f.*

Proof. By Corollary 15, the process v defined by

$$v(t) = u(t) - u(0) - \int_0^t \mathcal{D}u(\tau)d\tau$$

is a martingale. Therefore, $\mathcal{D}u$ satisfies the definition of extended infinitesimal generator of f. $\square$

Up to minor adjustments, the extended infinitesimal generator has been used by Hansen and Scheinkman (2009) for the analysis of pricing operators. In fact, the extended infinitesimal generator keeps the most relevant features of the infinitesimal generator (in particular the nullity for martingales) but it does not require the Feller property of the underlying process.

To recap, the weak time-derivative is a general instrument that allows the differentiation of adapted processes and it nicely compares with both the infinitesimal generator and the extended infinitesimal generator. First, the infinitesimal generator applies to Feller processes and its extended version involves Markov processes, while the weak time-derivative is defined for a wide class of special semimartingales. Note, moreover, that the weak time-derivative mostly relies on measure theoretical assumptions, while the infinitesimal generator requires more restrictive topological assumptions.

3 No arbitrage pricing

We consider a continuous-time market over the time window $[0, T]$ with N risky assets, whose prices are collected in the vectorial process X, where $X_t = \left[X_t^{(1)}, \ldots, X_t^{(N)}\right]'$. A risk-free security with price B such that $B_t = e^{rt}$ for all $t \in [0, T]$ is also traded. Following for instance Björk (2004), for any $t \in [0, T]$ we define the vector of relative asset prices $Z_t = X_t/B_t$. We then consider as environment the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by Z and P is the physical probability.

We assume that our price system satisfies the no free lunch with vanishing risk (NFLVR) condition and that relative asset prices Z are semimartingales. The dynamics of these processes are, indeed, compatible with NFLVR restrictions, as explained by Delbaen and Schachermayer (1994).

A portfolio strategy is given by any adapted $(N + 1)$-dimensional process ϑ and its discounted value process is V^ϑ defined by

$$V_t^\vartheta = \vartheta_t^{(0)} + \sum_{i=1}^{N} \vartheta_t^{(i)} Z_t^{(i)}, \qquad t \in [0, T].$$

All portfolios under consideration are supposed to be *admissible*, namely there is $w \geqslant 0$ such that

$$\int_0^t \sum_{i=1}^{N} \vartheta_t^{(i)} dZ_t^{(i)} \geqslant -w$$

for all $t \in [0, T]$, and *self-financing*, i.e.

$$dV_t^\vartheta = \sum_{i=1}^{N} \vartheta_t^{(i)} dZ_t^{(i)}, \qquad t \in [0, T].$$

According to the First Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer (1998), NFLVR ensures that there exists a probability measure Q equivalent to P such that Z is a *sigma-martingale*. In other words, Z is the martingale transform of some martingale via an integrable predictable process (see Émery (1980)). Moreover, the measure Q does not need to be unique.

Remark. We assume that at least one of the measures Q inferred by Delbaen and Schachermayer (1998) is an *equivalent martingale measure*, i.e. it makes Z a martingale process.

From now on we consider one of these risk-neutral measures Q for the valuation of securities in the market. In particular, we move to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Integrability conditions, expectations and convergences are computed with respect to Q.

Since relative prices are martingales under Q, each $Z^{(i)}$ belongs to $\mathcal{W}$. In the following, we look for discounted price processes of attainable payoffs in this space of special semimartingales. Moreover, any process u belongs to $\mathcal{W}$ if and only if u/B belongs to $\mathcal{W}$, a fact that will be apparent from the proofs of Proposition 17 and Theorem 18. Therefore, we concentrate directly on (non-discounted) payoff prices in $\mathcal{W}$. In general, special semimartingale prices are compatible with no free lunch conditions, as in Föllmer and Schweizer (1991) and Ansel and Stricker (1992), and they are included in the generic semimartingales required by Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998).

We consider a payoff h_T in $L^1(\mathcal{F}_T)$ associated, for instance, with a traded European derivative. We want to determine a price process π for h_T in $\mathcal{W}$, which is consistent with our arbitrage-free market, namely π/B is a martingale under Q.

Proposition 17. *Let Q be an equivalent martingale measure for Z and let $\pi \in \mathcal{W}$. The following are equivalent:*

(i) *π is a no arbitrage price process;*

(ii) $\mathcal{D}(\pi/B) = 0$*;*

(iii) $\mathcal{D}\pi = r\pi$*.*

Proof. We first prove $(i) \Leftrightarrow (ii)$ and then $(ii) \Leftrightarrow (iii)$.

$(i) \Leftrightarrow (ii)$ π is a no arbitrage price process if and only if π/B is a martingale under Q. By Proposition 4, this is equivalent to claim that the weak time-derivative of π/B is null.

$(ii) \Leftrightarrow (iii)$ For any $t \in [0,T]$, $\pi(t)/B_t = e^{-rt}\pi(t)$. For any $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1([t,T],\mathbb{R})$,

$$\begin{aligned}\int_t^T \mathbb{E}^Q\left[\mathcal{D}\pi(\tau)\mathbf{1}_{A_t}\right]e^{-r\tau}\varphi(\tau)d\tau &= -\int_t^T \mathbb{E}^Q\left[\pi(\tau)\mathbf{1}_{A_t}\right]\left(e^{-rt}\varphi(t)\right)'(\tau)d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[\pi(\tau)\mathbf{1}_{A_t}\right]e^{-r\tau}\varphi'(\tau)d\tau \\ &\quad + \int_t^T \mathbb{E}^Q\left[\pi(\tau)\mathbf{1}_{A_t}\right]e^{-r\tau}r\varphi(\tau)d\tau,\end{aligned}$$

namely

$$\begin{aligned}&\int_t^T \mathbb{E}^Q\left[e^{-r\tau}\left(\mathcal{D}\pi(\tau) - r\pi(\tau)\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\ &\qquad = -\int_t^T \mathbb{E}^Q\left[e^{-r\tau}\pi(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau.\end{aligned}$$

As a result, the weak time-derivative of $e^{-rt}\pi(t)$ is such that

$$\mathcal{D}\left(e^{-rt}\pi(t)\right) = e^{-rt}\left(\mathcal{D}\pi(t) - r\pi(t)\right).$$

Therefore, the weak time-derivative of π/B is null if and only if $\mathcal{D}\pi = r\pi$.

$\square$

In fact, Proposition 17 may be re-read as a characterization of equivalent martingale measures. Specifically, Q is an equivalent martingale measure if and only if it is equivalent to P and, employed in the weak time-derivative, it ensures that $\mathcal{D}\pi = r\pi$ for all asset prices π in the market. Intuitively, this property is reminiscent of the outcome of Girsanov Theorem in Black-Scholes model (illustrated in Subsection 3.1), which makes the drift of the stock price proportional to r under Q.

When the price process is deterministic, point (iii) constitutes the usual differential equation solved by the price of a riskless bond. In particular, the bond price satisfies the boundary problem

$$\begin{cases} \frac{dB}{dt}(t) = rB(t) & t \in [0,T) \\ B(T) = e^{rT} \end{cases}$$

where the classical time-derivative is employed. In words, with continuous compounding, the rate of change of $B(t)$ is proportional to $B(t)$ and the coefficient of proportionality coincides with r.

Proposition 17, in fact, establishes a differential relation which is satisfied by the no arbitrage pricing function π of any risky payoff. For instance, in the case of a European derivative, we formulate the boundary problem

$$\begin{cases} \mathcal{D}\pi(t) = r\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases} \tag{6}$$

where $h_T \in L^1(\mathcal{F}_T)$.

The financial interpretation of the problem is straightforward once we recall that $\mathcal{D}\pi$ is an adapted process and that the filtration $\mathbb{F}$ is right-continuous. In fact, for any $t \in [0,T]$, the infinitesimal variation $\mathcal{D}\pi(t)$ is known at time t and so the no arbitrage condition imposes that $\mathcal{D}\pi$ instantaneously behaves as the deterministic bond. The riskiness of π locally becomes immaterial. In other words, the rate of change of $\pi(t)$ must be proportional to $\pi(t)$ as it is for the riskless asset price. Equivalently, instantaneous returns of h_T, i.e. $\mathcal{D}\pi/\pi$, coincide with the risk-free rate r when arbitrages are forbidden.

We now show that there exists a unique solution of Problem (6) that coincides with the risk-neutral valuation formula of a payoff at time T under the measure Q.

Theorem 18. *There exists a unique solution π of Problem* (6) *in $\mathcal{W}$ and, for every t in* $[0,T]$

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t^Q[h_T]. \tag{7}$$

Proof. • EXISTENCE

In order to show that $\pi \in \mathcal{W}$, we prove that π belongs to $\mathcal{V}$ and that it is weakly time-differentiable.

First, for all $\tau \in [0,T]$, $\pi(\tau) \in L^1(\mathcal{F}_\tau)$ because

$$\mathbb{E}^Q[|\pi(\tau)|] = e^{-r(T-\tau)}\mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q[h_T]\right|\right] \leqslant e^{-r(T-\tau)}\mathbb{E}^Q[|h_T|],$$

which is finite since $h_T \in L^1(\mathcal{F}_T)$. In addition,

$$\begin{aligned}\int_0^T \mathbb{E}^Q[|\pi(\tau)|]\,d\tau &\leqslant \int_0^T e^{-r(T-\tau)}\mathbb{E}^Q[|h_T|]\,d\tau \\ &= \left(\int_0^T e^{-r(T-\tau)}d\tau\right)\mathbb{E}^Q[|h_T|],\end{aligned}$$

which is finite, too.

As for the L^1-continuity, we check that, for any $t \in [0,T)$, $\mathbb{E}^Q[|\pi(\tau)-\pi(t)|]$ tends to zero as $\tau \longrightarrow t^+$. We have

$$\begin{aligned}
\mathbb{E}^Q\left[|\pi(\tau)-\pi(t)|\right] &= \mathbb{E}^Q\left[\left|e^{-r(T-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-e^{-r(T-t)}\mathbb{E}^Q_t\left[h_T\right]\right|\right] \\
&= e^{-r(T-t)}\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right] \\
&\leqslant e^{-r(T-t)}\Bigg\{\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_\tau\left[h_T\right]\right|\right] \\
&\quad +\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right]\Bigg\} \\
&\leqslant e^{-r(T-t)}\Bigg\{\left|e^{-r(t-\tau)}-1\right|\mathbb{E}^Q\left[|h_T|\right] \\
&\quad +\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right]\Bigg\}.
\end{aligned}$$

In the last expression, both addends go to zero as τ approaches t from the right. In particular, the convergence of the second one is ensured by Lévy's Downward Theorem, which guarantees that

$$\mathbb{E}^Q_\tau\left[h_T\right]\xrightarrow{L^1}\mathbb{E}^Q_{t^+}\left[h_T\right]=\mathbb{E}^Q_t\left[h_T\right] \qquad \tau\longrightarrow t^+.$$

Similarly, when $\tau \longrightarrow T^-$,

$$\mathbb{E}^Q\left[|\pi(\tau)-h_T|\right]\leqslant\left\{\left|e^{-r(T-\tau)}-1\right|\mathbb{E}^Q\left[|h_T|\right]+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-h_T\right|\right]\right\}.$$

Here, the convergence of the second term is due to Lévy's Upward Theorem:

$$\mathbb{E}^Q_\tau\left[h_T\right]\xrightarrow{L^1}\mathbb{E}^Q_{T^-}\left[h_T\right]=\mathbb{E}^Q_T\left[h_T\right]=h_T \qquad \tau\longrightarrow T^-.$$

Therefore π belongs to $\mathcal{V}$.

Now we look for the weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and $\varphi \in C^1_c\left([t,T],\mathbb{R}\right)$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau\in[t,T]$, we have

$$\begin{aligned}
-\int_t^T\mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= -\int_t^T\mathbb{E}^Q\left[e^{-r(T-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= -\int_t^T e^{-r(T-\tau)}\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\int_t^T e^{-r(T-\tau)}\varphi'(\tau)d\tau \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\int_t^T re^{-r(T-\tau)}\varphi(\tau)d\tau \\
&= \int_t^T r\mathbb{E}^Q\left[e^{-r(T-\tau)}h_T\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\
&= \int_t^T\mathbb{E}^Q\left[r\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}$$

Therefore, the candidate weak time-derivative of π is $r\pi$. Since $r\pi$ belongs to $\mathcal{V}$, $r\pi$ is effectively the weak time-derivative of π:

$$\mathcal{D}\pi = r\pi.$$

Of course, $\pi(T) = h_T$ and so $\pi \in \mathcal{W}$ solves Problem (6).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}$ be two solutions of Problem (6), that is

$$\mathcal{D}\pi_i = r\pi_i \qquad i = 1, 2,$$

with the boundary condition $\pi_i(T) = h_T$. By defining $z = \pi_1 - \pi_2 \in \mathcal{W}$, we have that

$$\mathcal{D}z = rz$$

and $z(T) = 0$. As in the proof of Proposition 17, the weak time-derivative of $e^{-rt}z(t)$ is

$$e^{-rt}\left(\mathcal{D}z(t) - rz(t)\right).$$

However this process is null. Therefore, $e^{-rt}z(t)$ has null weak time-derivative. Consequently, by Proposition 4, $e^{-rt}z(t)$ defines a martingale and so, for any $t \in [0, T]$ and $\tau \in [t, T]$

$$\mathbb{E}_t^Q[z(\tau)] = e^{r(\tau - t)}z(t).$$

Letting τ go to T^-, we have that

$$\mathbb{E}_t^Q[z(\tau)] \longrightarrow e^{r(T-t)}z(t) \qquad \textit{pointwise}.$$

In addition, $z(\tau)$ converges to $z(T) = 0$ in $L^1(Q)$ as τ approaches T^- and so $\mathbb{E}_t^Q[z(\tau)]$ tends to zero in $L^1(Q)$. By uniqueness of the L^1-limit, we infer that $z(t) = 0$ for all $t \in [0, T]$. This proves uniqueness of the solution of Problem (6). □

To be coherent with the no arbitrage setting, Q must be an equivalent martingale measure for the extended market made by the securities with discounted prices Z and π/B. Therefore, the only possible no arbitrage price process is given by eq. (7) because it satisfies

$$\frac{\pi(t)}{B_t} = \mathbb{E}_t^Q\left[\frac{h_T}{B_T}\right] \tag{8}$$

for all $t \in [0, T]$. We refer to $\pi(t)$ as the *no arbitrage pricing function* (or *risk-neutral pricing function*) of any payoff h_T under Q.

The additional value of Theorem 18 with respect to the existing theory consists in the fact that π belongs to the space $\mathcal{W}$ and is characterized by its dynamics $\mathcal{D}\pi = r\pi$.

Suppose now that the interest rate is deterministic but time-dependent. Moreover, $r(t)$ is assumed to be Lebesgue measurable (over time) and bounded. Then, under additional technical assumptions, the no arbitrage pricing function

$$\pi(t) = e^{-\int_t^T r(s)ds} \mathbb{E}_t^Q [h_T]$$

is the unique solution of problem

$$\begin{cases} \mathcal{D}\pi(t) = r(t)\pi(t) & t \in [0, T) \\ \pi(T) = h_T. \end{cases}$$

We provide a detailed solution of the problem in Section 5, in the more general case of stochastic interest rates.

3.1 Example: Black-Scholes model

Black and Scholes (1973) model involves a continuous-time financial market with a riskless bond with price B and a risky asset with price X. In the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the filtration $\mathbb{F}$ is generated by a P-Wiener process $\mathbf{W}$. The bond and stock prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t d\overline{W}_t,$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and $r \in \mathbb{R}$ is the risk-free rate. Girsanov Theorem ensures that there exists a probability measure Q equivalent to P under which the discounted stock price process is a martingale. According to the First Fundamental Theorem of Asset Pricing the market is, then, arbitrage-free. In particular, the dynamics of the stock price under Q are

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

where W is a Q-Wiener process. Hence, in this geometric Brownian motion setting, the risky security and the bond must share the same drift coefficient given by the interest rate r in order to exclude any arbitrage possibility (see, e.g., Björk (2004)). Specifically, the stock price is

$$X_t = X_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

and it is included in $\mathcal{V}$. Its integral representation is

$$X_t = X_0 + \int_0^t rX_s ds + \int_0^t \sigma X_s dW_s,$$

which individuates a continuous Itô semimartingale. Then, as in Example 8, the weak time-derivative captures the drift, that is

$$\mathcal{D}X_t = rX_t.$$

This relation turns out to be a restatement of the no arbitrage pricing equation of Problem (6) for the stock price under Q.

As described in the initial part of the section, the no arbitrage price process of a European derivative with $\mathcal{F}_T$-measurable payoff h_T is π given by eq. (7). In addition, in Black-Scholes model $\pi(t)$ is a deterministic function of t and X_t. Since the discounted price process $\left\{e^{-rt}\pi(t)\right\}_{t\in[0,T]}$ is also a Q-martingale, the drift of π is equal to r, too. This is the crucial outcome of no arbitrage, which is captured, more in general, by Problem (6), where no specific price dynamics are assumed. This is also the intuition that drives Cox and Ross (1976) derivation of Black-Scholes equation, which is based on a hedging strategy that exploits a locally riskless portfolio.

3.2 Valuation of cashflows

The payoff of a European derivative with maturity T can be seen as a special cashflow in which there is a unique random payment at time T. Indeed, the no arbitrage theory described so far generalizes to the pricing of payoff streams.

In particular, we consider an adapted cashflow h such that $h : [0,T] \longrightarrow L^1(\mathcal{F}_T)$. We assume that h is Bochner integrable with respect to a finite measure μ on $[0,T]$ that weighs cashflows over time.

Given an equivalent martingale measure Q, the no arbitrage price process π of h is the expected discounted value of future cashflows under Q, i.e.

$$\pi(t) = \mathbb{E}_t^Q\left[\int_t^T e^{-r(m-t)}h_m\mu(dm)\right] \tag{9}$$

for all $t \in [0,T]$. For example, if μ is a counting measure, the previous formula evaluates a finite number or a sequence of future payments. In case μ is absolutely continuous, we are pricing instead a continuous stream of payoffs. In the next statement we assume that μ is absolutely continuous with respect to Lebesgue measure on $[0,T]$ and we write $\mu(dm) = p_m dm$, denoting the Radon-Nikodym derivative by p_m.

We also assume that the process $h \cdot p$ belongs to $\mathcal{V}$. Hence, we are able to show that the risk-neutral pricing formula for cashflows satisfies the differential equation

$$\mathcal{D}\pi(t) = r\pi(t) - h_t p_t \qquad t \in [0,T],$$

where $\mathcal{D}\pi$ is the weak time-derivative of π under Q.

Proposition 19. *Let $h \cdot p \in \mathcal{V}$ and $\pi(t) = \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m p_m dm\right]$. Then, π belongs to $\mathcal{W}$ and it solves the equation*

$$\mathcal{D}\pi(t) = r\pi(t) - h_t p_t \qquad t \in [0, T].$$

Proof. We denote $\mu(dm) = p_m dm$. We show that $\pi(t) = \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]$ belongs to $\mathcal{V}$ and it is weakly time-differentiable.

First, for all $\tau \in [0, T]$, $\pi(\tau) \in L^1(\mathcal{F}_\tau)$:

$$\begin{aligned}
\mathbb{E}\left[|\pi(\tau)|\right] &= \mathbb{E}^Q \left[\left|\mathbb{E}_\tau^Q \left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm)\right]\right|\right] \\
&\leqslant \mathbb{E}^Q \left[\int_\tau^T e^{-r(m-\tau)} |h_m| \mu(dm)\right] \\
&\leqslant K\mathbb{E}^Q \left[\int_\tau^T |h_m| \mu(dm)\right] < +\infty
\end{aligned}$$

with $K > 0$. The last quantity is finite because h is Bochner integrable with respect to μ. Similarly,

$$\int_0^T \mathbb{E}^Q \left[|\pi(\tau)|\right] d\tau \leqslant TK\mathbb{E}^Q \left[\int_0^T |h_m| \mu(dm)\right] < +\infty.$$

To establish L^1-continuity, for any $t \in [0, T)$, consider $\tau \longrightarrow t^+$. We have

$$\begin{aligned}
&\mathbb{E}^Q \left[|\pi(\tau) - \pi(t)|\right] \\
&\quad = \mathbb{E}^Q \left[\left|\mathbb{E}_\tau^Q \left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm)\right] - \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\quad \leqslant \mathbb{E}^Q \left[\left|\mathbb{E}_\tau^Q \left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm)\right] - \mathbb{E}_\tau^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\qquad + \mathbb{E}^Q \left[\left|\mathbb{E}_\tau^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\quad = \mathbb{E}^Q \left[\left|e^{-r(t-\tau)}\mathbb{E}_\tau^Q \left[\int_\tau^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_\tau^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\qquad + \mathbb{E}^Q \left[\left|\mathbb{E}_\tau^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right]
\end{aligned}$$

In the last sum, the second term converges to zero as τ approaches t by Lévy's Downward

Theorem. Next, we show that also the first term vanishes:

$$
\begin{aligned}
\mathbb{E}^Q & \left[\left| e^{-r(t-\tau)} \mathbb{E}^Q_\tau \left[\int_\tau^T e^{-r(m-t)} h_m \mu(dm) \right] - \mathbb{E}^Q_\tau \left[\int_t^T e^{-r(m-t)} h_m \mu(dm) \right] \right| \right] \\
& \leqslant \left| e^{-r(t-\tau)} - 1 \right| \mathbb{E}^Q \left[\int_\tau^T e^{-r(m-t)} \left| h_m \right| \mu(dm) \right] \\
& \quad + \mathbb{E}^Q \left[\int_t^\tau e^{-r(m-t)} \left| h_m \right| \mu(dm) \right] \\
& \leqslant \left| e^{-r(t-\tau)} - 1 \right| \mathbb{E}^Q \left[\int_t^T e^{-r(m-t)} \left| h_m \right| \mu(dm) \right] \\
& \quad + \mathbb{E}^Q \left[\int_t^\tau e^{-r(m-t)} \left| h_m \right| \mu(dm) \right].
\end{aligned}
$$

Bochner integrability of h ensures that the last quantities are well-defined and convergent to zero as $\tau \longrightarrow t^+$.

As for L^1-convergence from the left in T, we consider $\tau \longrightarrow T^-$. Since $\pi(T) = 0$, we get

$$
\begin{aligned}
\mathbb{E}^Q \left[|\pi(\tau) - \pi(T)| \right] &= \mathbb{E}^Q \left[\left| e^{-r(T-\tau)} \mathbb{E}^Q_\tau \left[\int_\tau^T e^{-r(m-T)} h_m \mu(dm) \right] \right| \right] \\
&\leqslant e^{-r(T-\tau)} \mathbb{E}^Q \left[\int_\tau^T e^{-r(m-T)} \left| h_m \right| \mu(dm) \right],
\end{aligned}
$$

where the last quantity goes to zero as τ tends to T from the left because of Bochner integrability of h.

In consequence, π belongs to $\mathcal{V}$.

Now we compute the weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and any $\varphi \in C^1_c\left([t,T], \mathbb{R}\right)$. The indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$ and so

$$
\begin{aligned}
- \int_t^T & \mathbb{E}^Q \left[\pi(\tau) \mathbf{1}_{A_t} \right] \varphi'(\tau) d\tau \\
&= - \int_t^T \mathbb{E}^Q \left[\mathbb{E}^Q_\tau \left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm) \right] \mathbf{1}_{A_t} \right] \varphi'(\tau) d\tau \\
&= - \int_t^T \mathbb{E}^Q \left[\int_\tau^T e^{-r(m-\tau)} h_m \mathbf{1}_{A_t} p_m dm \right] \varphi'(\tau) d\tau.
\end{aligned}
$$

because $\mu(dm) = p_m dm$. Since the expectation is a bounded operator, by Lemma 11.45 in Aliprantis and Border (2006) we can exchange it with the integral. Later we apply integration by parts:

$$
\begin{aligned}
& -\int_t^T \mathbb{E}^Q[\pi(\tau)\,\mathbf{1}_{A_t}]\varphi'(\tau)d\tau \\
& = -\int_t^T \left(\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi'(\tau)d\tau \\
& = 0 + \int_t^T \frac{d}{d\tau}\left(\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi(\tau)d\tau \\
& = \int_t^T \left(0 - e^{-r(\tau-\tau)}\mathbb{E}^Q\left[h_\tau p_\tau \mathbf{1}_{A_t}\right] + r\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi(\tau)d\tau \\
& = \int_t^T \left(\mathbb{E}^Q\left[\left(-h_\tau p_\tau\right)\mathbf{1}_{A_t}\right] + \mathbb{E}^Q\left[r\int_\tau^T e^{-r(m-\tau)}h_m p_m \mathbf{1}_{A_t}dm\right]\right)\varphi(\tau)d\tau \\
& = \int_t^T \mathbb{E}^Q\left[\left(-h_\tau p_\tau\right)\mathbf{1}_{A_t} + r\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)}h_m\mu(dm)\right]\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\
& = \int_t^T \mathbb{E}^Q\left[\left(r\pi(\tau) - h_\tau p_\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}
$$

Since both π and $h \cdot p$ belong to $\mathcal{V}$, it follows that $\pi - h \cdot p$ is included in $\mathcal{V}$. Therefore, the latter is the weak time-derivative of π:

$$
\mathcal{D}\pi(t) = r\pi(t) - h_t p_t, \qquad t \in [0,T].
$$

□

Observe that a term analogous to $-h_t p_t$ is added in Feynman-Kac equation when a stream of dividends is present (see, e.g., Duffie (2010)).

Intuitively, if h_t is null except for the time T and μ has mass concentrated in T, we retrieve as special case the differential equation of Problem (6) about individual payoffs T. However, a formal claim about this case requires the theory of distributions and it is beyond the scope of Proposition 19.

4 An operator approach

In this section we define the spaces and operators that allow us to formalize Problem (6) as an eigenvalue-eigenvector problem.

We first introduce some notation, still in the framework of Section 3. We denote by L_T the Radon-Nikodym derivative of the risk-neutral measure Q with respect to the physical measure P. Setting $L_t = \mathbb{E}_t[L_T]$ for all $t \in [0,T]$, we rewrite the no arbitrage price at any time t as

$$
\pi(t) = e^{-r(T-t)}\mathbb{E}_t\left[\frac{L_T}{L_t}h_T\right].
$$

We can also restate the martingale property of discounted prices under Q by saying that the process $\left\{e^{-rt}L_t\pi\left(t\right)\right\}_{t\in[0,T]}$ is a martingale under the physical measure. In addition, the measure Q induces a stochastic discount factor process S that, at any $t\in[0,T]$, takes the form $S_t=e^{-rt}L_t$. See, for instance, Hansen and Richard (1987) and Björk (2004) as general references on risk-neutral pricing.

The starting point of our derivation is the observation that the no arbitrage pricing function π is weakly time-differentiable infinitely many times. Indeed, $\mathcal{D}\pi$ belongs to $\mathcal{V}$ and it equals the original π except for the multiplicative constant r. Hence, $\mathcal{D}\pi$ is weakly time-differentiable, too. By defining the subspace of $\mathcal{W}$

$$\mathcal{Z}=\left\{\text{infinitely weakly time-differentiable } u\in\mathcal{V}\right\},$$

we have that $\pi\in\mathcal{Z}$. Moreover, the weak time-derivative defines a linear operator $\mathcal{D}:\mathcal{Z}\longrightarrow\mathcal{Z}$ that maps any $u\in\mathcal{Z}$ to $\mathcal{D}u$. Therefore, the differential equation of Problem (6) delivers the eigenvalue-eigenvector problem

$$\mathcal{D}\pi=r\pi,\qquad \pi\in\mathcal{Z}, \tag{10}$$

which rephrases the one faced by Hansen and Scheinkman (2009) where, instead of $\mathcal{D}$, the *extended generator* of the underlying Markov process is involved. In our setting Markovianity is not required and the no arbitrage pricing function π is an eigenfunction of the operator $\mathcal{D}$, defined through weak time-derivatives. Moreover, the process $\left\{e^{-rt}L_t\pi\left(t\right)\right\}_t$ is a martingale under P.

Following Hansen and Scheinkman (2009), we choose a positive payoff h_T. The positivity of h_T is related to the requirement of π to be an eigenfunction related to the *principal eigenvalue* in Hansen and Scheinkman (2009). Indeed, Hansen and Scheinkman generalize the Perron-Frobenius theory (see Meyer (2000)) from the finite-state Markov chain setting to more abstract frameworks.

Then, we define

$$\hat{L}_t=e^{-rt}L_t\frac{\pi\left(t\right)}{\pi\left(0\right)},$$

which still satisfies the martingale property. In addition, the stochastic discount factor S_t decomposes as

$$S_t=\hat{L}_t\frac{\pi\left(0\right)}{\pi\left(t\right)}=e^{-rt}\hat{L}_t\frac{\tilde{\pi}\left(0\right)}{\tilde{\pi}\left(t\right)},$$

where we define $\tilde{\pi}\left(t\right)=\mathbb{E}_t\left[L_{t,T}h_T\right]$. In the last decomposition $-r$ is referred to as the *growth rate* of S_t, $\hat{L}_t$ is the *martingale component* and $\tilde{\pi}\left(0\right)/\tilde{\pi}\left(t\right)$ is the *transient component*. However, the decomposition is not unique.

This kind of results has proved to be fruitful in the macro-financial literature. For instance, Alvarez and Jermann (2005) employ the last decomposition to quantify the dynamics of stochastic discount factors. Moreover, an application to the study of long-term risk-return trade-off for the valuation of cash flows is described in Hansen, Heaton, and Li (2008).

The use of weak time-derivatives allows to extend the applicability of Hansen-Scheinkman decomposition to a wide class of special semimartingales. The crucial point of the construction is the identification of martingales through null weak time-derivatives.

4.1 Comparison with the infinitesimal generator

As we saw in Proposition 13, the weak time-derivative provides a way to differentiate random processes which generalizes the infinitesimal generator for Feller processes X and the extended infinitesimal generator for Markov processes. By focusing on the first one, if the infinitesimal generator of f is null, then the process $\{f(X_t)\}_{t\in[0,T]}$ is a martingale, a fact that parallels Proposition 4. In particular, simple computations show that the no arbitrage pricing function of eq. (7) satisfies the eigenvalue-eigenvector problem $\mathcal{A}\pi = r\pi$. Hence, we can refer to $\mathcal{A}\pi = r\pi$ as a *strong form* eigenvalue-eigenvector problem, while Problem (6), rewritten as (10), defines a *generalized form.*

In addition, it holds that $\mathcal{A}\left(e^{-rt}\pi(t)\right) = 0$, hence the discounted price process $\left\{e^{-rt}\pi(t)\right\}_{t\in[0,T]}$ is a martingale under Q. By exploiting the terminal condition $\pi(T) = h_T$, this fact ensures that π is the unique solution of the problem in strong form.

In fact, we followed a parallel path of reasoning with weak time-derivatives, but with relevant differences: the class of processes involved and the continuity required (L^1 instead of uniform topology). An analogous remark is valid for the extended infinitesimal generator.

5 No arbitrage pricing with stochastic interest rates

We provide a refinement of our theory in order to solve the no arbitrage pricing differential equation when interest rates are stochastic.

In this case we have two sources of randomness described by processes X and r defined on the probability space $(\Omega, \mathcal{F}, P)$ over the time interval $[0,T]$. As in Section 3, X is associated with the N-dimensional process of underlying stock prices. In addition, r is one-dimensional and represents stochastic instantaneous rates. Here, $B_t = e^{\int_0^t r(s)ds}$ for all $t \in [0,T]$ and normalized prices are defined by $Z_t = e^{-\int_0^t r(s)ds} X_t$. Hence, we consider the filtration generated by the pair (Z, r), which we denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$.

In this section we assign a stronger meaning to weak time-differentiability. To distinguish the new definition from that of Section 2 we add *r-*, which stays for *robust*. Indeed, *r-weak time-differentiability* involves a larger set of test functions than weak time-differentiability of Definition 1. Specifically, we employ as test functions all adapted processes in $C_c^1([t,T], L^\infty(\mathcal{F}_T))$. This space is composed of functions $\varphi : [0,T] \longrightarrow L^\infty(\mathcal{F}_T)$ with compact support that are continuously differentiable in the following sense: for any $t \in [0,T]$, $\varphi(t) = \int_0^t \psi(\tau)d\tau$ with $\psi : [0,T] \longrightarrow L^\infty(\mathcal{F}_T)$ adapted, continuous and with compact support. Hence, r-weak time-differentiability is a stronger requirement than weak time-differentiability.

Definition 20. *Given $u \in \mathcal{V}$, we say that u is* r-weakly time-differentiable *when there exists $w \in \mathcal{V}$ such that for every $t \in [0,T]$*

$$\int_t^T \mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau = -\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T)) \quad \textit{adapted.}$$

In this case we call w a r-weak time-derivative *of u.*

Definition 20 is well-posed because the integrals

$$\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau \quad \text{and} \quad \int_t^T \mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau$$

are finite for any choice of A_t and φ, as required. Indeed, φ and φ' are continuous functions that take values in $L^\infty(\mathcal{F}_T)$, so their image is bounded.

We finally define the space

$$\mathcal{W}_r = \{\text{r-weakly time-differentiable } u \in \mathcal{V}\}.$$

If u is r-weakly time-differentiable, it is also weakly time-differentiable because the test functions φ, that are random processes, may specialize to deterministic functions. This simple observation allows us to inherit some of the results of Section 2. For instance, the r-weak time-derivative is still unique. Moreover, if a process $u \in \mathcal{V}$ is r-weakly time-differentiable with $\mathcal{D}u = 0$, then it is a martingale.

In our financial application, we assume that instantaneous rates define an adapted process $r : [0,T] \longrightarrow L^\infty(\mathcal{F}_T)$. In addition, we impose that interest rates are uniformly bounded over time, i.e. there is a positive $\tilde{R}$ such that

$$|r(t)| \leqslant \tilde{R} \qquad \forall t \in [0,T]$$

and that they are L^2-right-continuous in any $t \in [0,T)$ and L^2-left-continuous in T. Progressive measurability (which holds up to modifications) and boundedness ensure the Bochner integrability of r. As a result, the Bochner integral $\int_0^T r(\tau)d\tau$ is a well-defined object in $L^\infty(\mathcal{F}_T)$.

By following the same line of reasoning of Section 3, from NFLVR we infer the existence of a sigma-martingale measure Q, which we assume to be an equivalent martingale measure. Thus, we move to the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Since now interest rates are stochastic, under the measure Q the no arbitrage pricing differential equation is

$$\begin{cases} \mathcal{D}\pi(t) = r(t)\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases} \tag{11}$$

where we assume $h_T \in L^2(\mathcal{F}_T)$. Differently from Problem (6), now each $r(t) \in L^\infty(\mathcal{F}_t)$ and $\mathcal{D}$ represents the r-weak time-derivative. We now show the unique solution of this problem in $\mathcal{W}_r$.

Theorem 21. *Under the previous assumptions on r, there exists a unique solution π of Problem (11) in $\mathcal{W}_r$ and, for every t in $[0,T]$*

$$\pi(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds} h_T\right]. \tag{12}$$

Proof. • EXISTENCE

In order to show that $\pi \in \mathcal{W}_r$, we prove that π belongs to $\mathcal{V}$ and that it is r-weakly time-differentiable.

First, for all $\tau \in [0,T]$, $\pi(\tau) \in L^1(\mathcal{F}_\tau)$ because r is uniformly bounded. Hence, for some $K > 0$,

$$\mathbb{E}^Q[|\pi(\tau)|] \leqslant \mathbb{E}^Q\left[e^{-\int_\tau^T r(s)ds}|h_T|\right] \leqslant K\mathbb{E}^Q[|h_T|]$$

and the last quantity is finite because $h_T \in L^1(\mathcal{F}_T)$. In addition,

$$\int_0^T \mathbb{E}^Q[|\pi(\tau)|]\,d\tau \leqslant KT\mathbb{E}^Q[|h_T|],$$

which is finite.

As for L^1-continuity, for all $t \in [0,T)$, consider $\tau \longrightarrow t^+$. Then,

$$\begin{aligned}
\mathbb{E}^Q\left[|\pi(\tau)-\pi(t)|\right] &= \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]\\
&\leqslant \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right]-\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]\\
&= \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\left(e^{-\int_\tau^t r(s)ds}-1\right)\right]\right|\right]\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]\\
&\leqslant \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds}\left|h_T\right|\left|e^{-\int_\tau^t r(s)ds}-1\right|\right]\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]\\
&\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|e^{-\int_\tau^t r(s)ds}-1\right|\right]\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]
\end{aligned}$$

because r is uniformly bounded by $\tilde{R}$. Moreover, we can apply Lagrange's Theorem to the continuously differentiable function $\sigma \longmapsto e^{-\int_\sigma^t r(s)ds}$ for all $\sigma \in [t,\tau]$. Hence, we find $\hat{t} \in (t,\tau)$ such that

$$e^{-\int_\tau^t r(s)ds}-1 = r\left(\hat{t}\right)e^{-\int_{\hat{t}}^t r(s)ds}(\tau-t), \tag{13}$$

so that

$$\left|e^{-\int_\tau^t r(s)ds}-1\right| = \left|r\left(\hat{t}\right)\right|e^{-\int_{\hat{t}}^t r(s)ds}(\tau-t).$$

As a result,

$$\begin{aligned}
e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|e^{-\int_\tau^t r(s)ds}-1\right|\right] &= e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r\left(\hat{t}\right)\right|e^{-\int_{\hat{t}}^t r(s)ds}\right](\tau-t)\\
&\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\right]\tilde{R}e^{(t-\hat{t})\tilde{R}}(\tau-t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}^Q\left[|\pi(\tau)-\pi(t)|\right] &\leqslant e^{(T-\hat{t})\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\right](\tau-t)\tilde{R}\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right].
\end{aligned}$$

In the last expression, both terms tend to zero as τ reaches t^+. Specifically, the convergence of the second one is guaranteed by Lévy's Downward Theorem.

Now we concentrate on L^1-convergence in T from the left and we take $\tau \longrightarrow T^-$. Similarly to before, we get

$$\begin{aligned}
\mathbb{E}^Q\left[|\pi(\tau)-\pi(T)|\right] &\leqslant e^{(T-\hat{t})\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\right](T-\tau)\tilde{R}\\
&\quad+\mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-h_T\right|\right]
\end{aligned}$$

and the convergence of the last term is due to Lévy's Upward Theorem.

Therefore π belongs to $\mathcal{V}$.

Now we look for the r-weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and any adapted $\varphi \in C_c^1\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right)$. Recall that indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since φ' is adapted too, we deduce that

$$\begin{aligned}
-\int_t^T & \mathbb{E}^Q\Big[\pi\left(\tau\right)\mathbf{1}_{A_t}\varphi'(\tau)\Big] d\tau \\
&= -\int_t^T \mathbb{E}^Q\left[\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi'(\tau)\right] d\tau \\
&= -\int_t^T \mathbb{E}^Q\left[e^{-\int_\tau^T r(s)ds}h_T\mathbf{1}_{A_t}\varphi'(\tau)\right] d\tau.
\end{aligned}$$

$e^{-\int_\tau^T r(s)ds}\varphi'(\tau)$ is a continuous function of $\tau \in [t,T]$, hence it is Bochner integrable. The expectation is a bounded operator, so Lemma 11.45 in Aliprantis and Border (2006) allows us to exchange expectation and integral. Therefore,

$$\begin{aligned}
-\int_t^T \mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\varphi'(\tau)\right] d\tau &= -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T e^{-\int_\tau^T r(s)ds}\varphi'(\tau)d\tau\right] \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r(s)ds}\right)\varphi'(\tau)d\tau\right] \\
&\quad - \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\varphi'(\tau)d\tau\right] \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r(s)ds}\right)\varphi'(\tau)d\tau\right]
\end{aligned}$$

because φ has compact support. Now consider the function $u \longmapsto r(u)e^{-\int_u^T r(s)ds}$. This function is Bochner integrable (because r is uniformly bounded) and its Bochner integral coincides almost surely with the pathwise Lebesgue integral. For any state $\omega \in \Omega$ the restriction r_ω of r satisfies:

$$\int_\tau^T r_\omega(u)e^{-\int_u^T r_\omega(s)ds}du = \left[e^{-\int_u^T r_\omega(s)ds}\right]_\tau^T = 1 - e^{-\int_\tau^T r_\omega(s)ds}.$$

In consequence, the Bochner integral is

$$\int_\tau^T r(u)e^{-\int_u^T r(s)ds}du = 1 - e^{-\int_\tau^T r(s)ds}.$$

By exploiting integration by parts (see Craven (1970)), we obtain

$$
\begin{aligned}
&-\int_t^T \mathbb{E}^Q\Big[\pi\left(\tau\right)\mathbf{1}_{A_t}\varphi'(\tau)\Big]d\tau \\
&\qquad\qquad = \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(\int_\tau^T r(u)e^{-\int_u^T r(s)ds}du\right)\varphi'(\tau)d\tau\right] \\
&\qquad\qquad = \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)d\tau\right] \\
&\qquad\qquad = \int_t^T \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)\right]d\tau \\
&\qquad\qquad = \int_t^T \mathbb{E}^Q\left[r(\tau)\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&\qquad\qquad = \int_t^T \mathbb{E}^Q\left[r(\tau)\pi(\tau)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau.
\end{aligned}
$$

Therefore, the candidate r-weak time-derivative of π is $r\pi$ and $r(t)\pi(t)$ belongs to $L^1\left(\mathcal{F}_t\right)$ for all t because r is bounded. As for L^1-continuity, let τ go to t^+ for any $t \in [0,T)$. Then,

$$
\begin{aligned}
&\mathbb{E}^Q\Big[\Big|r(\tau)\pi(\tau) - r(t)\pi(t)\Big|\Big] \\
&\qquad = \mathbb{E}^Q\left[\left|r(\tau)\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right] - r(t)\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad \leqslant \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(\tau)e^{-\int_\tau^T r(s)ds}h_T\right] - \mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}^Q_t\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad \leqslant \mathbb{E}^Q\left[\left|r(\tau)e^{-\int_\tau^T r(s)ds}h_T - r(t)e^{-\int_t^T r(s)ds}h_T\right|\right] \\
&\qquad\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}^Q_t\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right].
\end{aligned}
$$

By exploiting the uniform boundedness of r and eq. (13), we find that the first addend in the last expression satisfies

$$
\begin{aligned}
&\mathbb{E}^Q\left[\left|r(\tau)e^{-\int_\tau^T r(s)ds}h_T - r(t)e^{-\int_t^T r(s)ds}h_T\right|\right] \\
&\qquad = \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds}\left|h_T\right|\left|r(\tau)e^{-\int_\tau^t r(s)ds} - r(t)\right|\right] \\
&\qquad \leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau)e^{-\int_\tau^t r(s)ds} - r(t)\right|\right] \\
&\qquad = e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau) + r(\tau)r\left(\hat{t}\right)e^{-\int_{\hat{t}}^t r(s)ds}(\tau - t) - r(t)\right|\right] \\
&\qquad \leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau) - r(t)\right|\right] \\
&\qquad\quad + e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau)r\left(\hat{t}\right)\right|e^{-\int_{\hat{t}}^t r(s)ds}\right]\left|\tau - t\right|.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathbb{E}^{Q}&\left[\left|r(\tau)\pi(\tau)-r(t)\pi(t)\right|\right] \\
&\leqslant e^{(T-t)\tilde{R}}\left(\mathbb{E}^{Q}\left[h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^{Q}\left[\left|r(\tau)-r(t)\right|^2\right]\right)^{\frac{1}{2}} \\
&\quad+e^{(T-\hat{t})\tilde{R}}\tilde{R}^2\mathbb{E}^{Q}\left[\left|h_T\right|\right]\left|\tau-t\right| \\
&\quad+\mathbb{E}^{Q}\left[\left|\mathbb{E}_{\tau}^{Q}\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]-\mathbb{E}_{t}^{Q}\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right].
\end{aligned}
$$

As τ approaches t from the right, the first term goes to zero because $h_T \in L^2\left(\mathcal{F}_T\right)$ and r is L^2-right-continuous; the second term tends to zero because r is uniformly bounded; the last term is convergent to zero by Lévy's Downward Theorem. Therefore, the L^1-right-continuity is proved.

In order to establish L^1-left-continuity in T, we observe that, by analogous steps, we obtain

$$
\begin{aligned}
\mathbb{E}^{Q}&\left[\left|r(\tau)\pi(\tau)-r(T)\pi(T)\right|\right] \\
&\leqslant\left(\mathbb{E}^{Q}\left[h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^{Q}\left[\left|r(\tau)-r(T)\right|^2\right]\right)^{\frac{1}{2}} \\
&\quad+\tilde{R}^2\mathbb{E}^{Q}\left[\left|h_T\right|\right]\left|\tau-T\right|,
\end{aligned}
$$

which goes to zero by L^2-left-continuity of r in T.

Hence, $r\pi$ belongs to $\mathcal{V}$ and it is the r-weak time-derivative of π:

$$\mathcal{D}\pi\left(t\right)=r(t)\pi\left(t\right).$$

Summing up, we showed that $\pi \in \mathcal{W}_r$ and it solves Problem (11).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}_r$ be two solutions of Problem (11), that is for every $t \in [0,T]$

$$\mathcal{D}\pi_i\left(t\right)=r(t)\pi_i\left(t\right) \qquad i=1,2,$$

with $\pi_i\left(T\right)=h_T$. By defining $z=\pi_1-\pi_2 \in \mathcal{W}_r$, we have that, for every $t \in [0,T]$,

$$\mathcal{D}z\left(t\right)=r(t)z\left(t\right)$$

and $z\left(T\right)=0$.

The process r is Bochner integrable over time. Reasoning state by state, we have

$$\int_t^T r_\omega(s)ds=R_\omega(T)-R_\omega(t),$$

where R_ω is a primitive of r_ω. By denoting with R the random variable that collects all R_ω, it follows that the Bochner integral of r is

$$\int_t^T r(s)ds = R(T) - R(t).$$

Now we show that the r-weak time-derivative of the process $e^{-R(t)}z(t)$ is

$$e^{-R(t)}\left(\mathcal{D}z(t) - r(t)z(t)\right).$$

For any adapted $\varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T))$, consider the function

$$u \longmapsto e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\varphi'(u).$$

Since r is bounded, this function is Bochner integrable. By reasoning pathwise, it follows that

$$\int_\tau^T \left(e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\varphi'(u)\right)du = e^{-R(\tau)}\varphi(\tau).$$

Hence, $e^{-R}\varphi$ is adapted, it belongs to $C_c^1([t,T], L^\infty(\mathcal{F}_T))$ and so we can use it as test function in the definition of r-weak time-derivative of z:

$$\begin{aligned}
&\int_t^T \mathbb{E}^Q\left[\mathcal{D}z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}\varphi(\tau)\right]d\tau \\
&\qquad = -\int_t^T \mathbb{E}^Q\left[z(\tau)\mathbf{1}_{A_t}\left(e^{-R(\tau)}\varphi'(\tau) - e^{-R(\tau)}r(\tau)\varphi(\tau)\right)\right]d\tau \\
&\qquad = -\int_t^T \mathbb{E}^Q\left[z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}\varphi'(\tau)\right]d\tau \\
&\qquad\quad + \int_t^T \mathbb{E}^Q\left[z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}r(\tau)\varphi(\tau)\right]d\tau
\end{aligned}$$

that is

$$\begin{aligned}
&\int_t^T \mathbb{E}^Q\left[e^{-R(\tau)}\Big(\mathcal{D}z(\tau) - r(\tau)z(\tau)\Big)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&\qquad\qquad = -\int_t^T \mathbb{E}^Q\left[e^{-R(\tau)}z(\tau)\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau.
\end{aligned}$$

This means that the r-weak time-derivative of $e^{-R(t)}z(t)$ is

$$e^{-R(t)}\left(\mathcal{D}z(t) - r(t)z(t)\right).$$

However this process is null. Therefore, $e^{-R(t)}z(t)$ has null r-weak time-derivative. Consequently, by following the proof of Proposition 4 for test functions in $C_c^1([t,T], L^\infty(\mathcal{F}_T))$, $e^{-R(t)}z(t)$ constitutes a martingale: for every $t \in [0,T]$ and $\tau \in [t,T]$

$$\mathbb{E}_t^Q\left[e^{-R(\tau)}z(\tau)\right] = e^{-R(t)}z(t).$$

As τ approaches T from the left, $\mathbb{E}_t^Q\left[e^{-R(\tau)}z\left(\tau\right)\right]$ goes to zero in $L^1(Q)$. Indeed, since $e^{-R(\tau)}$ is bounded,

$$\begin{aligned}\mathbb{E}^Q\left[\left|\mathbb{E}_t^Q\left[e^{-R(\tau)}z(\tau)\right]-0\right|\right] &\leqslant \mathbb{E}^Q\left[\mathbb{E}_t^Q\left[\left|z(\tau)\right|e^{-R(\tau)}\right]\right]\\ &\leqslant C\mathbb{E}^Q\left[\left|z(\tau)\right|\right]\end{aligned}$$

for some $C > 0$. However, the last term converges to zero because $z(\tau)$ tends to $z(T) = 0$ in $L^1(Q)$ as τ approaches T^-.

By uniqueness of the L^1-limit, we infer that $e^{-R(t)}z(t) = 0$. As a result, $z(t) = 0$ for all $t \in [0, T]$ and uniqueness of the solution of Problem (11) is established. □

6 Conclusions

We introduced the weak time-derivative, a novel mathematical tool that allows us to differentiate stochastic processes in a more general way than infinitesimal generators. It provides easy characterizations of martingales and permits to formulate differential equations for random processes in weak form. Therefore, we expect this instrument to be suitable for different kinds of differential problems, beyond the ones discussed in this work.

As we described in the paper, a fruitful application of the weak time-derivative involves the solution of the no arbitrage pricing equation for random payoffs. In particular, the generalized form that we solve clarifies the central role of interest rates in driving the asset prices, with both deterministic and stochastic short-term rates. In addition, constant interest rates deliver an eigenvalue-eigenvector formulation of the risk-neutral pricing equation in full agreement with the long-term risk literature. Nevertheless, how to set up the analogous eigenvalue-eigenvector problem when interest rates are time-varying or stochastic still remains an open problem. Indeed, the candidate eigenvalue would be a function or a random process. Moreover, such a formulation should be able to generate a term structure of interest rates. We leave this question for future research.

Another promising direction of research comes from the analysis of price dynamics through the lenses of different risk-neutral measures, associated with specific numéraire changes (see Geman et al (1995)). Indeed, the risk-free rate r is an eigenvalue when the measure Q is employed. Under a different equivalent martingale measure (as, for instance, the forward measure in the context of stochastic rates), the first question to answer is whether a differential equation with the structure of eq. (6) is still valid. A plausible possibility is that the same dynamics are present, but the driving parameter is not r. Hence, the second question is the identification of the proper eigenvalue according to the employed measure. This approach, made possible by the flexibility of weak time-derivatives, opens a promising perspective on payoff valuation in various contexts, while remaining within the boundaries of no arbitrage theory. We will deal with these aspects in future works.

A Lemma

Lemma 1. *Let* $f : [t, T] \longrightarrow \mathbb{R}$.

(i) *If* f *is bounded, nonnegative, with compact support and* $\int_t^T f(\tau)g(\tau)d\tau = 0$ *for any* $g \in C_c([t, T], \mathbb{R})$, *then* $f = 0$ *a.e.*

(ii) *If* f *is measurable and* $\int_t^T f(\tau)g(\tau)d\tau = 0$ *for any* $g \in C_c([t, T], \mathbb{R})$, *then* $f = 0$ *a.e.*

Proof. (i) If f is strictly positive on a set A with positive measure, consider the indicator function $\mathbf{1}_A$ and a sequence $\{U_n\}_n$ of continuous positive approximations of $\mathbf{1}_A$, obtained by convolution with a smooth positive kernel. As U_n converges to $\mathbf{1}_A$ in L^2,

$$0 \leqslant \int_t^T f(\tau)\mathbf{1}_A(\tau)d\tau = \lim_n \int_t^T f(\tau)U_n(\tau)d\tau = 0.$$

In consequence, f is null a.e.

(ii) Suppose that f is positive with compact support. For any $N > 0$ consider $f_N(s) = \min\{f(\tau), N\}$. Then

$$0 \leqslant \int_t^T f_N(\tau)g(\tau)d\tau \leqslant \int_t^T f(\tau)g(\tau)d\tau = 0.$$

Therefore, each f_N is null a.e. by (i) and so f is.

$\square$

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