Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles*

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Abstract

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Abstract

We model consumption and dividend growth rates as containing (i) a small long-run predictable component and (ii) fluctuating economic uncertainty. The magnitudes of the predictable variation and changing volatility in growth rates are quite small. These growth rate dynamics, for which we provide empirical support, in conjunction with plausible parameter configurations of the Epstein and Zin (1989) preferences, can explain key observed asset markets phenomena. Our model captures the intuition that financial markets dislike economic uncertainty and that economic growth prospects are important for asset prices. We show that the model can justify the observed equity premium, the risk-free rate, and the ex-post volatilities of the market return, real risk-free rate, and the price-dividend ratio. As in the data, the model also implies that dividend yields predict returns and that market return volatility is stochastic.

1 Introduction

Several key aspects of asset market data pose a serious challenge to economic models.[1] It is difficult to justify the 6% equity premium and the low risk-free rate (see Mehra and Prescott (1985), Weil (1989), and Hansen and Jagannathan (1991)). The literature on variance bounds highlights the difficulty in justifying the market volatility of 19% per annum (see Shiller (1981) and LeRoy and Porter (1981)). The conditional variance of the market return, as shown by Bollerslev, Engle, and Wooldridge (1988), fluctuates across time and is very persistent. Price-dividend ratios seem to predict long horizon equity returns (see Campbell and Shiller (1988b)). In addition, as documented in this paper, realized volatility of consumption growth rates and price-dividend ratios are significantly negatively correlated at long leads and lags.

We present a model that helps explain the above features of asset market data. There are two main ingredients in the model. First, we rely on the standard Epstein and Zin (1989) preferences, which allow for a separation between the intertemporal elasticity of substitution (IES) and risk aversion, and consequently permit both parameters to be simultaneously larger than one. Second, we model consumption and dividend growth rates as containing (i) a small persistent expected growth rate component, and (ii) fluctuating volatility, which captures time-varying economic uncertainty. We show that this specification for consumption and dividends is consistent with observed annual consumption and dividend data. In our economy, when IES is larger than one, agents demand large equity risk premia because they fear that a reduction in economic growth prospects or a rise in economic uncertainty will lower asset prices. Our results show that risks related to varying growth prospects and fluctuating economic uncertainty can quantitatively justify many of the observed features of asset market data.

Why is persistence in the growth prospects important? In a partial equilibrium model, Barsky and DeLong (1993) show that persistence in expected dividend growth rates is an important source of volatility of price-dividend ratios. In our equilibrium model the degree of persistence in expected growth rate news affects the volatility of the price-dividend ratio and also determines the risk premium on the asset. News regarding future expected growth rates leads to large reactions in the price-dividend ratio and the ex-post equity return; these reactions positively co-vary with the marginal rate of substitution of the representative agent, and hence lead to large equity risk premia. The dividend elasticity of asset prices and the

[1] Notable papers addressing asset market anomalies include Abel (1990), Abel (1999), Bansal and Coleman (1997), Barberis, Huang, and Santos (2001), Campbell (1996), Campbell and Cochrane (1999), Cecchetti, Lam, and Mark (1990), Chapman (2002), Constantinides (1990), Constantinides and Duffie (1996), Hansen, Sargent, and Tallarini (1999), Heaton (1995), Heaton and Lucas (1996), and Kandel and Stambaugh (1991).

risk premia on assets rise as the degree of permanence of expected dividend growth rates increases. We formalize this intuition in section 2 with a simple version of the model that incorporates only fluctuations in growth prospects.

To allow for time-varying risk premia we incorporate changes in the conditional volatility of future growth rates. Fluctuating economic uncertainty (conditional volatility of consumption) directly affects price-dividend ratios and a rise in economic uncertainty leads to a fall in asset prices. In our model, shocks to economic uncertainty carry a positive risk premium. About half of the volatility of price-dividend ratios in the model can be attributed to variation in expected growth rates and the remaining to variation in economic uncertainty. This is distinct from models where growth rates are *i.i.d.*, and consequently, all the variation in price-dividend ratio is attributed to changing cost of capital.

Our specification for growth rates emphasizes persistent movements in expected growth rates and fluctuations in economic uncertainty. For these channels to have a significant quantitative impact on the risk premium and volatility of asset prices, the persistence in expected growth rate has to be quite large, close to 0.98.[2] A pertinent question is whether this is consistent with growth rate data as observed autocorrelations in realized growth rates of consumption and dividends are small. Shephard and Harvey (1990) and Barsky and DeLong (1993) show that in finite samples it is very difficult to distinguish between a purely *i.i.d.* process and one which incorporates a small persistent component. While it is hard to econometrically distinguish between the two alternative processes, the asset pricing implications across them are very different. We show that our specification for the consumption and dividend growth rates, which incorporates the persistent component, is consistent with the growth rate data and helps justify several puzzling aspects of asset market data.

We provide direct empirical evidence for fluctuating consumption volatility, which motivates our time-varying economic uncertainty channel. In annual data, price-dividend ratios are significantly correlated with consumption volatility at long leads and lags. The variance ratios of realized consumption volatility increase up to 10 years. If residuals of consumption growth are *i.i.d.*, then the absolute value of these residuals will not be predictable and the variance ratios will be flat across different horizons.

In terms of preferences, all our results are based on a risk aversion of less than 10 and an IES that is somewhat larger than 1. There is considerable debate about what are reasonable magnitudes for these parameters. Mehra and Prescott (1985) argue that risk aversion of 10 and below seems reasonable. As discussed below, our estimates of the IES are consistent

[2] Barsky and DeLong (1993) choose a value of 1. Our choice ensures that the growth rate process is stationary.

with the findings of Hansen and Singleton (1982) and many other authors. In addition, we show that the presence of fluctuating economic uncertainty leads to a serious downward bias in estimating the IES when using the regression approach used in Hall (1988). This bias may help interpret Hall's small estimates of the IES.

The remainder of the paper is organized as follows. In section 2 we formalize this intuition and present the economics behind our model. The data and the model's quantitative implications are described in section 3. The last section provides concluding comments.

2 An Economic Model for Asset Markets

We consider a representative agent with the Epstein and Zin (1989) - Weil (1989) recursive preferences. For these preferences, Epstein and Zin (1989) show that the asset pricing restrictions for gross return $R_{i,t+1}$ satisfy

$$E_t[\delta^{\theta} G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1, \tag{1}$$

where G_{t+1} is the aggregate gross growth rate of consumption, R_a is the gross return on an asset paying off aggregate consumption, $0 < \delta < 1$ is the time discount factor, $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$, where $\gamma \geq 0$ is the risk-aversion (sensitivity) parameter, and $\psi \geq 0$ is the intertemporal elasticity of substitution. The sign of θ is determined by the magnitudes of the risk aversion and the elasticity of substitution.[3]

We distinguish between the *unobservable* return on a claim to aggregate consumption, $R_{a,t+1}$, and the *observable* return on the market portfolio $R_{m,t+1}$; the latter is the return on the aggregate dividend claim. As in Campbell (1996), we model aggregate consumption and aggregate dividends as two separate processes; the agent is implicitly assumed to have access to labor income.

Although we solve our model numerically, we demonstrate the mechanisms working in our model via approximate analytical solutions. To derive these solutions for the model we use the standard approximations utilized in Campbell and Shiller (1988a),

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{2}$$

where lowercase letters refer to logs, so that $r_{a,t+1} = \log(R_{a,t+1})$ is the continuous return, $z_t = \log(P_t/C_t)$ is the log price-consumption ratio, and κ_0 and κ_1 are approximating

[3]In particular, if $\psi > 1$ and $\gamma > 1$ then θ will be negative. Note that when $\theta = 1$, that is $\gamma = (1/\psi)$, the above recursive preferences collapse to the standard case of expected utility. Further, when $\theta = 1$ and in addition $\gamma = 1$, we get the standard case of log utility.

constants that both depend only on the average level of z.[4] Analogously, $r_{m,t+1}$ and $z_{m,t}$ correspond to the market return and its log price-dividend ratio.

The logarithm of the Intertemporal Marginal Rate of Substitution (IMRS) is

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}.$$

It follows that the innovation in m_{t+1} is driven by the innovations in g_{t+1} and $r_{a,t+1}$. Covariation with the innovation in m_{t+1} determines the risk premium for any asset. When θ equals one, the above IMRS collapses to the usual case of power utility. To present the intuition of our model in a simple manner we first discuss the case (Case I) in which there are fluctuations only in the expected growth rates. Subsequently, we present the complete model (Case II), which also includes fluctuating economic uncertainty.

2.1 Case I: Fluctuating Expected Growth Rates Only

We first solve for the consumption return $r_{a,t+1}$, as this determines the pricing kernel and consequently risk premia on the market portfolio, $r_{m,t+1}$, as well as all other assets. To do so we first specify the dynamics for consumption and dividend growth rates. We model consumption and dividend growth rates, g_{t+1} and $g_{d,t+1}$, respectively, as containing a small persistent predictable component x_t, which determines the conditional expectation of consumption growth,

$$\begin{aligned} x_{t+1} &= \rho x_t + \varphi_e \sigma e_{t+1} \\ g_{t+1} &= \mu + x_t + \sigma \eta_{t+1} \\ g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma u_{t+1} \\ & \qquad e_{t+1}, u_{t+1}, \eta_{t+1} \sim N.i.i.d.(0,1) \end{aligned} \tag{3}$$

with all three shocks, e_{t+1}, u_{t+1} and η_{t+1}, mutually independent. The above growth rate dynamics are also utilized by Campbell (1999), Cecchetti, Lam, and Mark (1993), and Wachter (2002) to model consumption growth rate. $\phi > 1$ and $\varphi_d > 1$ are two additional parameters that allow us to calibrate the overall volatility of dividends (which in the data is significantly larger than that of consumption) and its correlation with consumption. The parameter ϕ can be interpreted, as in Abel (1999), as the leverage ratio on expected

[4] Note that $\kappa_1 = exp(\bar{z})/(1 + exp(\bar{z}))$. κ_1 is approximately 0.997, which is consistent with the magnitude of $\bar{z}$ in our sample and with magnitudes used in Campbell and Shiller (1988a).

consumption growth.[5] It is straightforward to allow the three shocks to be correlated; however, to maintain parsimony in the number of parameters we have assumed they are independent.

The parameter ρ determines the persistence of the expected growth rate process. First, note that when $\varphi_e = 0$, the processes g_t and $g_{d,t+1}$ are *i.i.d.* Second, if $e_{t+1} = \eta_{t+1}$, the process for consumption is the ARMA(1,1) used in Bansal and Yaron (2000). Additionally, if $\varphi_e = \rho$, then consumption growth is an AR(1) process, as in Mehra and Prescott (1985).

Since g and g_d are exogenous processes, a solution for the log price-consumption ratio z_t and the log price-dividend ratio $z_{m,t}$ leads to a complete characterization of the returns $r_{a,t+1}$ and $r_{m,t+1}$ (using equation (2)). The relevant state variable for deriving the solution for z_t and $z_{m,t}$ is the expected growth rate of consumption x_t. Exploiting the Euler equation (1), the solution for the log price-consumption z_t has the form $z_t = A_0 + A_1 x_t$. An analogous expression holds for the log price-dividend ratio $z_{m,t}$. Details of both derivations are provided in the Appendix.

The solution coefficients for the effect of expected growth rate x_t on the price-consumption ratio, A_1, and the price-dividend ratio, $A_{1,m}$, respectively, are

$$A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho} \qquad A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} \ . \tag{4}$$

It immediately follows that A_1 is positive if IES, ψ, is greater than one. In this case the intertemporal substitution effect dominates the wealth effect. In response to higher expected growth (higher expected rates of return), agents buy more assets, and consequently the wealth to consumption ratio rises. In the standard power utility model, the need to have risk aversion larger than one also implies that $\psi < 1$, and hence A_1 is negative. Consequently, the wealth effect dominates the substitution effect.[6] In addition, note that $A_{1,m} > A_1$ when $\phi > 1$; consequently, expected growth rate news leads to a larger reaction in the price of the dividend claim than in the price of the consumption claim.

Substituting the equilibrium return for $r_{a,t+1}$ into the IMRS, it is straightforward to show

[5] Note that we have modelled the growth rates of consumption (nondurables plus services) and dividends. In our model, as with many papers in the literature (e.g., Campbell and Cochrane (1999)), consumption and dividends are not cointegrated. It is an empirical question if these series (and possibly labor income) are cointegrated or not. We do not explore these empirical and any ensuing theoretical issues in this paper.

[6] An alternative interpretation with the power utility model is that higher expected growth rates increase the risk-free rate to an extent that discounting dominates the effects of higher expected growth rates. This leads to a fall in asset prices.

that the innovation to the pricing kernel is (see equation (15) in the Appendix)

$$
\begin{aligned}
m_{t+1} - E_t(m_{t+1}) &= [-\frac{\theta}{\psi} + \theta - 1]\sigma\eta_{t+1} - (1-\theta)[\kappa_1(1-\frac{1}{\psi})\frac{\varphi_e}{1-\kappa_1\rho}]\sigma e_{t+1} \\
&= \lambda_{m,\eta}\sigma\eta_{t+1} - \lambda_{m,e}\sigma e_{t+1}. \qquad (5)
\end{aligned}
$$

$\lambda_{m,e}$ and $\lambda_{m,\eta}$ capture the pricing kernel's exposure to the expected growth rate and the independent consumption shocks, η_{t+1}. The key observation is that the exposure to expected growth rate shocks $\lambda_{m,e}$ rises as the permanence parameter ρ rises. The conditional volatility of the pricing kernel is constant, as all risk sources have constant conditional variances.

As asset returns and the pricing kernel in this model economy are conditionally log-normal, the continuous risk premium on any asset i is $E_t[r_{i,t+1} - r_{f,t}] = -cov_t(m_{t+1}, r_{i,t+1}) - 0.5\sigma^2_{r_i,t}$. Given the solutions for A_1 and $A_{1,m}$, it is straightforward to derive the equity premium on the market portfolio (see section 5.1.3 in the Appendix),

$$
E(r_{m,t+1} - r_{f,t}) = \beta_{m,e}\lambda_{m,e}\sigma^2 - 0.5Var(r_{m,t}), \qquad (6)
$$

where $\beta_{m,e} \equiv [\kappa_{1,m}(\phi - \frac{1}{\psi})\frac{\varphi_e}{1-\kappa_{1,m}\rho}]$ and $Var_t(r_{m,t+1}) = [\beta^2_{m,e} + \varphi^2_d]\sigma^2$. The exposure of the market return to expected growth rate news is $\beta_{m,e}$, and the price of expected growth risk is determined by $\lambda_{m,e}$. The expressions for these parameters reveal that a rise in ρ increases both $\beta_{m,e}$ and $\lambda_{m,e}$. Consequently, the risk premium on the asset also increases with ρ. Similarly, the volatility of the market return also increases with ρ (see equation (23) in the Appendix).

Because of our assumption of a constant σ, the conditional risk premium on the market portfolio in (6) is constant, and so is its conditional volatility. Hence, the ratio of the two, namely the Sharpe ratio, is also constant. In order to address issues that pertain to time-varying risk premia and predictability of risk premia, we augment our model in the next section and introduce time-varying economic uncertainty.

2.2 Case II: Incorporating Fluctuating Economic Uncertainty

We model fluctuating economic uncertainty as time-varying volatility of consumption growth. The dynamics for the system (3) that incorporate stochastic volatility are:

$$
\begin{aligned}
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\
\sigma^2_{t+1} &= \sigma^2 + \nu_1(\sigma^2_t - \sigma^2) + \sigma_w w_{t+1} \\
& \quad e_{t+1}, u_{t+1}, \eta_{t+1}, w_{t+1} \sim N.i.i.d.(0,1),
\end{aligned}
\tag{7}
$$

where σ_{t+1} represents the time-varying economic uncertainty incorporated in consumption growth rate and σ^2 is its unconditional mean. To maintain parsimony, we assume that the shocks are uncorrelated, and allow for only one source of economic uncertainty to affect consumption and dividends.

The relevant state variables in solving for the equilibrium price-consumption (and price-dividend) ratio are now x_t and σ^2_t. Thus, the approximate solution for the price-consumption ratio is $z_t = A_0 + A_1 x_t + A_2 \sigma^2_t$. The solution for A_1 is unchanged (equation (4)). The solution coefficient A_2 for measuring the sensitivity of price-consumption ratios to volatility fluctuations is

$$
A_2 = \frac{0.5[(\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi_e)^2]}{\theta(1 - \kappa_1 \nu_1)} . \tag{8}
$$

An analogous coefficient for the price-dividend ratio, $A_{2,m}$, is derived in the Appendix and has a similar form. Two features of this model specification are noteworthy. First, if IES and risk aversion are larger than one, then θ is negative, and a rise in volatility lowers the price-consumption ratio. Similarly, an increase in economic uncertainty raises risk premia and lowers the market price-dividend ratio. Second, an increase in the permanence of volatility shocks, that is ν_1, magnifies the effects of volatility shocks on valuation ratios, as changes in economic uncertainty are perceived as being long lasting.

As the price-consumption ratio is affected by volatility shocks, so is the return $r_{a,t+1}$. Consequently, the pricing kernel (IMRS) is also affected by volatility shocks. Specifically, the innovation in the pricing kernel is now:

$$
m_{t+1} - E_t(m_{t+1}) = \lambda_{m,\eta} \sigma_t \eta_{t+1} - \lambda_{m,e} \sigma_t e_{t+1} - \lambda_{m,w} \sigma_w w_{t+1}, \tag{9}
$$

where $\lambda_{m,w} \equiv (1-\theta) A_2 \kappa_1$, while $\lambda_{m,\eta}$ and $\lambda_{m,e}$ are defined in equation (5). This expression is similar to the earlier model (see equation (5)) save for the inclusion of w_{t+1}: shocks to consumption volatility. In the special case of power utility, where $\theta = 1$, these volatility innovations are not reflected in the innovation of the pricing kernel, as $\lambda_{m,w}$ equals zero.[7]

The equation for the equity premium will now have two sources of systematic risk. The first, as before, relates to fluctuations in expected consumption growth, and the second to fluctuations in consumption volatility. The equity premium in the presence of time-varying economic uncertainty is

$$E_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2 - 0.5Var_t(r_{m,t+1}), \tag{10}$$

where $\beta_{m,w} \equiv \kappa_{1,m}A_{2,m}$ and $Var_t(r_{m,t+1}) = \{\beta_{m,e}^2\sigma_t^2 + \varphi_d^2\sigma_t^2 + \beta_{m,w}^2\sigma_w^2\}$.

The market compensation for stochastic volatility risk in consumption is determined by $\lambda_{m,w}$. The risk premium on the market portfolio is time-varying as σ_t fluctuates. The ratio of the conditional risk premium to the conditional volatility of the market portfolio fluctuates with σ_t, and hence the Sharpe ratio is time-varying. The maximal Sharpe ratio in this model economy, which approximately equals the conditional volatility of the pricing kernel innovation (equation (9)), also varies with σ_t.[8] This means that during periods of high economic uncertainty, risk premia will rise. For further discussion on the specialization of the risk premia under expected utility see Bansal and Yaron (2000).

The first-order effects on the level of the risk-free rate (see equation (24) in the Appendix) are the rate of time preference and the average consumption growth rate, divided by the IES. Increasing the IES keeps the level low. In addition, the variance of the risk-free rate is primarily determined by the volatility of expected consumption growth rate and the IES. Increasing the IES lowers the volatility of the risk-free rate.

3 Data and Model Implications

To derive asset market implications of the model described in (7), we calibrate the model at the monthly frequency such that its time-aggregated annual growth rates of consumption and dividends match salient features of their annual counterparts in the data, and at the same time allow the model to reproduce many observed asset pricing features. As in Campbell and Cochrane (1999), Kandel and Stambaugh (1991), and many others, we calibrate the model.

[7] Recall that in our specification the conditional volatility and expected growth rate processes are independent. With power utility, the volatility shocks will not be reflected in the innovations of the IMRS. With Epstein and Zin (1989) preferences, in spite of this independence, volatility shocks influence the innovations in the pricing IMRS.

[8] As in Campbell and Cochrane (1999), given the normality of the growth rate dynamics, the maximal Sharpe ratio is simply given by the standard deviation of the log pricing kernel.

Further, as in these papers the decision interval of the agent is assumed to be monthly, but the targeted data to be matched are at an annual frequency.[9]

Our choices of the time series and preference parameters are designed to simultaneously match observed growth rate data and asset market data. In order to isolate the economic effects of persistent expected growth rates from those of fluctuating economic uncertainty, we report our results first for Case I where fluctuating economic uncertainty has been shut off (σ_w is set to zero), and then consider the model specification where both channels are operational.

3.1 Persistent Expected Growth

In Table 1 we display the time series properties of the model given in (3). The specific parameters are given below the table. In spite of a persistent growth component, the model's implied time series properties are largely consistent with the data.

Barsky and DeLong (1993) rely on a persistence parameter ρ equal to one. We calibrate ρ at 0.979, this ensures that expected consumption growth rates are stationary and permits the possibility of large dividend elasticity of equity prices and equity risk premia. Our choice of φ_e and σ is motivated to ensure that we match the unconditional variance and the autocorrelation function of annual consumption growth. The standard deviation of the one-step ahead innovation in consumption, that is σ, equals 0.0078. This parameter configuration implies that the predictable variation in monthly consumption growth, i.e., the R^2, is only 4.4%. Our choice of ϕ is very similar to that in Abel (1999) and captures the "levered" nature of dividends. The standard deviation of the monthly innovation in dividends, $\varphi_d\sigma$, is 0.0351. This parameter configuration allows us to match the unconditional variance of dividend growth and its annual correlation with consumption.

[9]The evidence regarding the model is based on numerical solutions using standard polynomial-based projection methods discussed in Judd (1998). The numerical results are quite close to those based on the approximate analytical solutions.

Table 1 : Annualized Time-Averaged Growth Rates

	Data		Model				
Variable	Estimate	S.E.	Mean	95%	5%	P-val	Pop
$\sigma(g)$	2.93	(0.69)	2.72	3.80	2.01	0.37	2.88
$AC(1)$	0.49	(0.14)	0.48	0.65	0.21	0.53	0.53
$AC(2)$	0.15	(0.22)	0.23	0.50	-0.17	0.70	0.27
$AC(5)$	-0.08	(0.10)	0.13	0.46	-0.13	0.93	0.09
$AC(10)$	0.05	(0.09)	0.01	0.32	-0.24	0.80	0.01
$VR(2)$	1.61	(0.34)	1.47	1.69	1.22	0.17	1.53
$VR(5)$	2.01	(1.23)	2.26	3.78	0.79	0.63	2.36
$VR(10)$	1.57	(2.07)	3.00	6.51	0.76	0.77	2.96
$\sigma(g_d)$	11.49	(1.98)	10.96	15.47	7.79	0.43	11.27
$AC(1)$	0.21	(0.13)	0.33	0.57	0.09	0.53	0.39
$corr(g, g_d)$	0.55	(0.34)	0.31	0.60	-0.03	0.07	0.35

The model parameters are based on the process given in equation (3). The parameters are $\mu = \mu_d = 0.0015$, $\rho = 0.979$, $\sigma = 0.0078$, $\phi = 3$, $\varphi_e = 0.044$, and $\varphi_d = 4.5$. The statistics for the data are based on annual observations from 1929 to 1998. Consumption is real non-durables and services (BEA); dividends are from the CRSP value weighted return. $AC(j)$ is the j^{th} autocorrelation and $VR(j)$ is the j^{th} variance ratio. Standard errors are Newey and West (1987) corrected using 10 lags. The statistics for the model are based on 1000 simulations each with 840 time-aggregated monthly observations. The *mean* displays the mean across the simulations. The *95%* and *5%* columns display the estimated percentiles of the simulated distribution. The *P-val* column denotes the number of times in the simulation the parameter of interest was larger than the corresponding estimate in the data. The *Pop* column refers to population value.

Since our model emphasizes the long horizon implications of the predictable component x_t, we first demonstrate that our proposed process for consumption is consistent with annual consumption data along a variety of dimensions. We use BEA data on real per capita annual consumption growth of non-durables and services for the period 1929 to 1998. This is the longest single data source of consumption data. Dividends and the value-weighted market return data are taken from CRSP. All nominal quantities are deflated using the CPI. To facilitate comparisons between the model, which is calibrated to a monthly decision interval, and the annual data, we time aggregate our monthly model and report its annual statistics. As there is considerable evidence of small sample biases in estimating autoregression coefficients and variance ratios (see Hurwicz (1950) and Ansley and Newbold

(1980)) we report statistics based on 1000 Monte-Carlo experiments each with 840 monthly observations — each experiment corresponding to the 70 annual observations available in our data set. Increasing the size of the Monte-Carlo makes little difference to the results.

The annualized real per-capita consumption growth mean is 1.8% and its standard deviation is about 2.9%. Note that this volatility is somewhat lower for our sample than for the period considered in Mehra and Prescott (1985), Kandel and Stambaugh (1991), and Abel (1999). Table 1 shows that, in the data, consumption growth has a large first autocorrelation coefficient and a small second-order one. The standard errors in the data for these autocorrelations are sizeable. An alternative way to view the long horizon properties of the model is to use variance ratios which are themselves determined by the autocorrelations (see Cochrane (1988)). In the data the variance ratios first rise significantly and at about 7 years out start to decline. The standard errors on these variance ratios, not surprisingly, are quite substantial.

The mean (across simulations) of the model's implied first autocorrelation is similar to that in the data. The second and tenth autocorrelations are within one standard error of the data. The fifth autocorrelation is just about two standard errors of the data. The empirical distribution of these estimates across the simulations as depicted by the *5th* and *95th* percentiles is wide and contains the point estimates from the data. The model's variance ratios mimic the pattern in the data. The point estimates are slightly larger than the data, but they are well within one standard error of the data. The point estimates from the data are clearly contained in the 5% confidence interval based on the empirical distribution of the simulated variance ratios. The unconditional volatility of consumption and dividend growth closely matches that in the data. In addition, the correlation of dividends with consumption of about 0.3 is somewhat lower, but within one standard error of its estimate in the data. This lower correlation is a conservative estimate, and increasing it helps the model generate a higher risk premium. Overall, Table 1 shows that allowing for a persistent predictable component produces consumption and dividend moments that are largely consistent with the data.

It is often argued that consumption growth is close to being *i.i.d.* As shown in Table 1 the consumption dynamics, which contain a persistent but small predictable component, are also largely consistent with the data. This evidence is consistent with Shephard and Harvey (1990) and Barsky and DeLong (1993) who show that in finite samples discrimination across the *i.i.d.* growth rate model and the one considered above is extremely difficult. While the financial market data are hard to interpret from the perspective of the *i.i.d.* dynamics, they are, as shown below, interpretable from the perspective of the growth rate dynamics considered above.

Before we discuss the asset pricing implications we highlight two additional issues related to the data. First, data for consumption, dividends, and asset returns pertain to the long sample from 1929. Clearly moments of these data will differ across sub-samples. Our choice of the long sample is similar to Mehra and Prescott (1985), Kandel and Stambaugh (1991), and Abel (1999) and is motivated to keep the estimation error on the moments small. The annual autocorrelations of consumption growth for our model are well within standard error bounds even when compared to those in the post-war annual consumption data.[10] Second, our dividend model is calibrated to cash dividends; this is similar to that used by many earlier studies. While it is common to use cash dividends, this measure of dividends may mismeasure total payouts, as it ignores other forms of payments made by corporations. Given the difficulties in accurately measuring total payouts of corporations and to maintain comparability with earlier work we have focused on cash dividends as well. Jagannathan, McGrattan, and Scherbina (2000) provide evidence pertaining to the issue of dividends, and show that alternative measures of dividends have even higher volatility.

3.1.1 Case I: Asset Pricing Implications

In Table 2 we display the asset pricing implications of the model for a variety of risk aversion and IES configurations. In panel A, we use the time series parameters from Table 1. In Panel B we increase ϕ, the dividend leverage parameter, to 3.5, and in Panel C we analyze the implications of an *i.i.d.* process. The table intentionally concentrates on a relatively narrow set of asset pricing moments, namely the mean risk-free rate, equity premium, the market and risk-free rate volatility, and the volatility of the log price-dividend ratio. These moments are the main focus of many asset pricing models. In section 3.3 we discuss additional model implications.

[10] The first autocorrelations for annual consumption growth in 1951-1999 and 1961-1999 are 0.38 and 0.44, respectively – hence the consumption growth autocorrelations vary with samples. Based on Table 1, both estimates are well within the model-based 5% confidence interval for the first autocorrelation. We have focused on annual data (consumption and dividends) to avoid dealing with seasonalities and other measurement problems discussed in Wilcox (1992).

Table 2 : Asset Pricing Implications – Case I

γ	ψ	$E(R_m - R_f)$	$E(R_f)$	$\sigma(R_m)$	$\sigma(R_f)$	$\sigma(p-d)$
			Panel A: $\phi = 3.0$, $\rho = 0.979$			
7.5	0.5	0.55	4.80	13.11	1.17	0.07
7.5	1.5	2.71	1.61	16.21	0.39	0.16
10.0	0.5	1.19	4.89	13.11	1.17	0.07
10.0	1.5	4.20	1.34	16.21	0.39	0.16
			Panel B: $\phi = 3.5$, $\rho = 0.979$			
7.5	0.5	1.11	4.80	14.17	1.17	0.10
7.5	1.5	3.29	1.61	18.23	0.39	0.19
10.0	0.5	2.07	4.89	14.17	1.17	0.10
10.0	1.5	5.10	1.34	18.23	0.39	0.19
			Panel C: $\phi = 3.0$, $\rho = \varphi_e = 0$			
7.5	0.5	-0.74	4.02	12.15	0.00	0.00
7.5	1.5	-0.74	1.93	12.15	0.00	0.00
10.0	0.5	-0.74	3.75	12.15	0.00	0.00
10.0	1.5	-0.74	1.78	12.15	0.00	0.00

All entries are based on $\delta = 0.998$ and the parameter configuration given in Table 1.

Our choice of parameters attempts to take economic considerations into account. In particular $\delta < 1$, and the risk aversion parameter γ is either 7.5 or 10. Mehra and Prescott (1985) argue that a reasonable upper bound for risk aversion is around 10. In this sense, our choice for risk aversion is reasonable. The magnitude for IES that we focus on is 1.5. Hansen and Singleton (1982) and Attanasio and Webber (1989) estimate the IES to be well in excess of 1.5. More recently, Vissing-Jorgensen (2002) and Guvenen (2001) also argue that the IES is well over one. However, Hall (1988) and Campbell (1999) estimate the IES to be well below one. Their results are based on a model without fluctuating economic uncertainty. In section 3.3.4, we show that ignoring the effects of time-varying consumption volatility leads to a serious downward bias in the estimates of the IES. To highlight the role of IES, we choose one value of IES less than one (IES= 0.5) and another larger than one (IES=1.5).

Table 2 shows that the model with persistent expected growth is able to generate sizeable risk premia, market volatility, and fluctuations in price-dividend ratios. Larger risk aversion clearly increases the equity premium; changing risk aversion mainly affects this dimension of the model. To qualitatively match key features of the data it is important for the IES to be larger than one. Lowering the IES lowers $A_{1,m}$, the dividend elasticity of asset prices, and the risk premia on the asset. As IES rises the volatility of the price-dividend ratio and asset returns rise along with $A_{1,m}$. At very low values of the IES, $A_{1,m}$ can become negative which would imply that a rise in dividends' growth rate expectations will lower asset prices (see the discussion in section 2). In addition, note that if the leverage parameter ϕ is increased, it increases the riskiness of dividends, and $A_{1,m}$ rises. The price-dividend ratio becomes more volatile, and the equity premium rises.

As discussed earlier we assumed that u_t, e_t, and η_t are independent. To give a sense of how the results change if we allow for correlations in the various shocks, consider the case with IES at 1.5 and risk aversion of 10. When we assume that the correlation between u_t and η_t is 0.25 and all other innovations are set at zero, then the equity premium rises to 5.02%. If the correlation between u_t and e_t is assumed to be 0.25, then the equity premium and the market return volatility rise to 5.21% and 17.22% respectively. There are virtually no other changes. As stated earlier, in Table 2, we have made the conservative assumption of zero correlations to maintain parsimony in the parameters that we have to calibrate.

It is also interesting to consider the case where consumption and dividend growth rates are assumed to be *i.i.d.*, that is $\varphi_e = 0$. In this case, the equity premium for the market is $E_t(r_{m,t+1} - r_{f,t}) = \gamma cov(g_{t+1}, g_{d,t+1}) - 0.5Var(r_{m,t+1})$. In our baseline model, dividend innovations are independent of consumption innovations, hence, with *i.i.d.* growth rates $cov(g_{t+1}, g_{d,t+1})$ equals zero, and the market equity premium is $-0.5Var(r_{m,t+1})$; this explains the negative equity premium in the *i.i.d.* case reported in Panel C of Table 2. If we assume that the correlation between monthly consumption and dividend growth is 0.25, then the equity premium is 0.08% per annum. This is similar to the evidence documented in Weil (1989) and Mehra and Prescott (1985). For comparable IES and risk-aversion values, shifting from the persistent growth rate process to *i.i.d.* growth rates lowers the volatility of the equity returns. In all, this evidence highlights the fact that although the time-series dynamics of the model with small persistent expected growth are difficult to distinguish from a pure *i.i.d.* model, its asset pricing implications are vastly different than those of the *i.i.d.* model. In what follows we use the parameters in Panel A, with an IES of 1.5 as our preferred configuration, and display the implications of adding fluctuating economic uncertainty.

3.2 Fluctuating Economic Uncertainty

Before displaying the asset pricing implications of adding fluctuating economic uncertainty we first briefly discuss evidence for the presence of fluctuating economic uncertainty.

Panel A of Table 3 documents that the variance ratios of the absolute value of residuals from regressing current consumption growth on five lags increase gradually out to 10 years. This suggests slow-moving predictable variation in this measure of realized volatility. Note that if realized volatility were *i.i.d.* these variance-ratios would be flat.[11]

Table 3 : Properties of Consumption's Realized Volatility

Horizon	Panel A: Variance Ratios		Panel B: Predicting $\lvert\epsilon_{g^a,t+j}\rvert$		
	$VR(j)$	S.E.	$B(j)$	S.E.	R^2
2	0.95	(0.38)	-0.11	(0.04)	0.06
5	1.26	(1.09)	-0.10	(0.05)	0.04
10	1.75	(2.46)	-0.08	(0.08)	0.03

The entries in Panel A are the variance ratios for $\lvert\epsilon_{g^a,t}\rvert$, which is the absolute value of the residual from regressing $g_t^a = \sum_{j=1}^{5} A_j g_{t-j}^a + \epsilon_{g^a,t}$, and g_t^a denotes annual consumption growth rate.

Panel B provides regression results of $\lvert\epsilon_{g^a,t+j}\rvert = \alpha + B(j)(p_t - d_t) + v_{t+j}$, and j indicates the forecast horizon in years. The statistics are based on annual observations from 1929 to 1998 of real non-durables and services consumption (BEA). The price-dividend ratio is based on CRSP value weighted return. Standard errors are Newey and West (1987) corrected using 10 lags.

In Panel B of Table 3 we provide evidence that future realized consumption volatility is predicted by current price-dividend ratios. The current price-dividend ratio predicts future realized volatility with negative coefficients, with robust t-statistics around 2 and R^2s around 5% (for horizons of up to 5 years). If consumption volatility were not time-varying, the slope coefficient on the price-dividend ratio would be zero. As suggested by our theoretical model, this evidence indicates that information regarding persistent fluctuations in economic uncertainty is contained in asset prices. Overall, the evidence in Table 3 lends support to the view that the conditional volatility of consumption is time-varying. Bansal, Khatchatrian, and Yaron (2002) document the evidence in favor of time-varying consumption volatility more extensively and show that this evidence holds up quite well across different samples and economies.

Given the evidence above, a large value of ν_1, the parameter governing the persistence of conditional volatility, allows the model to capture the slow-moving fluctuations in economic uncertainty. In Table 4 we provide the asset pricing implications based on the system (7) when in addition to the parameters given in Table 1 we activate the volatility parameters (given below the table). It is important to note that the time-series properties displayed in Table 1 are virtually unaltered once we introduce the fluctuations in economic uncertainty.

[11] Also note that it is difficult to detect high-frequency GARCH effects once the data is time aggregated (see Nelson (1991), Drost and Nijman (1993)).

Table 4 provides statistics for the asset market data and for the model which incorporates fluctuating economic uncertainty (i.e., Case II). Columns 2 and 3 provide the statistics and their respective standard errors for our data sample. Columns 4 and 5 provide the model's corresponding statistics for risk aversion of 7.5 and 10 respectively. In this table IES is always set at 1.5 and ϕ is set at 3.

Table 4 : Asset Pricing Implications – Case II

	Data		Model	
Variable	Estimate	S.E.	$\gamma = 7.5$	$\gamma = 10$
Returns				
$E[r_m - r_f]$	6.33	(2.15)	4.01	6.84
$E[r_f]$	0.86	(0.42)	1.44	0.93
$\sigma(r_m)$	19.42	(3.07)	17.81	18.65
$\sigma(r_f)$	0.97	(0.28)	0.44	0.57
Price Dividend				
$E[exp(p - d)]$	26.56	(2.53)	25.02	19.98
$\sigma(p - d)$	0.29	(0.04)	0.18	0.21
$AC1(p - d)$	0.81	(0.09)	0.80	0.82
$AC2(p - d)$	0.64	(0.15)	0.65	0.67

Model entries are population values based on the process in (7). In addition to the parameter values given in Table 1, the parameters of the stochastic volatility process are $\nu_1 = 0.987$ and $\sigma_w = 0.23 \times 10^{-5}$. The predictable variation of realized volatility is 5.5%. Standard errors are Newey and West (1987) corrected using 10 lags.

Column 5 of Table 4 shows that with $\gamma = 10$ the model generates an equity premium that is comparable to that in the data.[12] The mean of the risk-free rate, and the volatilities of the market return and of the risk-free rate, are by and large consistent with the data. The model essentially duplicates the volatility and persistence of observed log price-dividend ratio. Comparing columns 4 and 5 provides sensitivity of the results to the level of risk-aversion. Not surprisingly, higher risk aversion increases the equity premium and aligns the model closer to the data. A comparison of Table 4 with Table 2 shows that when risk aversion is 10 the equity risk premium is about 2.5% higher – this additional premium reflects the premium associated with fluctuating economic uncertainty as derived in equation (10). One

[12]To derive analytical expressions we have assumed that the volatility process is conditionally normal. When we solve the model numerically we ensure that the volatility is positive by replacing negative realizations with a very small number. This happens for about 5% of the realizations; hence, the possibility that volatility in equation (7) can become negative is primarily a technical issue.

could, as discussed earlier, modify the above model and also include correlation between the different shocks. The inclusion of these correlations as documented above typically helps to increase the equity premium. Hence, it would seem that these correlations would help the model generate the same equity premium with a lower risk-aversion parameter.

Weil (1989) and Kandel and Stambaugh (1991) also explore the implications of the Epstein and Zin (1989) preferences for asset market data. However, these papers find it difficult to quantitatively explain the aforementioned asset market features at our configuration of preference parameters. Why, then, do we succeed in capturing these asset market features with Epstein and Zin (1989) preferences? Weil (1989) uses *i.i.d.* consumption growth rates. As discussed earlier, with *i.i.d.* consumption and dividend growth rates the risks associated with fluctuating expected growth and economic uncertainty are absent. Consequently, the model has great difficulty in explaining the asset market data.

Kandel and Stambaugh (1991) consider a model in which there is predictable variation in consumption growth rates and volatility. However, at our preference parameters, the persistence in the expected growth and conditional volatility in their specification is not large enough to permit significant response of asset prices to news regarding expected consumption growth and volatility. In addition, Kandel and Stambaugh (1991) primarily focus on the case in which the IES is close to zero. At very low values of IES, $\lambda_{m,e}$ and $\beta_{m,e}$ are negative (see equations (5), (6)). This may still imply a sizeable equity premium. However, a parameter configuration with an IES less than one and a moderate level of risk aversion (for example 10 or less) leads to high levels of the risk-free rate and/or its volatility. In contrast, our IES, which is greater than one, ensures that the level and volatility of the risk-free rate are low and comparable to those in the data. Hence, with moderate levels of risk aversion, both the high persistence and an IES greater than one are important in order to capture key aspects of asset market data.

3.3 Additional Asset Pricing Implications

As noted earlier, in the model where we shut off fluctuating economic uncertainty (Case I), both risk premia and Sharpe ratios are constant – hence, this simple specification cannot address issues regarding predictability of risk premia. The model which incorporates fluctuating economic uncertainty (Case II) does permit risk premia to fluctuate. Henceforth, we focus entirely on this model specification with the parameter configuration stated in Table 4 with $\gamma = 10$.

3.3.1 Variability of the Pricing Kernel

The maximal Sharpe ratio, as shown in Hansen and Jagannathan (1991), is determined by the conditional volatility of the pricing kernel. This maximal Sharpe ratio for our model is the volatility of the pricing kernel innovation defined in equation (9). In Table 5, we quantify the contributions of different shocks to the variance of the pricing kernel innovations (see equation (9)). The maximal annualized Sharpe ratio for our model economy is 0.73, which is quite large. The maximal Sharpe ratio with *i.i.d.* growth rates is $\gamma\sigma$ and with our parameter configuration its annualized value equals 0.27. Consequently, the Epstein and Zin preferences and the departure from *i.i.d.* growth rates are responsible for this larger maximal Sharpe ratio. Additionally, for our model the maximal Sharpe ratio exceeds that of the market return, which is 0.33. The sources of risk in order of importance are shocks to the expected growth rate (i.e., e_{t+1}), followed by that of fluctuating economic uncertainty (i.e., w_{t+1}). While the variance of these shocks in themselves is small, their effects on the pricing kernel get magnified because of the long-lasting nature of these shocks (see discussion in section 2). Finally, the variance of high-frequency consumption news, η_{t+1}, is relatively large, but this risk source contributes little to the pricing kernel variability, as this shock is not long-lasting.

Table 5 : Decomposing the Variance of the Pricing Kernel

	Relative Variance of Shocks		
Volatility of Pricing Kernel	Independent Consumption	Expected Growth Rate	Fluctuating Economic Uncertainty
0.73	14%	47%	39%

Entries are the relative variance of different shocks to the variance of the pricing kernel. The volatility of the maximal Sharpe ratio is annualized in order to make it comparable to the Sharpe ratio on annualized returns.

3.3.2 Predictability

Dividend yields seem to predict multi-horizon returns. A rise in the current dividend yield predicts a rise in future expected returns. Our model performs quite well in capturing this feature of the data. However, it is important to recognize that these predictability results are quite sensitive to changing samples, estimation techniques, and data sets (see Hodrick (1992), Goyal and Welch (1999), and Ang and Bekaert (2001)). Further, most dimensions of the evidence related to predictability (be it growth rates or returns) are estimated with considerable sampling error. This in conjunction with the rather high persistence in the price-dividend ratio suggests that considerable caution should be exercised in interpreting the evidence regarding predictability based on price-dividend ratios.

In Panel A of Table 6 we report the predictability regressions of future excess returns for horizons of 1, 3, and 5 years for our sample data. In Column 4 we report the corresponding evidence from the perspective of the model. The model captures the positive relationship between expected returns and dividend yields. The absolute value of the slope coefficients and the corresponding R^2s rise with the return horizon, as in the data. The predictive slope coefficients and the R^2s in the model are somewhat lower than those in the data; however, the model's slope coefficients are within two standard errors of the estimated coefficients in the data.[13]

Table 6 : Predictability of Returns, Growth Rates, and Realized Volatility

	Panel A: Excess Returns			Panel B: Growth Rates			Panel C: Volatility		
Variable	Data	S.E.	Model	Data	S.E.	Model	Data	S.E.	Model
$B(1)$	-0.08	(0.07)	-0.18	0.04	(0.03)	0.06	-8.78	(3.58)	-3.74
$B(3)$	-0.37	(0.16)	-0.47	0.03	(0.05)	0.12	-8.32	(2.81)	-2.54
$B(5)$	-0.66	(0.21)	-0.66	0.02	(0.04)	0.15	-8.65	(2.67)	-1.56
$R^2(1)$	0.02	(0.04)	0.05	0.13	(0.09)	0.10	0.12	(0.05)	0.14
$R^2(3)$	0.19	(0.13)	0.10	0.02	(0.05)	0.12	0.11	(0.04)	0.08
$R^2(5)$	0.37	(0.15)	0.16	0.01	(0.02)	0.11	0.12	(0.04)	0.05

This table provides predictability regressions of future excess returns, growth rates, and realized volatility of consumption. The entries in Panel A correspond to regressing $r^e_{t+1}+r^e_{t+2}..+..r^e_{t+j} = \alpha(j)+B(j)\log(P_t/D_t)+v_{t+j}$, where r^e_{t+1} is the excess return, and j denotes the forecast horizon in years. The entries in Panel B correspond to regressing $g^a_{t+1} + g^a_{t+2}.. + ..g^a_{t+j} = \alpha(j) + B(j)\log(P_t/D_t) + v_{t+j}$, and g^a is annualized consumption growth. The entries in Panel C correspond to $\log(P_{t+j}/D_{t+j}) = \alpha(j) + B(j)|\epsilon_{g^a,t}| + v_{t+j}$, where $|\epsilon_{g^a,t}|$ is the realized volatility of consumption (see details in Table 3). Model parameters are based on the process in (7), with parameter values given in Table 2. The entries for the model are based on 1000 simulations each with 840 monthly observations that are time aggregated. Standard errors are Newey and West (1987) corrected using 10 lags.

In Panel B of Table 6 we provide regression results where the dependent variable is the sum of annual consumption growth rates. In the data it seems that price-dividend ratios have little predictive power, particularly at longer horizons. The slope coefficients and R^2s of

[13] Consistent with Lettau and Ludvigson (2001), predictability coefficients and R^2s based on the wealth-consumption ratio follow the same pattern and are slightly larger than those based on price-dividend ratios.

these regressions are quite low in both the data and the model. The R^2s are relatively small in the model for two reasons. First, price-dividend ratios are determined by expected growth rates, and the variation in expected growth rates is quite small. Recall that the monthly R^2 for consumption dynamics is less than 5%. Second, price-dividend ratios are also affected by independent movements in economic uncertainty, which lowers their ability to predict future growth rates. Overall, the model, as in the data, suggests that growth rates at long horizons are not predicted by price-dividend ratios in any economically sizeable manner.[14]

In Panel C of Table 6 we report how well current realized consumption volatility predicts future price-dividend ratios. First, note that there is strong evidence in the data for this relationship. The regression coefficients for predicting future price-dividend ratios with current volatility for 1, 3 and 5 years are all negative, have robust t-statistics that are well above 2, and have R^2s of about 10%. The model produces similar, albeit in absolute terms slightly smaller, negative coefficients. The R^2s are within two standard errors of the data. Taken together with the results in Panel B of Table 3, the evidence is consistent with the economics of the model; fluctuating economic uncertainty, captured via realized consumption volatility, predicts future price-dividend ratios and is predicted by lagged price-dividend ratios. The empirical evidence shows that asset markets dislike economic uncertainty – a feature that our model is capable of reproducing. Bansal, Khatchatrian, and Yaron (2002) show that this evidence is robust across many samples and frequencies, and is consistently found in many developed economies.

Some caution should be exercised in interpreting the links between dividend growth rates and price-dividend ratios. Evidence provided in other papers (see, e.g., Ang and Bekaert (2001)) shows that alternative measures of cash flows, such as earnings, are sharply predictable by valuation ratios. Cash dividends, as discussed earlier may not accurately measure the total payouts to equity holders and hence may distort the link between growth rates and asset valuations. However, given the practical difficulties in measuring the appropriate payouts, and to maintain comparability with other papers in the literature we, like others, continue to use cash dividends. With this caveat in mind, we also explore the model's implications by exploring how much of the variation in the price-dividend ratio is from growth rates and what part is due to variation in expected returns.

In the data, the majority of the variation in price-dividend ratios seems to be due to variation in expected returns. For our sample the point estimate for the percentage of the

[14]Our model can be easily modified to further lower the predictability of growth rates. Consider an augmented model (as in Cecchetti, Lam, and Mark (1993)) that allows for additional predictable movements in dividend growth rates that are unrelated to consumption. This will not affect the risk-free rate and the risk premia in the model but will additionally lower the ability of price-dividend ratios to predict future consumption growth rates.

variation in price-dividend ratio due to return fluctuations is 108%, with a standard error of 42%, while dividends' growth rates account for -6%, with a standard error of 31%.[15] Our model produces population estimates that attribute about 52% of the variation in price-dividend ratios to returns and 54% to fluctuations in expected dividend growth. Note that the standard errors of the point estimates of this decomposition in the data are very large. To account for any finite sample biases, we also conducted a Monte Carlo exercise using simulations from our model of sample sizes comparable to our data. This Monte Carlo evidence implies that in our model the returns account for about 70% of the variation in price-dividend ratio, thus aligning the model closer to the data. Given the large sampling variation in measuring these quantities in the data using cash dividends, and the sharp differences in predictability implications using alternative cash flow measures makes economic inference based on this decomposition quite difficult.

Two additional features of the model are worth highlighting. First, in the data the contemporaneous correlation between equity return and consumption is very small at the monthly frequency and is about 0.20 at the annual frequency. Our model produces comparable magnitudes, with correlations of 0.04 and 0.15 for the monthly and annual frequencies, respectively. Second, the term premium on nominal bonds, the average one-period excess return on an n-period discount bond, is small. This suggests that the equity premium in the data is not driven by a large term premium. The term premium (which in our model is on real bonds) is in fact small and slightly negative. Hence the large equity premium in the model is not a by-product of a large positive term premium.[16] In totality, the above evidence, in conjunction with the results pertaining to predictability, suggests that the model is capable of capturing several key aspects of asset markets data.

3.3.3 Conditional Volatility of Returns and the Leverage Effect

A large literature documents that market return volatility is very persistent (see, e.g., Bollerslev, Engle, and Wooldridge (1988)). This feature of the data is easily reproduced in our model. The market volatility process, as described in equation (18) in the Appendix, is a linear affine function of the conditional variance of the consumption growth rate process σ_t. As the conditional variance of the consumption growth rate process is an AR(1) process,

[15] For explicit details of this decomposition see Cochrane (1992). Specifically, these represent the percentage of $var(p-d)$ accounted for by returns and dividend growth rates: $100 * \sum_{j=1}^{15} \Omega^j \frac{cov(p_t-d_t, x_{t+j})}{Var(p_t-d_t)}$, where $x = -r$ and g_d respectively, and $\Omega = 1/(1+E(r))$.

[16] The explicit formulas for the real term structure and the term premia are presented in Bansal and Yaron (2000). The negative real term premia of our model are consistent with the evidence provided in Evans (1998), who documents that for inflation-indexed bonds in the U.K. (1983-1995) the term premia are significantly negative (less than -2% at the 1-year horizon), while the term premia for nominal bonds are very slightly positive.

it follows that the market volatility inherits this property. Note that the coefficient on the conditional variance of consumption in the market volatility process is quite large. This magnifies the conditional variance of the market portfolio relative to consumption volatility. The persistence in market volatility coincides with the persistence in the consumption volatility process. In the monthly market return data this persistence parameter is about 0.986 (see Bollerslev, Engle, and Wooldridge (1988)), and in the model it equals ν_1, 0.987. As consumption volatility is high during recessions, this implies that the market volatility also rises during recessions. Also note that during periods of high consumption volatility (e.g., recessions), in the model the equity premium also rises. This implication of the model is consistent with the evidence provided in Fama and French (1989) that risk premia are countercyclical.

Campbell and Hentschel (1992), Glosten, Jagannathan, and Runkle (1993), and others document the leverage effect. That is, return innovations are negatively correlated with innovations in market volatility. The model is capable of reproducing this negative correlation. The leverage effect arises within the model in spite of the fact that the volatility innovations are independent of the expected consumption growth process. The key feature that allows the model to capture this dimension is the Epstein-Zin preferences in which volatility risk is priced (see the discussion in section 2.2). Using the analytical expressions for the innovation in the market return (see equation (17) in the Appendix) and the expression for the innovation in the market volatility, it is straightforward to show that the conditional covariance

$$cov_t((r_{m,t+1} - E_t r_{m,t+1}), var_{t+1}(r_{m,t+2}) - E_t[var_{t+1}(r_{m,t+2})]) = \beta_{m,w}(\beta_{m,e}^2 + \varphi_d^2)\sigma_w^2$$

where $\beta_{m,w} \equiv \kappa_1 A_{2,m} < 0$ as $A_{2,m}$ is negative. The correlation between market return innovations and market volatility innovations for our model is, -0.32.

An additional issue pertains to the relation between the expected return on the market portfolio and the market volatility. Glosten, Jagannathan, and Runkle (1993) and Whitelaw (1994) document that the expected market return and the market volatility are negatively related. French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992) argue that this relation is likely to be positive. In our model, theoretically, the relation between expected market return and market volatility is positive and is not consistent with the negative relation between expected returns and market volatility. Whitelaw (2000) shows that a standard power utility model with regime shifts in consumption growth can accommodate the negative relation between expected returns and market volatility. The unconditional correlation in our model between ex-post excess returns on the market and the ex-ante market volatility is a small positive number, 0.04. The model cannot generate the negative relation between expected returns and market volatility. To do so, we conjecture, will require significant changes, perhaps along the lines pursued in Whitelaw (2000). This departure is well outside the scope of this paper, and we leave this exploration for future work.

3.3.4 Bias in Estimating IES

As in Hall (1988), the IES is typically measured by the slope coefficient from regressing date $t + 1$ consumption growth rate on the date t risk-free rate. This projection would indeed recover the IES if no fluctuating uncertainty affected the risk-free rate. However, the risk-free rate in our model fluctuates as a result of both changing expected growth rate and independent fluctuations in the volatility of consumption. Thus, the above projection is misspecified and creates a downward bias. This bias is quite significant, as inside our model, where the value of IES is set at 1.5, Hall's (1988) regression would estimate the IES parameter to be 0.62. Our model is a simple one, and there may be alternative instrumental variable approaches to undo this bias. However, we view this result of the downward bias as suggestive of the difficulties in accurately pinning down the IES. As discussed in section 3.1, several papers report an estimated IES that is well over one. This evidence, along with the potential downward bias in estimating IES, makes our choice of an IES larger than one quite reasonable.

4 Conclusions

In this paper we explore the idea that news about growth rates and economic uncertainty continuously alters perceptions regarding long-term expected growth rates and economic uncertainty, and that this feature is important for explaining various asset market phenomena. If news about consumption and dividends has a non-trivial impact on long-term expected growth rates or economic uncertainty, then asset prices will be fairly sensitive to small growth rate and economic uncertainty shocks. Anderson, Hansen, and Sargent (2002) utilize features of our growth rate dynamics for motivating economic models that incorporate robust control.

We provide empirical support for the view that the observed aggregate consumption-dividend growth process contains a small persistent expected growth rate and conditional volatility component. We document that the interaction between such growth rate dynamics, in conjunction with Epstein and Zin (1989)-Weil (1989) preferences, can indeed explain many outstanding asset market puzzles. In particular, we show that such a model is capable of justifying the observed magnitudes of the equity premium, the low risk-free rate, and the volatility of market return, dividend-yield, and the risk-free rate. In addition, the model is also capable of justifying the predictive relation between dividend yields and returns, and the well documented GARCH-type stochastic volatility in ex-post equity returns. In our model approximately half of the variability in equity prices is due to fluctuation in expected growth rates, and the remainder is due to fluctuations in costs of capital.

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5 Appendix

The consumption and dividend process given in (7) is

$$\begin{aligned} g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\ \sigma^2_{t+1} &= \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1} \\ g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ & \quad w_{t+1}, e_{t+1}, u_{t+1}, \eta_{t+1} \sim N.i.i.d.(0,1). \end{aligned} \tag{11}$$

The IMRS (Intertemporal Marginal Rate of Substitution) for this economy is given by

$$\ln M_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}.$$

We derive asset prices using this IMRS and the standard asset pricing condition $E_t[M_{t+1}R_{i,t+1}] = 1$, so that

$$E_t[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} + r_{i,t+})] = 1 \tag{12}$$

for any asset $r_{i,t+1} \equiv \log(R_{i,t+1})$. We first start by solving the special case where $r_{i,t+1}$ is $r_{a,t+1}$ – the return on the consumption portfolio, and then solve for market return $r_{m,t+1}$, and the risk-free rate r_f.

5.1 The return on consumption portfolio, R_a

We conjecture that the log price-consumption ratio follows, $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$. Armed with the endogenous variable z_t we substitute the approximation $r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}$ into the Euler equation (12).

Since g, x, and σ_t^2 are conditionally normal, $r_{a,t+1}$ and $\ln M_{t+1}$ are also normal. Exploiting the normality of $r_{a,t+1}$ and $\ln M_{t+1}$, we can write down the Euler equation (12) in terms of the state variables x_t and σ_t. As the Euler condition has to hold for all values of the state variables, it follows that all terms involving x_t must satisfy the following:

$$-\frac{\theta}{\psi} x_t + \theta[\kappa_1 A_1 \rho x_t - A_1 x_t + x_t] = 0.$$

It immediately follows that

$$A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho},$$

which is (4) in the main text.

Similarly, collecting all the σ_t^2 terms leads to the solution for A_2,

$$\theta[\kappa_1 \nu_1 A_2 \sigma_t^2 - A_2 \sigma_t^2] + \frac{1}{2}[(\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi_e)^2]\sigma_t^2 = 0,$$

which implies that

$$A_2 = \frac{0.5[(\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi_e)^2]}{\theta(1 - \kappa_1 \nu_1)},$$

the solution given in (8).

Given the solution above for z_t it is possible to derive the innovation to the return r_a as a function of the evolution of the state variables and the parameters of the model.

$$r_{a,t+1} - E_t(r_{a,t+1}) = \sigma_t \eta_{t+1} + B \sigma_t e_{t+1} + A_2 \kappa_1 \sigma_w w_{t+1}, \tag{13}$$

where $B = \kappa_1 A_1 \varphi_e = \kappa_1 \frac{\varphi_e}{1 - \kappa_1 \rho}(1 - \frac{1}{\psi})$. Further it follows that the conditional variance of $r_{a,t+1}$ is

$$Var_t(r_{a,t+1}) = (1 + B^2)\sigma_t^2 + (A_2 \kappa_1)^2 \sigma_w^2. \tag{14}$$

5.1.1 IMRSs

Now substituting for $r_{a,t+1}$ and the dynamics of g_{t+1}, we can re-write the IMRS in terms of the state variables — referring to this as the pricing kernel. Suppressing all the constants in the pricing kernel,

$$\begin{aligned}
m_{t+1} \equiv lnM_{t+1} &= \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} \\
E_t[m_{t+1}] &= m_0 - \frac{x_t}{\psi} + A_2(\kappa_1 \nu_1 - 1)(\theta - 1)\sigma_t^2 \\
m_{t+1} - E_t(m_{t+1}) &= (-\frac{\theta}{\psi} + \theta - 1)\sigma_t \eta_{t+1} + (\theta - 1)(A_1 \kappa_1 \varphi_e)\sigma_t e_{t+1} + (\theta - 1) A_2 \kappa_1 \sigma_w w_{t+1} \\
&= \lambda_{m,\eta} \sigma_t \eta_{t+1} - \lambda_{m,e} \sigma_t e_{t+1} - \lambda_{m,w} \sigma_w w_{t+1}, \qquad (15)
\end{aligned}$$

where $\lambda_{m,\eta} \equiv [-\frac{\theta}{\psi} + (\theta - 1)] = -\gamma$, $\lambda_{m,e} \equiv (1-\theta)B$, $\lambda_{m,w} \equiv (1-\theta)A_2\kappa_1$, and B and A_2 are defined above. Note that the λs represent the market price of risk for each source of risk, namely η_{t+1}, e_{t+1}, and w_{t+1}.

5.1.2 Risk Premia for $r_{a,t+1}$

The risk premium for any asset is determined by the conditional covariance between the return and m_{t+1}. Thus the risk premium for $r_{a,t+1}$ is equal to $E_t(r_{a,t+1} - r_{f,t}) = -cov_t[m_{t+1} - E_t(m_{t+1}), r_{a,t+1} - E_t(r_{a,t+1})] - 0.5var_t(r_{a,t+1})$.

Exploiting the innovations in (13) and (15) it follows that,

$$E_t[r_{a,t+1} - r_{f,t}] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,e} B \sigma_t^2 + \kappa_1 A_2 \lambda_{m,w} \sigma_w^2 - 0.5Var_t(r_{a,t+1}), \qquad (16)$$

where $Var_t(r_{a,t+1})$ is defined in equation (14).

5.1.3 Equity Premium and Market Return Volatility

The risk premium for any asset is determined by the conditional covariance between the return and m_{t+1}. Thus the risk premium for the market portfolio $r_{m,t+1}$ is equal to $E_t(r_{m,t+1} - r_{f,t}) = -cov_t[m_{t+1} - E_t(m_{t+1}), r_{m,t+1} - E_t(r_{m,t+1})] - 0.5var_t(r_{m,t+1})$.

Equation (15) already provides the innovation in m_{t+1}. We now proceed to derive the innovation in the market return. The price-dividend ratio for the claim on dividends is $z_{m,t} = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2$. Consequently, the market return is

$$r_{m,t+1} = g_{d,t+1} + \kappa_{1,m} A_{1,m} x_{t+1} - A_{1,m} x_t + \kappa_{1,m} A_{2,m} \sigma_{t+1}^2 - A_{2,m}\sigma_t^2.$$

Exploiting the Euler condition $E_t[\exp(m_{t+1} + r_{m,t+1})] = 1$, and collecting all the x_t terms, we find that

$$-\frac{x}{\psi} + x\kappa_{1,m} A_{1,m} \rho - A_{1,m} x + \phi x = 0,$$

which implies that

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho}.$$

It follows that

$$\begin{aligned}
r_{m,t+1} &= g_{d,t+1} + \kappa_1 A_{1,m} x_{t+1} - A_{1,m} x_t + \kappa_{1,m} A_{2,m} \sigma_{t+1}^2 - A_{2,m}\sigma_t^2 \\
r_{m,t+1} - E_t(r_{m,t+1}) &= \varphi_d \sigma_t u_{t+1} + \kappa_1 A_{1,m} \varphi_e \sigma_t e_{t+1} + \kappa_1 A_{2,m} \sigma_w w_{t+1} \\
&= \varphi_d \sigma_t u_{t+1} + \beta_{m,e} \sigma_t e_{t+1} + \beta_{m,w} \sigma_w w_{t+1}, \qquad (17)
\end{aligned}$$

where $\beta_{m,e} \equiv \kappa_{1,m} A_{1,m} \varphi_e$, and $\beta_{m,w} \equiv \kappa_1 A_{2,m}$.

It follows that

$$Var_t(r_{m,t+1}) = (\beta^2_{m,e} + \varphi^2_d)\sigma^2_t + \beta^2_{m,w}\sigma^2_w. \tag{18}$$

The solution for $A_{2,m}$ follows from exploiting the asset pricing condition,

$$exp\{E_t(m_{t+1}) + E_t(r_{m,t+1}) + 0.5Var_t(m_{t+1} + r_{m,t+1})\} = 1, \tag{19}$$

and collecting all σ_t terms. Note that

$$\begin{aligned} Var_t(m_{t+1} + r_{m,t+1}) &= Var_t[\lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,w}\sigma_w w_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} + \beta_{m,e}\sigma_t e_{t+1} + \varphi_d\sigma_t u_{t+1} + \beta_{m,w}\sigma_w w_{t+1}] \\ &= [\lambda^2_{m,\eta} + (-\lambda_{m,e} + \beta_{m,e})^2 + \varphi^2_d]\sigma^2_t + [-\lambda_{m,w} + \beta_{m,w}]^2\sigma^2_w \end{aligned}$$

where $H_m \equiv [\lambda^2_{m,\eta} + (-\lambda_{m,e} + \beta_{m,e})^2 + \varphi^2_d]$. Now collect all the σ^2_t terms in equation (19). Note that σ_t appears in $E_t(r_{m,t+1})$ as well as $E_t(m_{t+1})$. This leads to the following restriction,

$$(\theta - 1)A_2(\kappa_1\nu_1 - 1) + A_{2,m}(\kappa_{1,m}\nu_1 - 1) + \frac{H_m}{2} = 0, \tag{20}$$

which implies that

$$A_{2,m} = \frac{(1-\theta)A_2(1 - \kappa_1\nu_1) + 0.5H_m}{(1 - \kappa_{1,m}\nu_1)}. \tag{21}$$

We now derive the expression for the equity premium.

$$E_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e}\lambda_{m,e}\sigma^2_t + \beta_{m,w}\lambda_{m,w}\sigma^2_w - 0.5Var_t(r_{m,t+1}), \tag{22}$$

where $Var_t(r_{m,t+1})$ is defined in equation (18).

To derive the unconditional variance of the market return, note that

$$\begin{aligned} r_{m,t+1} - E(r_{m,t+1}) &= -\frac{x_t}{\psi} + \beta_{m,e}\sigma_t e_{t+1} + \varphi_d\sigma_t u_{t+1} + A_{2,m}(\nu_1\kappa_{1,m} - 1)[\sigma^2_t - E(\sigma^2_t)] + \beta_{m,w}\sigma_w w_{t+1} \\ &= -\frac{x_t}{\psi} + \beta_{m,e}\sigma_t e_{t+1} + \varphi_d\sigma_t u_{t+1} + A_{2,m}(\nu_1\kappa_1 - 1)[\sigma^2_t - E(\sigma^2_t)] + \beta_{m,w}\sigma_w w_{t+1}. \end{aligned}$$

Hence, the unconditional variance is

$$Var(r_m) = \frac{\sigma^2_x}{\psi^2} + [\beta^2_{m,e} + \varphi^2_d]\sigma^2 + [A_{2,m}(\nu_1\kappa_1 - 1)]^2 Var(\sigma^2_t) + \beta^2_{m,w}\sigma^2_w. \tag{23}$$

The unconditional variance of $z_{m,t}$, the price dividend ratio for the market portfolio, can be derived as follows

$$Var(z_{m,t}) = A^2_{1,m}Var(x_t) + A^2_{2,m}Var(\sigma^2_t).$$

Finally, note that the innovation to the market return volatility follows from equation (17) and is

$$var_{t+1}(r_{m,t+2}) - E_t[var_{t+1}(r_{m,t+2})] = (\beta^2_{m,e} + \varphi^2_d)\sigma_w w_{t+1}$$

5.2 The Risk-Free Rate and Its Volatility

To derive the risk-free rate, start with (12) and plug in the risk-free rate for r_i:

$$r_{f,t} = -\theta\log(\delta) + \frac{\theta}{\psi}E_t[g_{t+1}] + (1-\theta)E_t r_{a,t+1} - \frac{1}{2}Var_t[\frac{\theta}{\psi}g_{t+1} + (1-\theta)r_{a,t+1}],$$

subtract $(1-\theta)r_{f,t}$ from both sides and divide by θ, where it is assumed that $\theta \neq 0$. It then follows that

$$r_{f,t} = -\log(\delta) + \frac{1}{\psi}E_t[g_{t+1}] + \frac{(1-\theta)}{\theta}E_t[r_{a,t+1} - r_t] - \frac{1}{2\theta}Var_t[\frac{\theta}{\psi}g_{t+1} + (1-\theta)r_{a,t+1}]. \tag{24}$$

To solve the above expression note that $Var_t[\frac{\theta}{\psi}g_{t+1} + (1-\theta)r_{a,t+1}] \equiv Var_t(m_{t+1})$. Thus,

$$Var_t(m_{t+1}) \quad = \quad (\lambda_{m,\eta}^2 + \lambda_{m,e}^2)\sigma_t^2 + \lambda_{m,w}^2\sigma_w^2. \tag{25}$$

Further, if the innovation in the growth rate process is homoskedastic, the above expression simplifies as $\sigma_w^2 = 0$. The unconditional mean of $r_{f,t}$ is derived by substituting the expression for the risk premium for $r_{a,t+1}$ given in (16) and (25) into (24). This substitution yields

$$E(r_{f,t}) = -\log(\delta) + \frac{1}{\psi}E(g) + \frac{(1-\theta)}{\theta}E[r_{a,t+1} - r_t] - \frac{1}{2\theta}[(\lambda_{m,\eta}^2 + \lambda_{m,e}^2)E[\sigma_t^2] + \lambda_{m,w}^2\sigma_w^2],$$

where note that $E[\sigma_t^2] = Var(\eta)$.

The unconditional variance of $r_{f,t}$ is,

$$Var(r_{f,t}) = (\frac{1}{\psi})^2 Var(x_t) + \left\{\frac{1-\theta}{\theta}Q_1 - Q_2\frac{1}{2\theta}\right\}^2 Var(\sigma_t^2), \tag{26}$$

where $Q_2 = (\lambda_{m,\eta}^2 + \lambda_{m,e}^2)$, and $Q_1 = (-\lambda_{m,\eta} + (1-\theta)B^2 - 0.5(1+B^2))$, where B is defined above. Note that Q_1 determines the time-varying portion of the risk premium on $r_{a,t+1}$. For all practical purposes the variance of the risk-free rate is driven by the first term.