Cooperation in Stochastic OLG Games*

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This paper builds on Cremer's [3] seminal analysis which shows that (almost) complete cooperation can be achieved as an equilibrium in a game played by overlapping generations of players if the institution in which players cooperate is infinitely lived.

We analyze a similar model in which the costs of cooperation are subject to random shocks. Even if these random shocks are very small, the range of parameters for which cooperation can be sustained is decreased considerably in comparison to the deterministic case. Furthermore, we show how the efficient outcome can be approximated if the level of cooperation can be varied continuously and the cooperation technology has decreasing or constant returns to scale, while this is not possible in the case of increasing returns to scale.

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1. INTRODUCTION

In an important paper, Cremer [3] has explored the following setting: Overlapping generations of finitely lived players play a prisoners' dilemma type game. Even though every player is finitely lived, there is an equilibrium in which all but the oldest player cooperate (this is true provided cooperation is sufficiently efficiency-enhancing). The cooperative equilibrium is sustained by the threat that cooperation would end as soon as one player deviated; the promise that cooperation will continue is credible since the "organization" in which the game is played lives forever and so there is no last period; future generations could always punish deviators who are still alive in the next period.

In this note, we explore the robustness of Cremer's result if we introduce a small amount of uncertainty in the model. This extension is important because it is very unlikely that in economically relevant settings, the parameters of the game remain constant forever. Rather, tomorrow's parameters are a random variable for today's players. We find that, for a very large class of stochastic processes, even a very small uncertainty may decrease substantially the range of parameters for which cooperation can be sustained as an equilibrium. This is true even if it is extremely unlikely that noncooperation becomes more profitable than cooperation in the future.

Suppose for example that the costs of cooperation follow a random walk.[1] If the costs of cooperation become too high, then the young player will not cooperate. This implies that there will always be a "threshold cooperator", a person who is supposed to cooperate, but if the costs increase next period, the next young player will not cooperate, and so the first player will not receive any benefit from cooperating. For the threshold cooperator to have incentives to keep cooperating, the payoff if everything goes well and cooperation does not terminate in the next period must be quite high. This is possible only if the threshold is reached for parameters for which cooperation would be very efficient. Consequently, there are other parameters for which cooperation would still be efficient, but is not an equilibrium.

Our result relates to one of the central questions in modelling repeated games. The vast majority of papers which use repeated games assume that players are infinitely lived and play the same stage game in every period. Both assumptions are certainly not appealing because they depict reality, but are made rather because they facilitate the analysis and appear quite innocuous: Adding sufficiently small uncertainty to a game with infinitely lived players does not change the equilibrium considerably; also, a model of overlapping generations of finitely lived players generates (under certain conditions) results which are very similar to the standard case of infinitely

[1] Our results in the main text hold for a considerably larger class of stochastic processes.

lived players and no uncertainty. Only when we change *both* of these assumptions, we can see that in an OLG model, the equilibrium in games with uncertainty does not necessarily converge to the equilibrium in the same game under certainty, as the uncertainty becomes small.[2]

While our first result casts some doubts on the robustness of cooperation in OLG models, our second major result is more positive: Under certain conditions, efficient cooperation can (almost) be restored if the level of cooperation can be adjusted continuously, and costs and benefits of cooperation decrease proportionally. To uphold efficient cooperation, society has to use flexible cooperation rules: When cooperation is only slightly better than non-cooperation, the level of cooperation that is required from an individual in order not to trigger punishment, decreases. This ensures that there is no single "last" cooperator who would suffer a lot if cooperation were to end in the next period and who has to be compensated accordingly in case cooperation does not end. We show that the cooperation technology is decisive: Full cooperation as in the deterministic case can be approximated if there are what we call decreasing or constant returns to cooperation, while the same is not possible for increasing returns.

The first analysis of cooperation in OLG models using noncooperative game theory is Hammond [7].[3] Several other papers have analyzed cooperation in OLG games. Kandori [8] proves a folk theorem for these games: If the players' lives overlap sufficiently long and there is no discounting, any individually rational payoff can be (approximately) achieved as average payoff in an equilibrium of the OLG game. A similar result is also obtained by Smith [12] and Salant [10]. The difference between these papers and the present one is that they analyze a deterministic game while our game is stochastic: The costs of cooperation change every period by a small amount. Another difference between the present paper and the works of Kandori [8], Smith [12] and Dutta [4] is that we do not prove a folk (or anti-folk) theorem, but rather analyze in which cases the allocations, which can be obtained in the equilibrium of the deterministic OLG game and the stochastic OLG game with very little uncertainty, coincide.

Esteban and Sakovics [6] analyze the role of "institutions" in facilitating cooperation in OLG games. In their model, an institution creates a switching cost that must be paid by the young generation if they choose to alter the level of cooperation. In our model, there is no such switching cost that could stabilize cooperation.

Our paper proceeds as follows. The next section will present the basic model which is very near in spirit to Cremer's paper and shows that

[2]We will make all these notions more precise in the paper.

[3]The problem of cooperation in OLG economies has also been analyzed extensively from a cooperative point of view. For this approach, see Esteban [5] and the sources cited there.

the parameter space for which cooperation can be achieved in equilibrium shrinks discontinuously if a small uncertainty is introduced and is strictly smaller than the set of parameters for which cooperation can be achieved as an equilibrium in the deterministic setting. In Section 3, we analyze what happens if the level of cooperation can be varied in a continuous way; we define a notion of returns to scale in cooperation and show that the efficient level of cooperation can be approximated in the case of non–increasing returns to cooperation while the same is not possible for increasing returns. The last section concludes.

2. THE MODEL

2.1. Basic setup and results

Consider the following, very simple, cooperation game between two generations. The young player can either "cooperate" which costs γ and gives the old player a benefit normalized to 1, or he can "not cooperate" which costs nothing for the young player and also creates no benefit for the old player. In the next period, the old player leaves the stage and is replaced by the (formerly) young player; a new young player enters the system. There is no discounting between periods, so cooperation is efficient as long as $\gamma < 1$, and we assume that this inequality is satisfied. We assume that all players are risk neutral.[4]

With an infinite sequence of players who enter the system one after another, there is a subgame perfect equilibrium in which all players cooperate in their period as young player. If all players play the following strategy: "Cooperate, as long as every player before you has cooperated as a young player; otherwise, do not cooperate.", then it is in each young player's self interest to cooperate, because this now costs γ, but brings a benefit of 1 in the next period.[5]

We depart from Cremer's deterministic setting by assuming that γ is not constant for all times, but rather follows some quite general Markov process. The only condition we require for our main result in this section to hold is what we call "stochastic monotonicity" of the process: This assumption means that the probability that the process increases in the next period over some value $\overline{\gamma}$ is larger if the process has already reached that value than if today's value is smaller than $\overline{\gamma}$.

[4]In a previous version of this paper which is available from the authors upon request, we consider the case where players are risk averse.

[5]In general, the result of Cremer [3] is that only "almost" full cooperation can be achieved; there is no way to induce an old player to do something "cooperative" (do something which decreases his utility but benefits the society in general), as there is no further period in which this player could be punished for not cooperating. Here, the old player does not need to cooperate, and hence even the first best can be achieved as an equilibrium.

Definition 2.1. *A stochastic process is a stochastically monotone Markov process if it is a Markov process and the following condition holds for all* $\overline{\gamma}$ *and* $\underline{\gamma}$ *with* $\overline{\gamma} > \underline{\gamma}$*:*

$$Prob(\gamma_t > \overline{\gamma}|\gamma_{t-1} = \overline{\gamma}) \geq Prob(\gamma_t > \overline{\gamma}|\gamma_{t-1} = \underline{\gamma}) \tag{1}$$

This is a rather weak condition which is satisfied, for example, if γ follows a random walk or a mean reverting process.

In Cremer's game where γ is constant, cooperation can be enforced by the threat that, if a player deviates, cooperation terminates for all time afterwards. Here however, γ fluctuates and after increasing above the threshold (call it γ^T) can fall again below it. Hence, such a harsh punishment does not seem reasonable. For the formulation of the equilibrium, it is useful to define a "correct behavior" which does not lead to punishment, as follows:

Definition 2.2. *The "correct behavior" for the young player in the first period* $t = 1$ *is to cooperate if and only if* $\gamma_1 \leq \gamma^T$*. The correct behavior for the young player at time* $t > 1$ *is to cooperate if and only if* $\gamma_t \leq \gamma^T$ *and the young player in the previous period* $t - 1$ *behaved correctly.*

We will look for equilibria in which all players play the same strategy, "behave correctly (given γ^T)". We will be particularly interested in γ^*, the greatest γ^T such that the strategy profile in Definition 2.2 is still an equilibrium, and we will therefore in the following just focus on γ^* if no confusion can arise.

Correct behavior as defined here requires to punish a predecessor who stopped to cooperate although γ was smaller than γ^*, or a predecessor who did not punish his predecessor although this would have been his task. On the other hand, a player who did not cooperate because γ increased over γ^* will not be punished if γ decreases again when he is old. Note that a simple "tit for tat" strategy profile ("cooperate if and only if $\gamma_t \leq \gamma^*$ and your predecessor cooperated") might be a Nash equilibrium, but is not subgame perfect:[6] If all players keep to the tit for tat strategy profile, but one player deviated, then it is not in the self interest of his successor to actually perform the punishment, since this would mean that he would be punished in the next period, too. Hence we need the more complicated strategy defined above for a subgame perfect equilibrium.

The following proposition characterizes γ^* and shows that cooperation is not sustainable for all values for which it can be sustained in the deterministic case.

Proposition 2.1. *Suppose that the cost of cooperation develops according to a stochastically monotone Markov process, and has* $Prob(\gamma_t > 1|\gamma_0) >$

[6] This was shown by Bhaskar [1].

0 for all γ_0 and at least some t. Then the most cooperative subgame perfect equilibrium is defined implicitly by the greatest solution of

$$1 - Prob(\gamma_t > \gamma^* | \gamma_{t-1} = \gamma^*) = \gamma^*. \tag{2}$$

Moreover, $\gamma^* < 1$.

Proof. The threshold cooperator pays a cost of cooperation equal to γ^* and receives a benefit of 1 only in the case that $\gamma_t \leq \gamma^*$, that is, with probability $1 - Prob(\gamma_t > \gamma^* | \gamma_{t-1} = \gamma^*)$; the largest γ^* which gives the threshold cooperator a non-negative return for cooperation is then given by (2). Moreover, the monotonicity condition (1) implies that for $\gamma_{t-1} < \gamma^*$, $Prob(\gamma_t > \gamma^* | \gamma_{t-1})$ is smaller and hence an individual who cooperates in this case gets a positive payoff which is strictly greater than zero, the payoff which can be achieved through non-cooperation. Finally, the assumptions that $Prob(\gamma_t > 1|\gamma_0) > 0$ for all γ_0 and some t, and of stochastic monotonicity, imply that the greatest solution of (2) satisfies $\gamma^* < 1$. ▌

Note that the condition that for every γ, we have $Prob(\gamma_t > 1|\gamma_0 = \gamma) > 0$ for some t, is important: Suppose that the γ process has an upper reflecting barrier at $\bar{\bar{\gamma}}$ such that γ can never increase above $\bar{\bar{\gamma}}$. If $\bar{\bar{\gamma}} < 1$ then there is always an equilibrium in which people cooperate: If all players believe that cooperation will continue for every value of γ that can be reached, it is always in their interest to cooperate.

2.2. Examples

In this subsection, we analyze two examples of stochastically monotone Markov processes in order to illustrate proposition 2.1.

Random walk. Suppose the costs of cooperation evolve according to the following random walk:

$$\gamma_{t+1} = \begin{cases} \gamma_t + h & \text{with probability } p \\ \gamma_t - h & \text{with probability } 1-p \end{cases}. \tag{3}$$

Since the probability that γ increases is always constant for this process, the (unique) solution of (2) is $\gamma^* = 1 - p$. The intuition behind this result is simple: The threshold cooperator receives a reward in the next period only with probability $1 - p$, i.e. if costs go down in the next period; for him to be willing to cooperate, it must be true that $-\gamma + (1-p) \geq 0$.

There are some points worth mentioning here. First, note that the random walk is not stationary: For $p \geq 1/2$, it is eventually (almost) certain that there must be a date when cooperation breaks down. Notice how the result differs from a model where the cost of cooperation increase deterministically and will eventually become greater than 1. In the deterministic model, simple backwards induction shows that cooperation will break down immediately, because otherwise, there would be a last person who is supposed to cooperate, but does not receive any benefits. In the stochastic model, there is no single individual who is sure to be the last one who is supposed to cooperate. Even the threshold cooperator has a probability of $1 - p$ that cooperation continues in the next period. Therefore, cooperation can be sustained for low values of γ that make cooperation particularly beneficial.

On the other hand, if $p < 1/2$ and h is sufficiently small, the probability that γ increases above 1 goes to zero for any $\gamma_0 < 1$. Nevertheless, there are values of $\gamma < 1$ for which cooperation is not feasible; this is in contrast to a situation with deterministically decreasing costs where cooperation is feasible for all $\gamma \leq 1$. Consider a situation where $\gamma_0 = 0.7$ and $p = 0.4$. The cooperation threshold $\gamma^* = 1 - p = 0.6$ is independent of h. In particular, h could be very small. The probability that the stochastic process *ever* increases above 1 is then almost zero. Nevertheless, there would be no cooperation in the beginning, as long as $\gamma_t > 0.6$.

Finally, consider two random walks, one as the one given in (3) with $p = 1/2$, and the other one as follows:

$$\gamma_{t+1} = \begin{cases} \gamma_t + \sqrt{2}h & \text{with probability } 1/3 \\ \gamma_t - \sqrt{2}h/2 & \text{with probability } 2/3 \end{cases}.$$

These two processes have the same expected drift of zero and the same variance per period, h^2. The law of large numbers then implies that the distribution of, say, γ_{1000} given γ_0 is approximately normally distributed with expected value γ_0 and variance $1000h^2$ *for both processes*. Nevertheless, the two processes have very different cooperation thresholds: For the first process, $\gamma^* = 1/2$, for the second one, $\gamma^* = 2/3$. This indicates that in general, the exact size of the cooperation threshold is very sensitive to the particular stochastic process governing γ.

AR(1) process. We now consider the following AR(1) process:

$$\gamma_{t+1} = \alpha\gamma_t + \varepsilon_t, \tag{4}$$

where $\varepsilon_t \sim N(\mu, \sigma^2)$ for all t and $0 < \alpha < 1$. Other than the random walk above, this process has a stationary (long run) distribution, which is the normal distribution $N(\mu/(1-\alpha), \sigma^2/(1-\alpha^2))$.[7] This example will therefore illustrate that our results do not require non-stationarity of the stochastic process of γ.

[7]This claim is easy to verify: If γ_t has the claimed distribution, γ_{t+1} is again normally distributed, with expected value $\alpha\frac{\mu}{1-\alpha} + \mu = \mu/(1-\alpha)$ and variance $\alpha^2\frac{\sigma^2}{1-\alpha^2} + \sigma^2 = \frac{\sigma^2}{1-\alpha^2}$.

Proposition 2.2 characterizes how the cooperation threshold depends on the parameters of the stochastic process. Intuitively, both a larger α and a larger μ increase the probability of an upward move in the next period, for each possible state of the process. For the threshold cooperator, this means a higher risk of a cooperation breakdown and hence the cooperation threshold must be lower. The effect of σ is ambiguous.

PROPOSITION 2.2. *For all (α, μ, σ), $\gamma^* \in (0,1)$. The threshold γ^* is a decreasing function of α and μ, and may increase, decrease or stay constant as σ increases.*

Proof. Applying proposition 2.1 to the process given in (4), γ^* is the largest zero of

$$Z(\gamma) = \int_{-\infty}^{(1-\alpha)\gamma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx - \gamma. \tag{5}$$

Since Z is continuous, and $Z(\gamma) > 0$ for all $\gamma \leq 0$ and $Z(\gamma) < 0$ for all $\gamma \geq 1$, we have $0 < \gamma^* < 1$. For the second claim, observe that, although Z need not be decreasing everywhere, it certainly must be so at its largest zero, γ^*. We will now show that an increase of α or μ shifts the whole function Z downward, and therefore γ^* decreases in these two parameters. Differentiating (5) with respect to μ yields

$$\frac{\partial Z}{\partial \mu} = -\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{[(1-\alpha)\gamma-\mu]^2}{2\sigma^2}} < 0, \tag{6}$$

and differentiating (5) with respect to α yields $\frac{\partial Z}{\partial \alpha} = \gamma \frac{\partial Z}{\partial \mu} < 0$. Finally, the sign of the derivative of Z with respect to σ,

$$\frac{\partial Z}{\partial \sigma} = \frac{\mu - (1-\alpha)\gamma}{\sqrt{2\pi}\sigma^2} e^{-\frac{[(1-\alpha)\gamma-\mu]^2}{2\sigma^2}}, \tag{7}$$

changes from positive to negative at $\gamma = \mu/(1-\alpha)$. Therefore, no clear cut comparative static results with respect to σ can be obtained, and Figure 1 shows γ^* as function of σ, for $\alpha = 0.5$ and $\mu = 0.3$. ∎

As another example, consider the stochastic process with $\alpha = 3/4$, $\mu = 1/8$ and $\sigma = 0.1$. With this process, the stationary distribution of γ is normal with mean 0.5 and variance 16/700; it places a probability of about 99.95% on $\gamma < 1$, so cooperation would be efficient in most periods. However, the highest zero of (5) is $\gamma^* = 0.5$, the mean of the distribution, and hence cooperation can only be sustained in one half of the periods in the long run.

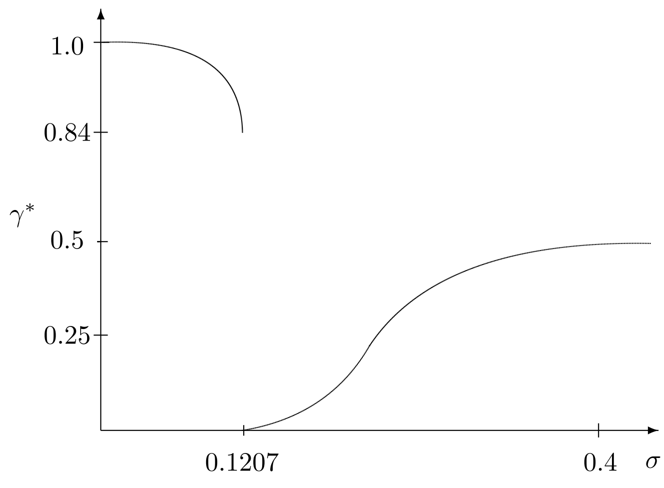

FIG. 1. γ^* as function of σ

3. RETURNS TO SCALE AND EQUILIBRIUM COOPERATION

The results of the last section have shown that cooperation ceases to be an equilibrium in the stochastic game for a lower threshold of cooperation costs than in the deterministic game. From this perspective, Cremer's result of almost complete cooperation appears to be very fragile. However, in this section, we will identify a setting in which complete cooperation can (almost) be achieved as an equilibrium in a stochastic OLG model through the use of "flexible cooperation rules". The most cooperative equilibrium in the stochastic game can approximate the most cooperative equilibrium in the deterministic game, if the size of the shock per period is small and the level of cooperation can be reduced to a positive level below full cooperation and the corresponding benefits decrease either proportionally or less than proportionally. However, approximately full cooperation is not possible, if benefits decrease more than proportionally once the level of cooperation is reduced.

The results of this section are limit results in the sense that the uncertainty with respect to γ is very small. We find it therefore useful to specify a particular stochastic process for γ, the symmetric random walk ($p = 1/2$ in equation (3)); the step size h provides a convenient measure of uncertainty, and we will let h go to 0. The loss in generality through this assumption is relatively small: In order to be comparable with the deterministic (i.e., static) setting, the stochastic process should have an expected increment of zero in each period. At the cost of additional notation, the result generalizes to other stochastic processes with a per period increment that has an expected value of zero and a sufficiently small variance.

The technology of the basic model is an extreme case of what we call *increasing returns to scale* in cooperation. Only two levels of cooperation (0 and 1) were feasible for the young player; economically, this is equivalent to saying that there are all levels of cooperation between 0 and 1 possible, but there is only a benefit for the old player if there is full cooperation. Of course, the scale economies are limited here by the fact that a level of cooperation greater than 1 is not possible.[8] Other possible cooperation technologies are *constant returns to scale (CRTS)* in cooperation: Decreasing the level of cooperation, measured by the costs incurred by the young player, decreases the benefit received by the old player proportionally. With *decreasing returns to scale (DRTS)*, decreasing the level of cooperation decreases the benefit received by the old player less than proportionally.

DEFINITION 3.3. *Let the cost of cooperation be γc, where $c \leq 1$ is the level of cooperation, and let $f(c)$ be the benefit of the old person, with $f(0) = 0$ and $f(1) = 1$.*[9] *If f is strictly convex, we speak of increasing returns to scale (IRTS) in cooperation; if f is linear, we have constant returns to scale (CRTS); and if f is strictly concave, we have decreasing returns to scale (DRTS).*

In a deterministic setting, let us define the *steady state efficient level of cooperation* $c_e(\gamma)$ as the one which maximizes a young player's expected lifetime utility:[10]

$$c_e(\gamma) := \arg\max_{z\in[0;1]} -\gamma z + f(z) \tag{8}$$

If f has IRTS or CRTS, $c_e = 1$ for $\gamma \leq 1$ and $c_e = 0$ for $\gamma > 1$. If f has DRTS, $c_e(\gamma)$ takes the value 1 for $\gamma < \gamma'$, is decreasing in $[\gamma'; \gamma'']$ and takes

[8]Perhaps, it would be more precise to speak of a technology which first exhibits increasing returns to scale (near zero cooperation) and then decreasing returns to scale. In the interest of brevity, we will be less precise.

[9]This is just a normalization, for comparison purposes.

[10]There are very many Pareto efficient plans, also in the deterministic setting; for example, to cooperate only in all even periods is clearly the best what can happen to all generations born in odd periods since they themselves do not have to contribute when they are young and receive the benefits when they are old. Even a restriction to individually rational plans does not yield a unique Pareto efficient level of cooperation since for example starting to cooperate only in period 2 is also a Pareto optimum because the generation born in period 1 cannot be made better off. The steady state efficient level of cooperation is hence only one particularly simple and appealing Pareto optimum. For more results on the notion of efficiency in stochastic OLG models, see Messner and Polborn [9].

the value 0 for $\gamma > \gamma''$, where γ' is defined by $-\gamma' + f'(1) = 0$ and γ'' is defined by $-\gamma'' + f'(0) = 0$.

In the following, we will analyze *cooperation functions* that specify the "correct behavior" for each γ. We require that a cooperation function must be incentive compatible; this means that it must always be in the self interest of the young player to transfer $c(\gamma_t)$ to the old player, given that the next player also keeps to the cooperation rules defined by c. If $c(\gamma) = 0$, the cooperation function is trivially incentive compatible at the point γ since "cooperating" costs nothing.

Definition 3.4. *A cooperation function* $c : \mathbf{R}^+ \to [0;1]$ *is called* incentive compatible at point γ *if and only if either* $c(\gamma) = 0$ *or* $c(\gamma) > 0$ *and*

$$\frac{1}{2} f(c(\gamma + h)) + \frac{1}{2} f(c(\gamma - h)) \geq \gamma c(\gamma). \tag{9}$$

A cooperation function is incentive compatible *if it is incentive compatible for all* γ.

There exists at least one incentive compatible cooperation function, since $c(\gamma) = 0$ for all γ is incentive compatible, independent of the functional form of $f(\cdot)$. Note that any incentive compatible cooperation function must have $c(\gamma) = 0$ for all $\gamma > 1$ in case of IRTS or CRTS and for all $\gamma \geq \gamma''$ in case of DRTS; otherwise, the young player could benefit by deviating. A cooperation function is called the *greatest* one if it enforces the highest possible level of cooperation *for all* γ. Formally:

Definition 3.5. *An incentive compatible cooperation function* c_G *is called the greatest cooperation function if for every incentive compatible cooperation function* $\tilde{c}$ *we have* $c_G(\gamma) \geq \tilde{c}(\gamma)$ *for all* γ.

While it is intuitive that the greatest cooperation function corresponds to the "best" equilibrium for the society since it enforces the maximum amount of cooperation feasible in this society under the constraint that all individuals must cooperate voluntarily, it is not immediately clear that a greatest cooperation function exists. However, Proposition 3.3 (1) shows that the pointwise supremum of two incentive compatible cooperation functions is itself incentive compatible, and hence a greatest cooperation function exists.

Under constant returns to scale, we can construct an incentive compatible cooperation function that decreases continuously from a cooperation level of one to a level of zero. For γ near to 1, this incentive compatible prescribes a level of cooperation of $c(\gamma) < 1$. If h is sufficiently small and c is continuous, a player who keeps to the stipulated equilibrium receives a utility of $-\gamma c(\gamma) + (1/2)c(\gamma + h) + (1/2)c(\gamma - h) \approx (1 - \gamma)c(\gamma)$. The key observation is that this utility is linear in the level of cooperation c;

hence, if a level of cooperation of 1 (that is certain in both periods of an individual's life) is better for individuals than what they can achieve by not cooperating, then any positive level of cooperation smaller than 1 is also better for them than what they would get by not cooperating. Even though partial cooperation is not optimal, it is yet better than no cooperation at all (for this parameter γ' near to 1). The same applies, a fortiori, if there are decreasing returns to cooperation.

The same is not necessarily true if there are IRTS: Then a reduction in the contribution reduces the benefits more than proportionally. Hence, $-\gamma+f(1)=1-\gamma>0$ does not imply that we must have $-\gamma c(\gamma)+f(c(\gamma))>0$ for $c(\gamma)<1$; partial cooperation can be worse than no cooperation at all, even though full cooperation would be worthwhile. This is the problem which prevents the construction of an incentive compatible cooperation function which is very near to the efficient level in the case of IRTS. These results are formally stated in Proposition 3.3.

PROPOSITION 3.3. *Consider the following technology: Cooperating at the level c costs the young player γc, and brings the old player a benefit of $f(c)$, where $f(0)=0$ and $f(1)=1$.*

1.A greatest cooperation function exists. It is the pointwise supremum of all incentive compatible cooperation functions.

2.If f is linear (CRTS), then, for every $\varepsilon>0$, there exists $\bar{h}>0$ and a cooperation function c such that c is incentive compatible for all $h\leq\bar{h}$, and $\mu\left(\{\gamma : c(\gamma)\neq c_e(\gamma)\}\right)\leq\varepsilon$, where $\mu(\cdot)$ is the Lebesgue measure.

3.If f is a strictly concave and twice differentiable function (DRTS), then for every $\varepsilon>0$ there exists $\bar{h}>0$ and a cooperation function c, such that c is incentive compatible for all $h\leq\bar{h}$, and $c_e(\gamma)-c(\gamma)\leq\varepsilon$ for all γ.

4.Let f be a strictly convex function (IRTS); then the efficient level of cooperation is $c_e=1$ for $\gamma\leq 1$ and $c_e=0$ otherwise. The greatest cooperation equilibrium has no cooperation for $\gamma\geq\gamma^*$, where $\gamma^*<1-\epsilon<1$.

Proof. See Appendix. ▌

A limitation of flexible cooperation rules as used in Proposition 3.3 is that with flexible cooperation rules, all players must be able to observe the preceding players' actions very closely. If individuals can only distinguish between "full cooperation" and "not full cooperation", the greatest incentive compatible cooperation function is given by the solution of the basic model. Hence, flexibility in the level of cooperation and non–increasing returns to scale in cooperation alone will not necessarily yield the result that the efficient level of cooperation can almost be achieved, but it is also necessary that all individuals are able to observe their predecessors and the state variables they faced exactly and that they are able to implement a rather complicated strategy profile.

An example that plausibly satisfies the assumption of non-increasing returns to scale in cooperation is the consumption-loan model introduced by Samuelson [11] and underlying the pay as you go pension scheme. Proposition 3.3 suggests that cooperation in this model is quite stable with regard to small uncertainty about the cost of cooperation.

On the other hand, increasing returns to scale in cooperation, as a result of fixed cost or a minimum efficient level of cooperation, will result in a breakdown of cooperation before this is socially optimal. It is hard to tell in general whether for example cooperation within a firm or within families is usually characterized by increasing or decreasing returns; both possibilities seem plausible for specific applications.

4. CONCLUSION

This paper analyzes an overlapping generations model of cooperation in the spirit of Cremer [3], with the addition that we assume that cooperation costs are not fixed but rather follow a random walk. We have shown that in the basic stochastic model, the introduction of even a small uncertainty in the cost of cooperation decreases considerably the range of parameters for which cooperation can be sustained in equilibrium in this economy.

Our second major result is to identify a setting in which almost the same level of cooperation can be sustained in equilibrium in the stochastic case as in the deterministic case. The assumption required for this result to go through is that the level of cooperation is flexible and there are non–increasing returns to scale in the cooperation technology. Our result hence indicates that cooperation will be more stable in environments where the level of cooperation can be adjusted continuously.

APPENDIX: PROOF OF PROPOSITION 3.3.

1. Let c_G be the pointwise supremum of all incentive compatible cooperation functions; we have to show that c_G is incentive compatible. For $\varepsilon > 0$ and $\gamma' \in [0;1]$, let $c^{\varepsilon}_{\gamma'}$ be an incentive compatible cooperation function which satisfies

$$\gamma'(c_G(\gamma') - c^{\varepsilon}_{\gamma'}(\gamma')) = \varepsilon. \tag{A.1}$$

(Note that $c^{\varepsilon}_{\gamma'}$ need not exist for every pair (γ', ε), but if c_G is not incentive compatible at γ', then there exist an infinite sequence of $\{\varepsilon\}$, converging to zero, such that all $c^{\varepsilon}_{\gamma'}$ exist.) Since c_G is the supremum of all incentive compatible cooperation functions and c^{ε}_{γ} is itself incentive compatible, we know that

$$\begin{aligned} -\gamma c^{\varepsilon}_{\gamma}(\gamma) + \frac{1}{2} f(c_G(\gamma+h)) + \frac{1}{2} f(c_G(\gamma-h)) \ &\geq \\ -\gamma c^{\varepsilon}_{\gamma}(\gamma) + \frac{1}{2} f(c^{\varepsilon}_{\gamma}(\gamma+h)) + \frac{1}{2} f(c^{\varepsilon}_{\gamma}(\gamma-h)) \ &\geq \ 0 \end{aligned} \tag{A.2}$$

must hold for all $\varepsilon > 0$. By construction, the term to the left of the first inequality sign converges for $\varepsilon \to 0$ to

$$-\gamma c_G(\gamma) + \frac{1}{2} f(c_G(\gamma-h)) + \frac{1}{2} f(c_G(\gamma+h)). \tag{A.3}$$

Hence, this term cannot be negative, and therefore c_G must be incentive compatible.

2. If f has constant returns to scale, the efficient level of cooperation is given by

$$c_e(\gamma) = \begin{cases} 1 & \text{for } 0 \leq \gamma \leq 1 \\ 0 & \text{for } \gamma > 1. \end{cases} \tag{A.4}$$

For a given step size h, let $n = \frac{1}{2}(-1 - \frac{\ln h}{\ln 2})$ such that $h = \frac{1}{2^{2n+1}}$. Define the following sequence of cooperation functions:

$$c_n(\gamma) = \begin{cases} 1 & \text{for } \gamma \leq \frac{2^{n+1}-1}{2^{n+1}} \\ 1 - \frac{m}{2^n} & \text{for } \frac{2^{n+1}-1}{2^{n+1}} + \frac{m-1}{2^{2n+1}} < \gamma \leq \frac{2^{n+1}-1}{2^{n+1}} + \frac{m}{2^{2n+1}}, m \in \{1, 2, ..., 2^n - 1\} \\ 0 & \text{for } \gamma \geq 1 \end{cases} \tag{A.5}$$

For $h \to 0$, this sequence converges to c_e in the sense described in the proposition. We have to show that it is also incentive compatible. It is clear that c_n is incentive compatible for $\gamma < \frac{2^{n+1}-1}{2^{n+1}} - h_n$ and $\gamma > 1$.

Next, consider the case that $\frac{2^{n+1}-1}{2^{n+1}} + \frac{m-1}{2^{2n+1}} < \gamma \leq \frac{2^{n+1}-1}{2^{n+1}} \frac{m}{2^{2n+1}}$ for $m \in \{1, 2, ..., 2^n - 1\}$. In that case, the expected payoff under cooperation is $(1-\gamma)(1 - \frac{m}{2^n}) \geq 0$, while not cooperating would yield only 0.

Finally, consider the last individual who is supposed to contribute the full amount of 1: $\frac{2^{n+1}-1}{2^{n+1}} - h_n < \gamma \leq \frac{2^{n+1}-1}{2^{n+1}}$. For him, the expected utility is

$$-\gamma + \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{2^n}\right) = 1 - \frac{1}{2^{n+1}} - \gamma \geq 0, \tag{A.6}$$

where equality is achieved only if $\gamma = \frac{2^{n+1}-1}{2^{n+1}}$. Hence, $c_n(\gamma)$ is incentive compatible.

3. We will prove this part of the proposition in two steps. First, fix ε, and define γ^* implicitly by $c_e(\gamma^*) = \varepsilon$. Define a set of piecewise linear

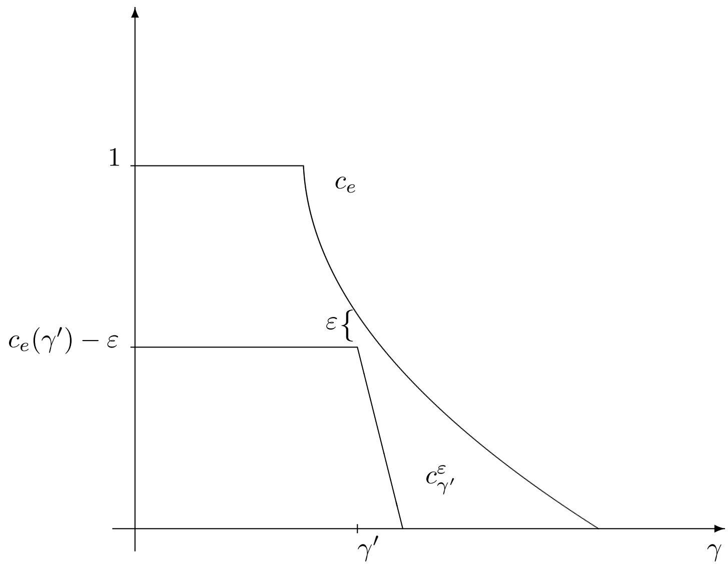

FIG. 2. Construction of $c^{\varepsilon}_{\gamma'}$

cooperation functions, parameterized by γ', as follows:

$$c_{\gamma'}(\gamma) = \begin{cases} c_e(\gamma') - \varepsilon & \text{for } \gamma \leq \gamma' \\ c_e(\gamma') - \varepsilon - a(\gamma - \gamma') & \text{for } \gamma' < \gamma \leq \hat{\gamma} \\ 0 & \text{for } \gamma > \hat{\gamma} \end{cases}, \tag{A.7}$$

where $a = \max_{\gamma \leq \gamma^*} -c'(\gamma)$ and $\hat{\gamma}$ is implicitly defined by $c_e(\gamma) - \varepsilon - a(\hat{\gamma} - \gamma) = 0$.[1] This function is composed of three linear parts and constructed such that it always lies below c_e (see Figure 2).

Claim 1: There exists $\bar{h} > 0$ such that for all $h < \bar{h}$ and all γ', the function $c_{\gamma'}$ is incentive compatible.

Proof of Claim 1: First, it should be clear that $c_{\gamma'}$ is incentive compatible for all $\gamma < \gamma' - h$ and $\gamma \geq \hat{\gamma}$. To save on notation in the following, we define $k \equiv c_{\gamma'}(\gamma)$. We now show incentive compatibility for $\gamma \in (\gamma' - h, \gamma')$ and

[1]Notice that a exists and is finite because the derivative of c_e is continuous for f strictly concave and twice differentiable.

$h \to 0$. In this interval, cooperation yields

$$-\gamma k + \frac{1}{2} f(k) + \frac{1}{2} f(c_{\gamma'}(\gamma + h)) \geq -\gamma k + \frac{1}{2} f(k) + \frac{1}{2} f(k - ah). \quad \text{(A.8)}$$

For $h \to 0$, this converges to $-\gamma k + f(k)$, which is strictly greater than 0, because of optimality considerations ($c_e(\gamma) > k > 0$, and the objective function is strictly concave).

For $\gamma \in [\gamma'; \gamma' + h]$, cooperation yields

$$-\gamma k + \frac{1}{2} f(c_{\gamma'}(\gamma - h)) + \frac{1}{2} f(c_{\gamma'}(\gamma + h)) \geq -\gamma k + \frac{1}{2} f(k) + \frac{1}{2} f(k - ah). \quad \text{(A.9)}$$

This converges for $h \to 0$ to $-\gamma k + f(k)$, which is strictly greater than 0, by the same argument as above. Hence, for sufficiently small h, $c_{\gamma'}$ is incentive compatible for $\gamma \in [\gamma'; \gamma' + h]$.

For $\gamma \in [\gamma' + h; \hat{\gamma} - h]$, cooperation yields

$$-\gamma k + \frac{1}{2} f(c_{\gamma'}(\gamma - h)) + \frac{1}{2} f(c'_{\gamma}(\gamma + h)) = -\gamma' k + f(k) + O(h^2) \quad \text{(A.10)}$$

which is strictly greater than 0 for $h \to 0$ (by the same argument as above), and hence, c'_{γ} is incentive compatible for $\gamma \in [\gamma' + h; \hat{\gamma} - h]$ and sufficiently small h.

For $\gamma' \in [\hat{\gamma} - h; \hat{\gamma}]$, cooperation yields[2]

$$\begin{aligned}
-\gamma k \ &+ \ \frac{1}{2} f(0) + \frac{1}{2} f(c_{\gamma'}(\gamma - h)) = -\gamma k + \frac{1}{2} f(k - ah - k + ah) + \frac{1}{2} f(k + ah) \\
&= \ -\gamma k + f(k) + \frac{1}{2} f'(k) \cdot (ah - k) + O(h^2) \\
&= \ -\tilde{\gamma} k + f(k) + (\tilde{\gamma} - \gamma) k + \frac{1}{2} f'(k) \cdot (ah - k) + O(h^2) \qquad \text{(A.11)}
\end{aligned}$$

where $\tilde{\gamma}$ is defined by $c_e(\tilde{\gamma}) = k$; The first two terms in (A.11) are greater than 0 by optimality ($c_e(\tilde{\gamma}) = k$ and not 0). So, there must exist a $\bar{h} > 0$ such that for $h < \bar{h}$ and all $k < ah$, the expression in (A.11) is positive, either because of the third or the fourth term. This completes the proof of claim 1.

Claim 2: The function $\tilde{c}(\gamma)$ is incentive compatible:

$$\tilde{c}(\gamma) = \begin{cases} c_e(\gamma) - \varepsilon & \text{for } 0 \leq \gamma < \gamma^* \\ 0 & \text{for } \gamma \geq \gamma^* \end{cases} .$$

Claim 1 has shown that all $\{c_\gamma\}$ are incentive compatible, and because $\tilde{c}(\gamma)$ is the pointwise supremum of all $\{c_\gamma\}$, the first part of this proposition implies that $\tilde{c}(\gamma)$ is incentive compatible, provided that $\bar{h}$ from the

[2] For this interval, the step downward leads to a cooperation level of 0.

preceding claim can be chosen independently of γ, so that all $\{c_\gamma\}$ are incentive compatible under the same step size $h < \bar{h}$.

Let $\bar{h}(\gamma)$ be the supremum over all stepsizes under which c_γ is incentive compatible. Continuity of the involved functions (f, c_e and c_γ) implies that $\bar{h}(\gamma)$ must vary continuously with γ. It follows therefore that $\min_{\gamma\in[0,\gamma^*]} \bar{h}(\gamma)$ exists, and because $\bar{h}(\gamma) > 0$ for all $\gamma \in [0, \gamma^*]$, it must be strictly positive.

4. It should be clear that it is efficient to cooperate fully if and only if $\gamma \leq 1$, since f is strictly convex and hence $f(c) < c$. We prove the remainder of this part in two steps, and by contradiction. First, we show that there cannot be a *continuous* incentive compatible cooperation function which is arbitrarily close to c_e (in the sense of the proposition). Hence, if the efficient cooperation function can be approximated at all, this has to be done by a function which decreases in discrete steps (rather than continuously). However, we then show that the ε-neighborhood of c_e (again as defined in the proposition) does not contain any incentive compatible step function either.

Claim 1: There exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon < \bar{\varepsilon}$, the ε-neighborhood of c_e does not contain a *continuous* cooperation function that is incentive compatible for $h \to 0$.

Proof of Claim 1: For each incentive compatible, continuous cooperation function c, the utility of a cooperator is given by

$$-\gamma c(\gamma) + \frac{1}{2} f(c(\gamma + h)) + \frac{1}{2} f(c(\gamma - h))$$

and converges to $-\gamma c(\gamma) + f(c(\gamma))$ for $h \to 0$. Suppose that, contrary to the hypothesis, there is a sequence $\langle c_k \rangle$ of continuous incentive compatible functions that converge to c_e in the sense specified in the proposition, and let $\tilde{c}$ be any number in $(0, 1)$. Then the sequence $\langle \gamma_k \rangle$ implicitly defined by the condition $c_k(\gamma_k) = \tilde{c}$ must converge to 1,[3] which in turn implies that $-\gamma_k c_k(\gamma_k) + f(c_k(\gamma_k)) \to -\tilde{c} + f(\tilde{c}) < 0$. Hence, there must be a $\bar{k}$ such that for $k > \bar{k}$ the functions c_k cannot be incentive compatible. By the same arguments, there cannot be an incentive compatible cooperation function near to c_e that is decreasing continuously only over some part of the range, either.

We will now consider an approximation of c_e by step functions. We will show that, the closer the approximation, the smaller must be the size of the steps in which the corresponding cooperation function decreases. We will show that the stepsize converges to zero and hence the limiting function must be continuous. It then follows from the result in Claim 1 that an

[3] Notice that γ_k is well defined, since c_k is (by assumption) continuous for every k.

arbitrary close approximation of c_e is not possible. In the following, denote the size of the ith step downward by s_i, so that the "first partial cooperator" is supposed to contribute $c(\gamma' + h) = 1 - s_1$, where $\gamma' = \max\{\gamma : c(\gamma) = 1\}$.

Claim 2: For $\gamma' \to 1$, $s_i \to 0$ for all i.

Proof of Claim 2: Since the player at $\gamma = \gamma'$ must be willing to cooperate, we have $-\gamma' + \frac{1}{2}f(1) + \frac{1}{2}f(1 - s_1) \geq 0$. As f is strictly convex, and hence $f(c) < c$ for all $0 < c < 1$, the left hand side of this inequality is smaller than $-\gamma' + 1 - \frac{1}{2}s_1$; this implies that $s_1 \leq 2(1 - \gamma)$.

The cooperator at $\gamma' + ih$ has expected utility $-(1 - \sum_{j=1}^{i} s_j)(\gamma' + ih) + \frac{1}{2}f(1 + s_i - \sum_{j=1}^{i} s_j) + f(1 - \sum_{j=1}^{i} s_j - s_{i+1})$. Since this is lower than $-(1 - \sum_{j=1}^{i} s_j)(\gamma' + ih) + 1 - \sum_{j=1}^{i} s_j + \frac{1}{2}(s_i - s_{i+1})$, a necessary condition for expected utility to be non-negative is $s_i - s_{i+1} \geq -(1 - \gamma' - ih)(1 - \sum_{j=1}^{i} s_j)$. Hence, if s_i goes to zero for $\gamma' \to 1$, so does s_{i+1}. This shows that, because $s_1 \to 0$, all other steps go to zero as well, as $\gamma' \to 0$.

REFERENCES

1. V. Bhaskar, Informational Constraints and the Overlapping Generations Model: Folk and Anti-Folk Theorems, *Rev. Econ. Studies* **65** (1998), 135-149.
2. S. Chattopadhyay and P. Gottardi, Stochastic OLG Models, Market Structure and Optimality", *J. Econ. Theory* **89** (1999), 21-67.
3. J. Cremer, Cooperation in Ongoing Organizations, *Quart. J. Econ.* **101** (1986), 33-49.
4. P. A. Dutta, Folk Theorem for Stochastic Games, *J. Econ. Theory* **66** (1995), 1-32.
5. J. M. Esteban, The Social Viability of Money. Competitive Equilibria and the Core of Overlapping Generations Economies, Springer, New York, 1991.
6. J. M.Esteban and J. Sakovics, Intertemporal Transfer Institutions, *J. Econ. Theory* **61** (1993), 189-205.
7. P. Hammond, Charity: Altruism or Cooperative Egoism?, in: "Altruism, Morality and Economic Theory" (E. Phelps, Ed.), pp. 115-131, Russell Sage Foundation, New York, 1975.
8. M. Kandori, Repeated Games Played by Overlapping Generations of Players, *Rev. Econ. Studies* **59** (1992), 81-92.
9. M. Messner and M. Polborn, Cooperation in Stochastic OLG Games, University of Western Ontario, mimeo, May 2000.
10. D. J. Salant, A Repeated Game with Finitely Lived Overlapping Generations of Players, *Games Econ. Behav.* **3** (1991), 244-259.
11. P. Samuelson, An Exact Consumption–Loan Model with or without the Social Contrivance of Money, *J. Polit. Economy*, **66** (1958), 467-482.
12. L. Smith, Folk Theorems in Overlapping Generations Games, *Games Econ. Behav.*, **4** (1992), 426-449.