Chapter 1

Introduction to Recursive Methods

These notes are targeted to advanced Master and Ph.D. students in economics. They can be of some use to researchers in macroeconomic theory. The material contained in these notes only assumes the reader to know basic math and static optimization, and an advanced undergraduate or basic graduate knowledge of economics.

Useful references are Stokey et al. (1991), Bellman (1957), Bertsekas (1976), and Chapters 6 and 19-20 in Mas-Colell et al. (1995), and Cugno and Montrucchio (1998).[1] For applications see Sargent (1987), Ljungqvist and Sargent (2003), and Adda and Cooper (2003). Chapters 1 and 2 in Stokey et al. (1991) represent a nice introduction to the topic. The student might find it useful reading these chapters before the course starts.

1.1 The Optimal Growth Model

Consider the following problem

$$\begin{aligned} V^*(k_0) &= \max_{\{k_{t+1},\, i_t,\, c_t,\, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1.1) \\ & \quad s.t. \\ k_{t+1} &= (1-\delta)k_t + i_t \\ c_t + i_t &\leq F(k_t, n_t) \\ c_t,\ k_{t+1} &\geq 0,\ n_t \in [0,1]\,;\ k_0 \text{ given.} \end{aligned}$$

where k_t is the capital stock available for period t production (i.e. accumulated past investment i_t), c_t is period t consumption, and n_t is period t labour. The utility function

[1] Cugno and Montrucchio (1998) is a very nice text in dynamic programming, written in Italian.

u is assumed to be strictly increasing, and F is assumed to have the characteristics of the *neoclassical* production function. [2] The above problem is know as the (deterministic neoclassical) optimal growth model, after the seminal works by Frank Ramsey (1928), Tjalling C. Koopmans (1963), and David Cass (1965). The first constraint in the above problem constitutes the is *law of motion* or *transition function* for capital, which describes the evolution of the *state variable* k. The optimal value of the problem $V^*(k_0)$ - seen as a function of the initial state k_0 - is denominated as the *value function*.

By the strict monotonicity of u, an optimal plan satisfies the feasibility constraint $c_t + i_t \leq F(k_t, n_t)$ with equality. Moreover, since the agent does not value leisure (and labor is productive), an optimal path is such that $n_t = 1$ for all t. Using the law of motion for k_t, one can rewrite the above problem by dropping the leisure and investment variables, and obtain:

$$\begin{aligned} V^*(k_0) &= \max_{\{k_{t+1},\, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \quad s.t. \\ c_t + k_{t+1} &= f(k_t); \\ c_t,\ k_{t+1} &\geq 0;\ k_0 \text{ given,} \end{aligned} \tag{1.2}$$

where $f(k_t) = F(k_t, 1) + (1-\delta)k_t$ is the total amount of resources available for consumption in period t. The problem can be further simplified and written - as a function of k alone - as follows [Check it!]:

$$\begin{aligned} V^*(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\ & \quad s.t. \\ 0 &\leq k_{t+1} \leq f(k_t);\ k_0 \text{ given.} \end{aligned}$$

This very simple framework includes a few interesting macroeconomic models (in addition to the optimal growth model). Set, for example, $f(k_t) = Ak_t$. If $A = 1$ the problem can be interpreted as the one of eating optimally a cake of initial size k_0. David Gale (1967)

[2] F is linearly homogeneous, with $F_1 > 0,\ F_{11} < 0$, and $F_2 > 0,\ F_{22} < 0$. In addition, commonly used assumptions are complementarity $F_{12} \geq 0$ and Inada conditions: $\lim_{k\to 0} F_1 = \infty$ and $\lim_{k\to\infty} F_1 = 0$. A typical example is the Cobb-Douglas production function

$$F(k, n) = Ak^{\alpha} n^{1-\alpha} \text{ with } \quad A > 0, \quad \alpha \in (0,1).$$

studied extensively the model with $f(k_t) = k_t$. For this reason, this example is known as `the Gale's Cake Eating problem`. If $A < 1$ the problem can still be interpreted as a cake eating problem, where the cake depreciates every period. Moreover, note that when $A > 1$ the problem becomes the simplest model of endogenous growth, the `AK-model`.[3] Finally, if we set $f(k_t) = (1 + r)\, k_t + y$, and interpret $r \geq 0$ as the interest rate and $y \geq 0$ as labor income, the model becomes one of the simplest problems of optimal saving (borrowing is not allowed since $k_{t+1} \geq 0$).

1.1.1 Sequential Formulation

We shall regard the above formulation as the *sequential form* of the problem. Notice that the objective function is an infinite sum, which should converge in order to have a well defined problem.[4] The max operator applied to the infinite dimensional object $\{k_{t+1}\}_{t=0}^{\infty}$ presents other complications (e.g., continuity and compactness are not straightforward concepts).

Once the problem is well defined, one could attack it by using appropriate generalizations to the well known tools of "static" maximization (such as Lagrange and Kuhn-Tucker theorems). Again, the fact that the choice variable belongs to an infinite dimensional space induces further technical complications, this is however a viable way.[5] If one takes the sequential approach, the infinite sum $\sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$ is a special case of the function(al) $W(k_0, k_1, k_2, \dots)$ defined on an appropriate infinite dimensional space. If the optimum is interior and W is sufficiently smooth, an optimal capital path $\left\{k_{t+1}^*\right\}_{t=0}^{\infty}$ starting from k_0 must satisfy the usual first order conditions $\frac{\partial W(k_0, k_1^*, k_2^*, \dots)}{\partial k_t} = 0$ for $t = 1, 2, \dots$ In our (time-separable) case, the first order conditions take the form

$$u'\left(f(k_{t-1}^*) - k_t^*\right) = \beta f'(k_t^*) u'\left(f(k_t^*) - k_{t+1}^*\right) \qquad t = 1, 2, \dots \tag{1.3}$$

with k_0 given. These conditions are known as Euler equations (or Euler-Lagrange conditions). In fact, conditions (1.3) are the differentiable version of a set of necessary conditions that one obtains by following a general approach to optimization problems which is denoted as the *Euler variational approach.* Unfortunately, even when the problem is

[3] This case can be seen as a Cobb-Douglas where the capital share α equals 1.

[4] The problem could, in fact, be tackled without imposing convergence of the series. This approach would however require the knowledge of advanced maximization concepts (such as overtaking), which are beyond the targets of these notes.

[5] An accessible text that introduces to the mathematics of optimization in infinite dimensional spaces is Luenberger (1969).

concave (i.e., when both u and f are concave) the Euler equations are not sufficient for an optimum. To detect on optimal path in our case one typically imposes the *transversality condition:* $\lim_{T\to\infty}\beta^T f'(k_T^*)u'\left(f(k_T^*)-k_{T+1}^*\right)k_T^*\leq 0$ as well. This will all be explained in Chapter 3.

The economic interpretation of the Euler equation is relatively easy, and quite important. Using the definition of $f(k_t)$, and recalling that $c_t = f(k_t)-k_{t+1}$, (1.3) becomes

$$\frac{u'(c_{t-1}^*)}{\beta u'(c_t^*)} = f'(k_t^*) = \frac{\partial F(k_t^*,1)}{\partial k_t} + (1-\delta), \tag{1.4}$$

which is a condition of dynamic efficiency. It asserts the equality between the marginal rate of substitution between $t-1$ and t consumption $\frac{u'(c_{t-1})}{\beta u'(c_t)}$, and its marginal rate of transformation $\frac{\partial F(k_t,1)}{\partial k_t}+(1-\delta)$: since consumption and investment are the same good, a marginal reduction of period $t-1$ consumption implies a one-to-one increase in i_{t-1}, which in turn increases by $\frac{\partial F(k_t,1)}{\partial k_t}+(1-\delta)$ the amount of goods available for period t consumption.

Under the assumptions mentioned in footnote 2, the economy allows for an interior steady state for capital and consumption: k^{ss}, and c^{ss}. In the steady state, the Euler equation implies $\frac{1}{\beta} = f'(k^{ss})$, which becomes - assuming a Cobb-Douglas production function -

$$\frac{1}{\beta} = \alpha A k^{\alpha-1} + (1-\delta) \implies k^{ss} = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$$

where $\rho = \beta^{-1}-1$ is the agent's discount rate [What is the intuition for the above expression?].

The model allows a simple graphical analysis of the dynamics in the (k_t, c_t) space. The steady state point can be identified as by the crossing point between two curves: the curve that describes the condition $k_{t+1}=k_t$ and that describing the condition $c_{t+1}=c_t$. From the feasibility constraint, we get: $k_{t+1}=k_t \Leftrightarrow c_t = f(k_t)-k_t$. The concave function $c^0(k) = f(k)-k$ hence describes the curve along which capital is time constant. Moreover notice, that given k_t next period capital k_{t+1} is decreasing in c_t. For consumption values above the curve c^0, capital must decrease, while when the chosen c is below this curve, one has $k_{t+1}>k_t$. If $f(0)=0$ this curve starts at the origin of the $\mathbb{R}_+^2$ diagram and - whenever $\lim_{k\to\infty} f'(k_t)<1$ - it intersects the zero consumption level at a finite $\bar{k}<\infty$.

The dynamics for c_t are described by condition (1.4). It is easy to see that by the strict concavity of f, and u there is only one interior point for k that generates constant consumption. This is the steady state level of capital. For capital levels above k^{ss} the value at the right hand side of the Euler equation $\alpha k^{\alpha-1}+(1-\delta)$ is lower than $\frac{1}{\beta}$, which implies a ratio $\frac{u'(c_{t-1})}{u'(c_t)}$ lower than one, hence $c_t > c_{t-1}$. The opposite happens when $k > k^{ss}$. This is hence represented by a vertical line in the (k, c) space. Of course, only one level of consumption is consistent with feasibility. But feasibility is guaranteed by the function c^0. At the crossing point of the two curves one finds the pair (k^{ss}, c^{ss}) which fully describes the steady state of the economy.

As we saw, the Euler equations alone do not guarantee optimality. One typically impose the transversality condition. It is easy to see, that any path satisfying the Euler equation and converging to the steady state satisfied the transversality condition, it is hence an optimal path [Check it].

1.1.2 Recursive Formulation

We now introduce the *Dynamic Programming* (DP) approach. To see how DP works, consider again the optimal growth model. One can show, under some quite general conditions, that the optimal growth problem can be analyzed by studying the following *functional equation*

$$V(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta V(k'), \tag{1.5}$$

where the unknown object of the equations is the function V. A function can be seen as in infinite dimensional vector. The number of equations in (1.5) is also infinite as we have one equation for every $k \geq 0$.

What are then the **advantages** of such a *recursive formulation* of this sort? First of all, realizing that a problem has a recursive structure quite often helps understand the nature of the problem we are facing. The recursive formulation reduces a single infinite-dimensional problem into a sequence of one-dimensional problems. This implies several computational advantages. Moreover, the study of the above functional equation delivers much richer conclusions than the study of the solution of a specific optimal growth problem. In dynamic programming we embed the original problem within a family of similar problems and study the general characteristics of this class of problems. Instead of deriving a specific path of optimal capital levels $\left\{k^*_{t+1}\right\}_{t=0}^{\infty}$, starting from a given k_0, a solution to the recursive problem delivers a *policy function* $k' = g(k)$ which determines tomorrow's optimal capital level k' for *all possible* levels of today's capital $k \geq 0$. The path $\left\{k^*_{t+1}\right\}_{t=0}^{\infty}$ can still be recovered by repeatedly applying the policy function starting from k_0 as: $k^*_1 = g(k_0)$; $k^*_2 = g(k^*_1)$, and so on. DP is however a much richer, concise, and elegant description of all solutions to the model.

Of course, solving a functional equation is typically a difficult task. However, once the function V is known, the derivation of the optimal choice k' is an easy exercise since it involves a simple 'static' maximization problem. Moreover, we will see that in order to study a given problem one does not always need to fully derive the exact functional form of V. Other advantages include the applied numerical computation facilities, the appealing philosophical idea of being able to summarize a complicate world (the infinite-dimensional problem) with a policy function, which is defined in terms of a small set of states.

In DP, state variables play a crucial role. But what are the state variable exactly?

> *"In some problems, the state variables and the transformations are forced upon us; in others there is a choice in these matters and the analytic solution stands or fall upon this choice; in still others, the state variables and sometimes the transformations must be artificially constructed. Experience alone, combines with often laborious trial and error, will yield suitable formulations of involved processes."* Bellman (1957), page 82.

In other terms, the detection of the state variables is sometimes very easy. In other cases it is quite complicated. Consider the following formulation of the model of optimal growth

$$\begin{aligned}
V(k_0) &= \max_{\{i_t,\, c_t,\, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& \quad s.t. \\
c_t + i_t &\leq F\left(\sum_{j=1}^{t} (1-\delta)^{t-j} i_{j-1} + (1-\delta)^t k_0, n_t\right) \\
c_t &\geq 0;\ \sum_{j=1}^{t} (1-\delta)^{t-j} i_{j-1} + (1-\delta)^t k_0 \geq 0;\ n_t \in [0,1]\,;\ k_0 \text{ given.}
\end{aligned}$$

The problem is now described in terms of *controls* alone, and can of course be solve in this form. However, "experience and trial and error" suggests that it convenient to first summarize some of the past controls $\{i_j\}_{j=0}^{t-1}$ by the state variable $k_t = \sum_{j=1}^{t}(1-\delta)^{t-j} i_{j-1} + (1-\delta)^t k_0$ and attack the problem using the (1.1)-(1.2) formulation. In this case it is also easier to detect the state as it was the predetermined variable in our problem (recall that k_0 was 'given'). Looking for predetermined variables is certainly useful, but the 'trick' does not always work.

In other cases, the purely sequential formulation can actually be very convenient. For example, one can easily check that the cake eating problem can be written in terms of controls alone as follows:

$$\begin{aligned} V(k_0) &= \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \quad s.t. \\ \sum_{t=0}^{\infty} c_t &\leq k_0;\ c_t \geq 0;\ k_0 \text{ given.} \end{aligned} \tag{1.6}$$

With the problem formulated in this form the solution is quite easy to find.

Exercise 1 *Consider the above problem and assume $u = \ln$. Use the usual Lagrangian approach to derive the optimal consumption plan for the cake eating problem.*

Let us be consistent with the main target of this chapter, and write the cake eating problem in recursive form. By analogy to (1.5), one should easily see that when u is logarithmic the functional equation associated to $(Cake)$ is

$$V(k) = \max_{0 \leq k' \leq k} \ln(k - k') + \beta V(k'). \tag{1.7}$$

As we said above, the unknown function V can be a quite complicate object to derive. However, sometimes one can recover a solution to the functional equation by first guessing a form for V and then verifying that the guess is right. Our guess for V in this specific example is an affine logarithmic form:

$$V(k) = A + B \ln(k).$$

Notice that the *guess and verify procedure* simplifies enormously the task. We now have to determine just two numbers: the coefficients A and $B > 0$. A task which can be accomplished by a simple static maximization exercise. Given our guess for V and the form of u, the problem is strictly concave, and first order conditions are necessary and sufficient for an optimum. Moreover, given that $\lim_{k \to 0} \ln(k) = -\infty$, the solution will be and interior one. We can hence disregard the constraints and solve a free maximization problem. The first order conditions are

$$\frac{1}{k - k'} = \beta \frac{B}{k'}$$

which imply the following policy rule $k' = g(k) = \frac{\beta B}{1 + \beta B} k$. Plugging it into our initial problem $V(k) = \ln(k - g(k)) + \beta V(g(k))$, we have

$$V(k) = A + \frac{1}{1 - \beta} \ln k,$$

so $B = \frac{1}{1-\beta}$, which implies the policy $g(k) = \beta k$.[6]

Exercise 2 *Visualize the feasibility set* $0 \leq k' \leq k$ *and the optimal solution* $g(k) = \beta k$ *into a two dimensional graph, where* $k \geq 0$ *is represented on the horizontal axis. [Hint: very easy!]*

Exercise 3 *Consider the optimal growth model with* $y = f(k) = k^{\alpha}$, $\alpha \in (0, 1)$ *(so* $\delta = 1$*) and log utility* ($u(c) = \ln c$)*. Guess the same class of functions as before:*

$$V(k) = A + B \ln k,$$

and show that for $B = \frac{\alpha}{1-\beta\alpha}$ *the functional equation has a solution with policy* $k' = g(k) = \alpha\beta k^{\alpha}$*. Use the policy to derive the expression for the steady state level of capital* k^{ss}*.*

Exercise 4 *Under the condition* $f(0) = 0$*, the dynamic system generated by the policy we described in the previous exercise has another non-interior steady state at zero capital and zero consumption. The non-interior steady state* $k = 0$ *is known to be a 'non-stable' stationary point. Explain the meaning of the term as formally as you can.*

Exercise 5 *Solve the Solow model with* $f(k) = k^{\alpha}$ *and* $\delta = 1$ *and show that given the saving rate* s*, the steady state level of capital is* $k^{ss} = s^{\frac{1}{1-\alpha}}$*. Show that the golden rule level of savings implies a policy of the form* $k' = \alpha k^{\alpha}$*. Compare this policy and the golden rule level of capital in the steady state, with the results you derived in the previous exercise. Discuss the possible differences between the two capital levels.*

Exercise 6 *Consider the following modification of the optimal growth model*

$$\begin{aligned}
V(k_0) &= \max_{\{k_{t+1},\, c_t\}_{t=0}^{\infty}} \widetilde{u}(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_{t-1}, c_t) \\
& \quad s.t. \\
c_t + k_{t+1} &= f(k_t) \\
c_t,\ k_{t+1} &\geq 0;\ k_0 \text{ given.}
\end{aligned}$$

Write the problem in recursive form by detecting the states, the controls, and the transition function for the states. [Hint: you might want to assume the existence of a non negative number $x \geq 0$ *such that* $\widetilde{u}(c_0) = u(x, c_0)$ *for any* c_0*].*

[6] Note that the actual value of the additive coefficient A is irrelevant for computing the optimal policy.

1.2 Finite Horizon Problems as Guided Guesses

Quite often an analytic form for V does not exist, or it is very hard to find (this is why numerical methods are useful). Moreover, even in the cases where a closed form does exist, it is not clear what a good guess is. Here is a way of getting 'good guesses'. Consider the one-period Gale's cake eating problem

$$V_1(k) := \max_{0 \leq k' \leq k} u\left(k - k^{'}\right). \tag{1.8}$$

The sub-index on the value function V indicates the time horizon of the problem. It says how many periods are left before the problem ends. In (1.8) there is only one period. Note importantly, that (1.8) describes a map from the initial function $V_0(k) \equiv 0$ and the new function V_1, as follows

$$V_1(k) := (TV_0)(k) = \max_{0 \leq k' \leq k} u\left(k - k^{'}\right) + \beta V_0(k') = \max_{0 \leq k' \leq k} u\left(k - k^{'}\right).$$

The operator T maps functions into functions. T is known as the *Bellman operator*. Given our particular choice for V_0, the n-th iteration of the Bellman operator delivers the function

$$V_n(k) := T^n V_0 = (TV_{n-1})(k) = \max_{0 \leq k' \leq k} u\left(k - k^{'}\right) + \beta V_{n-1}(k'),$$

which corresponds to the value function associated to the n-horizon problem. The solution to the functional equation (1.7) is the value function associated to the infinite horizon case, which can be see both as the limit function of the sequence $\{V_n\}$ and as a *fixed point* of the $T-$operator:

$$\lim_{n \to \infty} T^n V_0 = V_\infty = TV_\infty.$$

Below in these notes we will see that in the case of discounted sums the Bellman operator T typically describes *a contraction*. One important implication of the Contraction Mapping Theorem - which will be presented in the next chapter - is that, in the limit, we will obtain the same function V_∞, regardless our initial choice V_0.

As a second example, we can specify the cake eating problem assuming power utility: $u(c) = c^\gamma$, $\gamma \leq 1$. It is easy to show that the solution to the one-period problem (1.8) gives the policy $k' = g_0(k) = 0$. If the consumer likes the cake and there is no tomorrow, she will eat it all. This solution implies a value function of the form $V_1(k) = u\left(k - g_0(k)\right) = B_1 k^\gamma = k^\gamma$. Similarly, using first order condition, one easily gets the policy $g_1(k) = \frac{k}{1+\beta^{\frac{1}{\gamma-1}}}$ as an interior solution of the following two period problem

$$\begin{aligned}
V_2(k) &= T^2V_0 = (TV_1)(k) = \max_{0\le k'\le k} \left(k - k^{'}\right)^{\gamma} + \beta V_1(k') \\
&= (k - g_1(k))^{\gamma} + \beta V_1\left(g_1(k)\right) \\
&= \left(1 - \frac{1}{1+\beta^{\frac{1}{\gamma-1}}}\right)^{\gamma} k^{\gamma} + \beta \left(\frac{1}{1+\beta^{\frac{1}{\gamma-1}}}\right)^{\gamma} k^{\gamma} \\
&= B_2 k^{\gamma}
\end{aligned}$$

and so on. It should then be easy to guess a value function of the following form for the infinite horizon problem

$$V(k) = Bk^{\gamma} + A.$$

Exercise 7 ***(Cugno and Montrucchio, 1998)*** *Verify that the guess is the right one. In particular, show that with* $B = \left(1 - \beta^{\frac{1}{1-\gamma}}\right)^{\gamma-1}$ *the Bellman equation admits a solution, with implied policy* $k' = g(k) = \beta^{\frac{1}{1-\gamma}} k$*. Derive the expression for the constant* A*. What is the solution of the above problem when* $\gamma \to 1$*? Explain.*

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