Chapter 2

Useful Mathematics

In order to fully enjoy the beauty of dynamic programming modeling, we must first recall some useful mathematical concepts. Classical references for the material contained in this chapter are SLP, and Rudin (1964). More advanced references are Lipschutz (1994), and Rudin (1991).

2.1 Metric and Linear Spaces

Definition 1 *A metric space* (X, d) *is a set* X, *together with a metric (or distance function)* $d : X \times X \to \mathbb{R}$, *such that for all* $x, y, z \in X$ *we have: (i)* $d(x, y) \geq 0$, *with* $d(x, y) = 0$ *if and only if* $x = y$; *(ii)* $d(x, y) = d(y, x)$; *and (iii)* $d(x, z) \leq d(x, y) + d(y, z)$.

For example, the set of real numbers $X = \mathbb{R}$ together with the absolute value $d(x, y) = |x - y|$ is a metric space. Notice indeed that (i) is trivial and also (ii) is verified. To see (iii) make a picture. Remember that the absolute value is defined as follows

$$|x - y| = \begin{cases} x - y \text{ if } x \geq y \\ y - x \text{ otherwise.} \end{cases}$$

The previous discussion can be easily generalized to any n dimensional space, with $n < \infty$. The most natural metric for these spaces is the Euclidean distance.

Exercise 8 *Show that the absolute value represents a metric on the set* $\mathbb{N}$ *of the natural numbers.*

Exercise 9 *Consider the* $\mathbb{R}^n$ *space of the vectors* $\mathbf{x} = (x_1, x_2, .., x_n)$. *Show that the Euclidean distance* $d_E(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$ *defines a metric on* $\mathbb{R}^n$.

IR as an Ordered Field Notice we have implicitly defined a way of taking the difference between two real numbers and the fact that one real number can be greater than another one. This and other properties are common to any Ordered Field. In this section, we will also discuss what are the other properties that define IR as an Ordered Field.

- The fact we were able to take the difference between two real numbers is a combination of two properties. First, within a **Field** we can **A**dd up two elements being sure that the resulting element still belong to the field. Moreover, the addition satisfies the following properties: (A1) $a+b=b+a$ and (A2) $a+(b+c)=(a+b)+c$. Second, within a Field we can also **M**ultiply among them two real numbers. The multiplication satisfies two properties very similar to the ones of the addition, namely: (M1) $a\cdot b=b\cdot a$ and (M2) $a\cdot(b\cdot c)=(a\cdot b)\cdot c$. Usually we do not write explicitly the multiplication, so $a\cdot b=ab$. Moreover, a Field satisfies also a **mixed** property (AM) $a(b+c)=ab+ac$. Finally, we have the zero element 0 and the one element 1 as invariants of the two mentioned operations: namely $a+0=a$ and $a\cdot 1=a$. From these invariant elements we can define other two elements which are the inverse of the operations (and have to belong to the Field). Namely, given an element a, we can define the element s_a as $a+s_a=0$ and, when $a\neq 0$ we can also define the element q_a as $a\cdot q_a=1$. They can be also denoted as $s_a=-a$ and $q_a=a^{-1}$.

Exercise 10 *Show that the set of natural numbers $I\!N=\{1,2,3,...\}$ is not a Field, according to the previous discussion.*

Linear Spaces: IR as a Linear Space

Definition 2 *A (Real) vector space X (or **Linear Space**) is a set of elements on which we use the Field $I\!R$ to define two operations: addition and scalar multiplication. The important property of a Linear Space is that for any two elements $x,y\in X$ and real numbers $a,b\in I\!R$ we have that the vector $ax+by\in X$, where the vector ax is derived by the scalar multiplication between a and x, and the $+$ symbol stays for the addition law. The addition is defined as follows. For any three elements $x,y,z\in X$ and $a\in I\!R$ we have (i) $x+y=y+x$; (ii) $x+(y+z)=(x+y)+z$ and (iii) $a(x+y)=ax+ay$. So they are the usual low of algebra. The operations allowed between the scalars are the one we saw previously for a Field. Moreover, a vector space always contain the zero element θ, which is the invariant element of the sum, that is $x+\theta=x$. Finally, note that from the scalar multiplication we have that $0x=\theta$ and $1x=x$.*

Exercise 11 *(From Exercise 3.2 in SLP). Show that the following are vector spaces:*
(a) any finite-dimensional Euclidean space $I\!R^n$;
(b) the set of infinite sequences $(x_0,\ x_1,\ x_2,\ ...)$ *of real numbers;*
(c) the set of continuous functions on the interval $[0,1]$.
Is the set of integers $\{.....,\ -1,\ 0,\ 1,\ 2,\ ...\}$ *a vector space? Explain.*

Definition 3 *A normed linear space is a linear space* X, *together with a norm* $\|\cdot\| : X \to I\!R$, *such that for all* $x, y \in X$ *and* $a \in I\!R$ *we have: (i)* $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = \theta$; *(ii)* $\|ax\| \leq |a| \cdot \|x\|$; *and* $\|x+y\| \leq \|x\| + \|y\|$.

Note that from a norm, we can always define a metric (or distance), as follows $d(x,y) = \|x-y\|$. It is important to notice that this is a particular case of distance function, for example such that $d(x,y) = d(x-y,\theta)$. Indeed, in Linear Spaces, the zero element is very important.

The set of real numbers can also be seen as a Normed Linear space. Notice that - since we analyze *real* vector spaces - we use $\mathrm{I\!R}$ also for the scalar multiplication. However, this last use is very much different from the first one, namely the use of $\mathrm{I\!R}$ as a particular set of vectors of a Linear Space (that is we set $X = \mathrm{I\!R}$ in Definition 2). The absolute value can now be seen as a norm, and $\mathrm{I\!R}$ becomes a Normed Linear Space.

Exercise 12 *Solve Exercise 3.4 in SLP. Skip (a) and (f).*

2.2 The Infimum and the Supremum of a Set, the Completeness of $\mathrm{I\!R}$

Once we have an ordered set we can ask: what is the largest element? Does it exist? Take for example the open interval $(0,1)$ of real numbers. Common sense tells us that 0 should be the smallest element of the set and 1 should be the largest one. Moreover, we know that none of these two elements belongs to the set $(0,1)$.

Definition 4 *Given a set* $S \subseteq I\!R$ *we say that* $\mu = \sup\ S$ *if (i)* $\mu \geq x$ *for each* $x \in S$ *and (ii) for every* $\varepsilon > 0$ *there exists an element* $y \in S$ *such that* $y > \mu - \varepsilon$. *The infimum is defined symmetrically as follows:* $\lambda = \inf S$ *if (i)* $\lambda \leq x$ *for each* $x \in S$; *and (ii) for every* $\varepsilon > 0$ *there exists an element* $y \in S$ *such that* $y < \lambda + \varepsilon$.

According to the above definition, in order to derive the supremum of a set we first should consider all its upper bounds

Definition 5 *A real number M is an upper bound for the set $S \subseteq I\!R$ if $x \leq M$ for all $x \in S$.*

We then choose the smallest of such upper bounds. Similarly, the infimum of a set is its largest lower bound. In the example above we decided to take the number 1 (and not 1.2 for example) as representative of the largest element of the set, perhaps precisely because it has such a desirable property.

Notice that we introduced the concepts of sup and inf for sets of real numbers. One reason for this is that we will consider these concepts only for sets of real numbers. Another important reason is that the set of real numbers satisfies a very important property that qualifies it as *Complete,* which basically guarantees that both sup and inf are well defined concepts in $\mathbb{R}$. Before defining this property, and to understand its importance, consider first the set of Rational Numbers $\mathbf{Q}$:

$$\mathbf{Q} = \left\{ q : \exists\, n, m \in \mathbf{Z} \text{ such that } q = \frac{n}{m} = n \cdot m^{-1} = n : m \right\}.$$

Where $\mathbf{Z}$ is the set of integers, that is the natural numbers $\mathbb{N}$ with sign ($+$ or $-$) and the zero. With the usual $\geq$ operator, we can see $\mathbf{Q}$ as an ordered field.[1] Now consider the following subset $B = \{b \in \mathbf{Q} : b \cdot b \leq 2\}$. It is easy to see that the supremum (sup) of this set is the square root of 2. Moreover, we all know that the square root of 2 does not belong to $\mathbf{Q}$. This is not a nice property. In fact, this problem induced mathematicians to look for a new set of numbers. The set of real numbers does not have this problem.

> **Property C**. *The Set of Real Numbers* $\mathbb{R}$ *has the Completeness property, that is, each set of real numbers which is bounded above has the least upper bound (*$\sup$*), and each set of real numbers which is bounded below has the greatest lower bound (*$\inf$*). Where, we say that a set S is bounded above when it has an upper bound; that is, there exists a $U \in \mathbb{R}$ such that $x \leq U$ for all $x \in S$. Bounded below, if $\exists\, L \in \mathbb{R} : x \geq L\ \forall\, x \in S$.*

We will take the previous statement as an axiom. In fact, the set of real numbers can be defined as an ordered field satisfying **Property C**. So, if we consider sets of real numbers, we can always be sure that each bounded set has inf and sup. Sometimes, when the set $S \subseteq \mathbb{R}$ is not bounded, we will use the conventions $\sup S = \infty$ or/and $\inf S = -\infty$.

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[1] It is an useful exercise to check that it satisfies all the properties we introduced above for an Ordered Field.

2.3 Sequences: Convergence, liminf, limsup and the Cauchy Criterion

Definition 6 *A sequence from a set X is a mapping $f : \mathbb{N} \to X$, from the natural numbers to X. We will define each element $f(n) = x_n$, in turn, the whole mapping will be summarized as $\{x_n\}_{n\in\mathbb{N}}$.*

It is usual to denote it by $\{x_n\}_{n=0}^{\infty}$,[2] or, if does not create confusion, simply by $\{x_n\}$. Another important concept which is of common usage, and it is good to define formally is the one of subsequence.

Definition 7 *A sequence $\{y_k\}_{k\in\mathbb{N}}$ is a subsequence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that: (i) for every k we have $y_k = x_{g(k)}$, and (ii) for every n (index in the main sequence) there exists a N (index of the subsequence) such that $k \geq N$ implies $g(k) \geq n$.*

A typical notation for subsequences is $\{x_{n_k}\}$, where $n_k = g(k)$. Note that to define a subsequence we also need some sort of monotonicity property that has to be satisfied by the "renumbering" function g.

Definition 8 *A sequence of* ***real numbers*** *is said to be bounded if there exists a number M such that $|x_n| < M$ for all n.*

Definition 9 *Consider a metric space (X, d). We say that the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent to $y \in X$ if for each real number $\varepsilon > 0$ there exists a natural number N such that for all $n \geq N$ we have $d(x_n, y) < \varepsilon$. And we write $x_n \to y$, or*

$$\lim_{n\to\infty} x_n = y.$$

Notice that - since the distance function maps into $\mathbb{R}_+$ - we can equivalently say that the sequence $\{x_n\}_{n=0}^{\infty}$ in the generic metric space (X, d) converges to y, if (and only if) the sequence $\{d(x_n, y)\}_{n=0}^{\infty}$ of real numbers, converges to 0 (thus, in the familiar one-dimensional space $\mathbb{R}$, with the usual absolute value as distance function). So, this concept of convergence is the most usual one, and it easy to check .... Yes, it is easy to check, once we have the candidate y. But suppose we do not know y (and actually we do not care

[2] In fact, it is probably more common to write it as $\{x_n\}_{n=1}^{\infty}$, that is starting the sequence with x_1 not with x_0. We will see below the reason for our notational choice.

about y), but we still would like to know whether a given sequence is converging.[3] Then it can be very difficult to make the guess. Long time ago, the mathematician Cauchy defined an important concept for sequences, which is somehow close to convergence.

Definition 10 *We say that a sequence $\{x_n\}_{n=0}^{\infty}$ of elements from a metric space (X, d) is Cauchy if for each $\varepsilon > 0$ there exists a natural number N such that for all $n, m \geq N$ we have $d(x_n, x_m) < \varepsilon$.*

Below we will study the relationship between this concept and convergence. Intuitively, the difference between a Cauchy and converging sequence is that in the former, as both n and m increase the values x_n and x_m get closer and closer to each other. In a converging sequence, both x_n and x_m get closer and closer to the converging point y. Notice that if both x_n and x_m get close to y they must get close to each other as well. So convergence is a stronger concept than Cauchy. You will be asked to formally show this statement in Exercise 18 below.

We can also formally define divergence as follows.

Definition 11 *We say that a sequence of* ***real numbers*** *diverges, or converges to $+\infty$, if for each real number M there exists a natural number N such that for all $n \geq N$ we have $x_n > M$. The definition of divergence to $-\infty$ is trivially symmetric to this one.*

Notice that, of course, it is **not** true that every sequence of real numbers either converges or diverges. Indeed, the points of convergence are very special ones. In particular, they are accumulation points.

Definition 12 *Given a metric space (X, d) and a set $S \subset X$ an element x is an accumulation point for the set S if for each $\varepsilon > 0$ there exists an element $y \in S$, $y \neq x$ such that $d(x, y) < \varepsilon$.*

Notice that an accumulation point does not necessary belong to the considered set. However, we have the following result.[4]

[3] For example, we will see that in infinite horizon models this is the main problem. We just want to know whether the objective function (an infinite sum) converges somewhere or not.

[4] Here is a useful result about accumulation points, which is not crucial for the following analysis.

Remark 1 *Each bounded set $E \subset \mathbb{R}$ which contains infinitely many real numbers has at least one accumulation point.*

Proposition 1 *Consider a set S in (X, d). If a point y is an accumulation point of S then there exists a sequence of $\{y_n\}$, with $y_n \in S$ for every n, that converges to y.*

Proof From Definition 12 we can always choose a sequence of $\varepsilon_n = \frac{1}{n}$ and we can be sure that there exists a $y_n \in S$ such that $d(y_n, y) < \varepsilon_n = \frac{1}{n}$. But since $\lim_{n\to\infty} \varepsilon_n = 0$, $\lim_{n\to\infty} y_n = y$ and we are done. **Q.E.D.**

Now a useful concept that may be completely new to someone.

Definition 13 *Given a bounded sequence $\{x_n\}$ of* ***real numbers*** *we say that $-\infty < s \equiv \limsup_{n\to\infty} x_n < +\infty$ if (i) For every $\varepsilon > 0$ there is a N such that for all $n \geq N$ we have $x_n \leq s + \varepsilon$; and (ii) For every $\varepsilon > 0$ and N there is an $n > N$, such that $x_n \geq s - \varepsilon$. The concept of $\liminf$ is symmetric. We say $-\infty < l \equiv \liminf_{n\to\infty} x_n < +\infty$ if (i) For each $\varepsilon > 0$ there exists a N such that for every $n \geq N$ we have $x_n \geq l - \varepsilon$; and (ii) For each $\varepsilon > 0$ and N there exists an $n > N$, such that $x_n \leq l + \varepsilon$. We can write also*

$$\begin{aligned} s &= \inf_n \sup_{k\geq n} x_k \\ l &= \sup_n \inf_{k\geq n} x_k. \end{aligned}$$

And note that $s \geq l$ and that $\liminf x_n = -\limsup(-x)$.

Note that from the above definition both $\liminf$ and $\limsup$ must exist. For example, define $s = \inf_n y_n$ with $y_n = \sup_{k\geq n} x_k$. Since x_k are real numbers and the sequence is bounded, than by **Property C** each $y_n \in \mathbb{R}$ is well defined, an so is $s \in \mathbb{R}$. One can

Proof The proof is by construction. If $E \subset \mathbb{R}$ is bounded, then there exists an interval $[a, b]$ with $a, b \in \mathbb{R}$ such that $E \subset [a, b]$. First assume w.l.o.g. that $b > a$. We are going to split the interval in two and we will chose the one of the two semi-intervals that contain infinitely many points. Since the set contains infinitely many points this can always be done. So we have $\left[a, \frac{b-a}{2}\right]$ and $\left[\frac{b-a}{2}, b\right]$. We call the one we chose $[a_1, b_1]$. Notice that we can do this division infinitely many times. In this way, we generate at the same time two sequences and two sets of real numbers $A = \{a_n\}$ and $B = \{b_n\}$ such that $\sup A = \inf B = \xi$, note indeed that the way we constructed the sequence of real numbers,

$$0 \leq \inf B - \sup A \leq \frac{b-a}{2^n} \text{ for each } n.$$

Moreover it is easy to see that the real number ξ exists and has to belong to each of the intervals $[a_n, b_n]$, indeed if this were not true than $\sup A$ could not coincide with $\inf B$. Thus we have that for each $\varepsilon > 0$ we can construct an interval $I_\varepsilon(\xi)$ centered in ξ and such that $|x - \xi| < \varepsilon$ for all $x \in I_\varepsilon(\xi)$ and such that it contains infinitely many points different from ξ and belonging to E (so, we can always find at least one) and this proves that ξ is an accumulation point of E. **Q.E.D.**

actually show that $\liminf x_n$ is the smallest accumulation point of $\{x_n\}$ and $\limsup x_n$ is the greatest.

Consider now the lim sup concept in Definition 13. As a first approximation we can think it as the sup of the tail of the sequence. This is, of course not a well defined concept, since we do not know exactly what "the tail of the sequence" is yet. However, note that the number $s = \limsup$ has all the characteristics of a sup . Indeed, from (i) it is not possible to have values above it in the tail, and from (ii) for every $\varepsilon > 0$ we can find one element in the tail such that it is greater than $s - \varepsilon$. That is, we can show that the first definition implies the second. More formally, we will show that if s satisfies (i) and (ii), then $s = \inf_n \sup_{k\geq n} x_k$. First note that (i) implies $\forall\varepsilon > 0\ \exists N : y_N = \sup_{k\geq N}\ \{x_k\} < s+\varepsilon$, so $\inf_n\ y_n \leq y_N \leq s + \varepsilon$, but since ε is arbitrary, we can say that $\inf_n\ y_n \leq s$. Now from (ii) we have that $\forall\varepsilon > 0$ and $\forall N$ we have $y_N = \sup_{k\geq N}\ \{x_k\} > s - \varepsilon$, which further implies (since it is for any N) that actually $\inf_n\ y_n \geq s - \varepsilon$ (to see it suppose that it is not true and you will have a contradiction). Again, since the choice of ε was arbitrary, we have that $\inf_n\ y_n \geq s$ which, together with the previous result, gives $\inf_n \sup_{k\geq n} x_k = \inf_n\ y_n = s \equiv \limsup_{n\to\infty} x_k$.

Since the two concept coincide one should be able to show that also converse is true.

Exercise 13 *Consider a bounded sequence $\{x_n\}$ of real numbers. Show that if a real number is such that $s = \inf_n \sup_{k\geq n} x_k$ then it has both the properties (i) and (ii) stated for s at the beginning of Definition 13.*

Definition 14 *A point y is an accumulation point (or a limit point, or a cluster point) of a sequence $\{x_n\}$ from the metric space (X, d), if for every real number $\varepsilon > 0$ and natural number N there exists some $k > N$ such that $d(x_k, y) < \varepsilon$.*

Exercise 14 *State formally the relationship between the concept of an accumulation point in Definition 14 for sequences of real numbers and that in Definition 12 for sets.*

Exercise 15 *Show that a sequence $\{x_n\}$ of real numbers has limit L if and only if $\liminf x_n = \limsup x_n = L$.*

Thus, when a sequence converges to a point then the limit is the **unique** accumulation point for the sequence.

Theorem 1 *If a sequence $\{x_n\}$ has limit L, then every possible subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has limit L.*

Proof Straightforward from the definition of limit and subsequence. Indeed (sub)sequences can converge only to some accumulation point of the original sequence. **Q.E.D.**

Proposition 2 *Given a bounded sequence* $\{x_n\}$ *of* ***real numbers,*** *it is always possible to take a converging subsequence* $\{x_{n_k}\}$ *such that*

$$\lim_{n_k \to \infty} x_{n_k} = \limsup_{n \to \infty} x_n$$

Proof This is a particular case of accumulation point introduced in Definition 14. From Definition 13 we are sure that for every $\varepsilon > 0$ and $N > 0$ we can take an x_{n_k} such that $L - \varepsilon \leq x_{n_k} \leq L + \varepsilon$ (part (i) of Definition 13 is actually a stronger statement). Moreover, one can easily check that the sequence can be chosen to satisfy the monotone property for sequences requested in Definition 7. **Q.E.D.**

Exercise 16 *Following the lines of the previous proof, show Proposition 2 for the* $\liminf$.

Corollary 1 *From a bounded sequence* $\{x_n\}$ *of* ***real numbers,*** *it is always possible to extract a converging sequence.*

Proof This result was originally due to Weierstrass. From Proposition 2, we can always take a converging subsequence which converges to $\limsup$. **Q.E.D.**

Theorem 2 *Given a sequence* $\{x_n\}$ *of* ***real numbers*** *and* $y \in I\!R$. *If from any subsequence of* $\{x_n\}$ *it is possible to extract a converging sub-subsequence which converges to the same* y *then* $\lim_{n\to\infty} x_n = y$.

Proof From Proposition 2 (and Exercise 16) we know that among all the possible subsequences from $\{x_n\}$ there will be in particular a subsequence that converges to $\limsup \{x_n\}$ and another converging to $\liminf \{x_n\}$, but then $\liminf \{x_n\} = \limsup \{x_n\}$ and we are done by Exercise 15. **Q.E.D.**

Quick Review about Series

Definition 15 *A series is a couple of sequences* $\{x_n, s_n\}_{n=0}^{\infty}$ *where* s_n *are called the partial sums and are such that*

$$s_n = \sum_{t=0}^{n} x_t.$$

Sometimes, abusing in notation, a series is written $\sum_{t=0}^{\infty} x_t$ *or even simply* $\sum_t x_t$.

Definition 16 *A series is said to be convergent if exists finite the limit of the sequence of partial sums*

$$s = \lim_{n \to \infty} s_n < \infty,$$

*and the number s is said to be the **sum** of the series and we can write $s = \sum_{t=0}^{\infty} x_t$.*

Note that, in principle the above definition allows us to write the infinite sum only if the series converges. Actually a series can be of three *characters.* (i) Convergent, (ii) Divergent or (iii) Indeterminate. A series is *divergent* when the sequence of partial sums goes either to minus or to plus infinity.

2.4 Closed and Compact Sets: Maxima and Minima

We are now ready to define closed and compact sets. There are many concepts of closedness and compactness. Here we will use sequences to define them. So, to be precise we will speak about sequential closedness and sequential compactness, but for the spaces we will be interested in, these concepts will all coincide.

Definition 17 *A set S is closed if for each convergent sequence of elements in S, the point of convergence is in S, i.e.*

$$x_n \to y \text{ and } x_n \in S \ \forall n, \text{ implies } y \in S.$$

A set is open if its complement is closed. Where, in $\mathbb{R}$, the complement of S is defined $S^c \equiv \{x \in \mathbb{R} : x \notin S\}$.

Definition 18 *A set S is (sequentially) compact if for each sequence of elements in S we can take a subsequence which converges in S.*

Exercise 17 *Show that a set S is closed if and only if it contains all its accumulation points.*

Theorem 3 *A set $F \subset \mathbb{R}$ is (sequentially) compact if and only if it is closed and bounded.*

Proof ($\Rightarrow$) Assume F is compact. Then first it has to be limited otherwise it is possible to show that there must exist a subsequence going to either $+\infty$ or $-\infty$ (example). In this last case, by the definition of sequence converging to (say) $+\infty$ it will be impossible to find a sub-subsequence converging to any real number, since from Theorem 1 all of them will converge to $+\infty$, and this would contradict the assumption of compactness. Now we show that F has to be closed by showing that F contains all its accumulation points. So consider an accumulation point y. From Proposition 1 we know that there must be a sequence $\{y_n\}$ converging to y. Moreover, this sequence is such that $y_n \in F$ for every n. Now we know that by the definition of compactness it is possible to extract from $\{y_n\}$ a subsequence converging to a point $y' \in F$, but from Theorem 1 we know that each sub-sequence must converge to y, so $y' = y \in F$ and we are done.

($\Leftarrow$) Now assume F is closed and bounded, and consider a generic sequence $\{x_n\}$ with $x_n \in F$ for all n. Since F is bounded, this sequence must be bounded. Hence from Proposition 2 it is possible to extract a converging subsequence $\{x_{n_k}\}$. Moreover, since F is closed and $x_{n_k} \in F$ $\forall k$, the converging point must belong to F, and we are done. **Q.E.D.**

Definition 19 *An element is a maximum for S if $x = \sup S$ and $x \in S$. Similarly, y is the minimum of S if $y = \inf S$ and $y \in S$.*

Theorem 4 *Each closed and bounded set $F \subset \mathbb{R}$ has a Max and a Min.*

Proof Since it is bounded, it has $\sup F < \infty$. By the definition of $\sup F$, it is an accumulation point. Thus, by Proposition 1 we can always construct a sequence with elements in F that converges to $\sup F$. By the closedness of F the limit of this sequence must belong to F. **Q.E.D.**

2.5 Complete Metric Spaces and The Contraction Mapping Theorem

Complete Metric Spaces: $\mathbb{R}$ as a Complete Metric Space Recall that when we introduced the concept of a Cauchy sequence and we related it to the concept of a convergent sequence we also said that the latter is stronger then former. In the exercise below you are asked to show it formally.

Exercise 18 *Consider a generic metric space (X, d). Show that each convergent sequence is Cauchy.*

In fact, convergence is a strictly stronger concept only in some metric spaces. Metric spaces where the two concepts coincide are said to be *Complete*.

Definition 20 *A metric space (X, d) is said to be Complete if any Cauchy Sequence is convergent in X.*

The concept of completeness here seems very different from the one we saw in Section 2.2. However, one can show that there is a very close relationship between the two concepts of completeness.

Exercise 19 *Show that in the metric space $(\mathbb{R}, |\cdot|)$ every Cauchy sequence is bounded.*

Theorem 5 *$(\mathbb{R}, |\cdot|)$ is a Complete metric space.*

Proof Consider a generic Cauchy sequence $\{x_n\}$ in $\mathbb{R}$. From Exercise 19, we know it is bounded. Then from Corollary 1 of Proposition 2 there must be a subsequence converging to $y = \limsup x_n$. Using the triangular inequality, it is not difficult to see that if a Cauchy sequence has a converging subsequence, it is convergent. Hence we are done. **Q.E.D.**

To have an example of a non complete metric space consider again the set of real numbers, with the following metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \max\left\{\frac{1}{1+|x|}, \frac{1}{1+|y|}\right\} & \text{otherwise.} \end{cases}$$

[Check that d is actually a metric!]. Now consider the sequence $\{x_n\}_{n=0}^{\infty}$ of integers $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ \ldots\ x_n = n,\ \ldots$. It is easy to see that as m and n increase the distance $d(x_n, x_m) = d(n, m)$ goes to zero.[5] Hence the sequence is Cauchy. However, it is easy to see that the sequence $\{x_n\}_{n=0}^{\infty} = \{n\}_{n=0}^{\infty}$ does not converge to any real number x, since for any fixed $x < \infty$ we have $d(x, n) \geq \frac{1}{1+|x|} > 0$.

Another possibility is to keep the metric of the absolute value and change X. Assume for example $X = \mathbf{Q}$, the set of rational numbers. Consider now a sequence of rational numbers that converges to $\sqrt{2}$. It is clear that this sequence would satisfy the Cauchy criterion, but would not converge in $\mathbf{Q}$ by construction. Each time we guess a $q \in \mathbf{Q}$ at which the sequence converges, since $q \neq \sqrt{2}$, we must have $\left|q - \sqrt{2}\right| = \varepsilon > 0$ for some $\varepsilon > 0$, hence we can find a contradiction each time the elements of the sequence are sufficiently close to $\sqrt{2}$.

Exercise 20 *Show that if (X, d) is a complete metric space and S is a closed subset of X, then (S, d) is still a complete metric space.*

[5] If $n, m \geq N$, then $d(n, m) \leq \frac{1}{1+N}$.

Banach Space (Complete Normed Linear Space): $I\!R$ and C^1 as Banach Spaces

Definition 21 *A Complete Normed Linear Space is called Banach Space.*

Directly, from the definition is clear that $I\!R$ is a Banach space.

Theorem 6 *Let $X \subset I\!R$ and $\mathcal{C}(X)$ the set of bounded and continuous*[6] *real valued functions $f : X \to I\!R$, together with the "sup" norm $\|\cdot\|_\infty : \mathcal{C}(X) \to I\!R$ such that $\|f\|_\infty \equiv \sup_t |f(t)|$ is a Banach Space.*

Proof From Exercise 12 we know that this space is a normed linear space. We want to show that it is complete. Here is a sketch of the proof based on Theorem 3.1 in SLP (Page.47-49). One has to show that for any Cauchy sequence $\{f_n\}$ of functions there exists a limit function f^* such that $f_n \to f^*$, and $f^* \in \mathcal{C}(X)$. Our candidate f^* is defined as follows: $f^*(x) = \lim_n f_n(x)$ for any given x. Notice that f^* is well defined since for any given x, the sequence of real numbers $y_n = f_n(x)$ is Cauchy and, from the completeness of $I\!R$ it must converge. This type of convergence is called pointwise.

We have to first show that $\{f_n\}$ converge to our candidate in sup norm, or uniformly. Second, that f^* is bounded and continuous (we just show that f^* is real valued). The first part requires that for any $\varepsilon > 0$ exists a N_ε such that for all $n \geq N_\varepsilon$ we have

$$\sup_x |f_n(x) - f^*(x)| \leq \varepsilon.$$

Notice that if N_ε is such that for all $n, m \geq N_\varepsilon$ we have $\|f_n - f_m\| \leq \frac{\varepsilon}{2}$, then for any given x it must be that for $n \geq N_\varepsilon$

$$\begin{aligned} |f_n(x) - f^*(x)| &\leq |f_n(x) - f_{m_x}(x)| + |f_{m_x}(x) - f^*(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where $m_x \geq N_\varepsilon$ is possibly a different number for each x, but m_x must exist since we saw that for all x $f_m(x) \to f^*(x)$. For each x, there is a function $f_{m_x}(\cdot)$ that can be used as pivotal to show that f_n and f^* are close to each other at the point x. This function never appears in the left hand side however. Hence we are done. To show that f^* is bounded and continuous, let x be given and consider an arbitrary y in an proper neighborhood of x. If n is such that $\|f^* - f_n\| < \frac{\varepsilon}{3}$, we have

$$\begin{aligned} |f^*(x) - f^*(y)| &\leq |f^*(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f^*(y) - f_n(y)| \\ &\leq 2\|f^* - f_n\| + |f_n(x) - f_n(y)| \\ &< 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

[6]Continuity is intended with respect to the topology induced by the Euclidean norm in X.

This is true for all y belonging to the appropriate neighborhood of x. The second term in the second line can then be chosen small enough since f_n is continuous. **Q.E.D.**

The sup norm defines in the obvious way the sup distance function, which will be sometimes denoted by d_∞.

The Contraction Mapping Theorem

Definition 22 *Let* (X, d) *a metric space and* $T : X \to X$ *a function mapping* X *into itself.* T *is a Contraction (with modulus* β*) if for some* $\beta < 1$ *we have*

$$d(Tx, Ty) \leq \beta d(x, y), \text{ for all } x, y \in X.$$

To understand the idea, make a picture of a contraction from $\mathbb{R}_+$ to $\mathbb{R}_+$. For example draw a line with slope $\beta \in [0, 1)$ and nonnegative intercept.

Theorem 7 *If* (X, d) *is a complete metric space and* $T : X \to X$ *is a contraction with modulus* β*, then (i)* T *has exactly one fixed point* x^* *in* X*, i.e.* $x^* = Tx^*$*, and (ii) for all* $x_0 \in X$ *we have* $d(T^n x_0, T^n x^*) \leq \beta^n d(x_0, x^*)$, $n = 0, 1, 2, ...$

Proof The proof goes as follows. Start with a generic $x_0 \in X$ and construct a sequence as follows: $x_n = T^n x_0$. Since X is complete, to show existence in (i), it suffice to note that the so generated sequence is Cauchy. Let's show it.

Notice first that by repeatedly applying our map T one gets, $d(x_{n+1}, x_n) \leq \beta^n d(x_1, x_0)$. Consider now, w.l.o.g., $m = n + p + 1$; we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq \beta^p d(x_{n+1}, x_n) + \beta^{p-1} d(x_{n+1}, x_n) + ... + d(x_{n+1}, x_n) \\ &\leq \frac{1}{1-\beta} d(x_{n+1}, x_n) \leq \frac{\beta^n}{1-\beta} d(x_1, x_0) \end{aligned}$$

where, for the first inequality we used property (iii) in Definition 1, for the second and fourth we used the property of our sequence. The third inequality is trivial, since $d(x_{n+1}, x_n) \geq 0$ and $\beta < 1$.

As a consequence, for each $\varepsilon > 0$, we can choose an index N large enough and have $\frac{\beta^N}{1-\beta} d(x_1, x_0) < \varepsilon$ for all $n, m \geq N$ as it is required by the definition of Cauchy sequence. Since (X, d) is complete this sequence must converge, that is, there must exist a x^* such that

$$\lim_{n \to \infty} x_n = x^*.$$

By the continuity of T (in fact T is uniformly continuous) we have that the limit point of the sequence is our fixe fixed point of T[7]

$$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

It remains to show that the limit of the sequence $x^* = Tx^*$ is unique. Suppose the contrary, and call $x^{**} = Tx^{**}$ the second fixed point. Note that

$$d\left(x^{**}, x^*\right) = d\left(Tx^{**}, Tx^*\right) \leq \beta d\left(x^{**}, x^*\right),$$

which is a contradiction as long as $d\left(x^{**}, x^*\right) > 0,$ hence we must have $d\left(x^{**}, x^*\right) = 0,$ that is $x^*$and x^{**} must in fact be the same point. The (ii) part of the proof is simple and left as an exercise, see also Theorem 3.2 in SLP (page 50-52). **Q.E.D.**

Exercise 21 *Notice that we allow for $\beta = 0$. Draw a mapping $T : I\!R_+ \to I\!R_+$ which is a contraction of modulus $\beta = 0$ and show graphically that T must have a unique fixed point. Now formally show the statement, that is, show that if T is a contraction of modulus $\beta = 0$ in a complete metric space then T admits a unique fixed point. Is the completeness assumption on (X, d) crucial in this case?*

The following result is due to Blackwell and provides a couple of easy-to-verify conditions for a contraction.

Theorem 8 *Let $X \subset I\!R^l$, and let $\mathbf{B}(X)$ the space of bounded functions $f : X \to I\!R$, with the sup norm. Let $T : \mathbf{B}(X) \to \mathbf{B}(X)$ be an operator satisfying: (i) $f, g \in \mathbf{B}(X)$ and $f(x) \leq g(x)$ $\forall x \in X$, implies $(Tf)(x) \leq (Tg)(x)$ $\forall x \in X$, and (ii) there exists some $0 \leq \beta < 1$ such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \ \forall f \in \mathbf{B}(X), \ a \geq 0, \ x \in X.$$

Then T is a contraction with modulus β.

Proof Notice that for any two functions f and g in $\mathbf{B}(X)$ the sup norm $d_\infty(f, g) \equiv \sup |f(x) - g(x)|$ implies that $g(x) \geq f(x) - d_\infty(f, g)$ for all x; or $g + d_\infty(f, g) \geq f$. From the monotonicity of the operator T (property (i)) we have $T\left[g + d_\infty(f, g)\right](x) \geq Tf(x)$ for all x. Now using property (ii) (discounting) with $a \equiv d_\infty(f, g) \geq 0$, we have $Tg + \beta d_\infty(f, g) \geq Tf$ or $\beta d_\infty(f, g) \geq (Tf)(x) - (Tg)(x)$ for all x. But this implies $d_\infty(Tf, Tg) \leq \beta d_\infty(f, g)$, hence T is a contraction with modulus β. **Q.E.D.**

Assumptions (i) and (ii) in the above theorem are usually called *Blackwell sufficient conditions* for a contraction.

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[7]See also Exercise 25.

Exercise 22 *(a) Show that the Bellman operator of the Optimal Growth Model satisfies Blackwell's sufficient conditions if* u *is a bounded function. (b) One of the most commonly used utility functions in growth theory is the CRRA utility* $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$; $\sigma \geq 0$. *We know that in this case* u *is not a bounded function. Could you suggest a way of showing that Bellman operator of the optimal growth model with no technological improvement is a contraction when* $\sigma < 1$? *[Hint: Use the Inada conditions.]*

2.6 Continuity and the Maximum Theorem

Continuity and Uniform Continuity: The Weierstrass's Theorem We start by reviewing one of the most basic topological concepts: Continuity.

Definition 23 *A real valued function* $f : X \to I\!R$ *is continuous at* $x \in X$ *if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$d(x, y) < \delta \text{ implies } f(x) - \varepsilon < f(y) < f(x) + \varepsilon.$$

A function is continuous in a set $S \subset X$ *if it is continuous at every point* $x \in S$.

Definition 24 *A real value function is said to be* ***uniformly*** *continuous in* $S \subset X$ *if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for any two points* $x, y \in S$ *such that* $d(x, y) < \delta$ *we have* $f(x) - \varepsilon < f(y) < f(x) + \varepsilon$.

Notice that uniform continuity is stronger than continuity. Indeed, for any $\varepsilon > 0$ uniform continuity requires to find a δ, which is the same for each point $x \in X$. While the usual concept of continuity allows you to choose a different δ for each x.

Exercise 23 *Show that if* f *is uniformly continuous in* E, *then* f *is also continuous in* E.

Exercise 24 *Show that a continuous function on a compact set is uniformly continuous. [A bit difficult].*

Exercise 25 *Show that a real valued function* $f : I\!R \to I\!R$, *is continuous in* x *if and only if for every sequence* $\{x_n\}$ *converging to* x *we have that the implied sequence* $\{y_n = f(x_n)\}$ *converges to* $y = f(x)$.

We are now ready to show one of the most well known theorems in Real analysis.

Theorem 9 *(Weierstrass) A continuous real valued function f defined on a compact set S has maximum and minimum.*

Proof The simplest way to show this result is using sequences. We already saw that the supremum of a set of real numbers is an accumulation point. As a consequence, Proposition 1 guarantees that there exists a sequence y_n that converges to $y^* = \sup_x f(x)$. Since f is continuous, Exercise 25 implies that the induced sequence x_n such that $y_n = f(x_n)$ must also converge (say to the point x^*), hence $y^* = f(x^*)$. By the compactness of S, x^* must belong to it, so x^* is the maximum. **Q.E.D.**

Exercise 26 *Show that a continuous function on a compact set is bounded, i.e.* $\sup_x f(x) < \infty$.

Exercise 27 *(**Brower Fixed Point**) Show that a continuous function defined on the compact set $[0,1]$ and mapping values into $[0,1]$ has a fixed point, that is, a point x^* such that $f(x^*) = x^*$.*

Correspondences: Some Basic Concepts Correspondences are more complicated concepts than functions but the idea is similar.

Definition 25 *A map $\Gamma : X \to Y$ is said a correspondence if for any $x \in X$ assigns a set $\Gamma(x) \subset Y$.*

Sometimes many concepts are easy to understand if we consider the case where it fails to be satisfied.

Definition 26 *A non empty correspondence Γ is said to be **not** lower hemi-continuous (**not l.h.c.**) at x if for at least one converging sequence $x_n \to x$, it is* not *possible to reach a point $y \in \Gamma(x)$ with a converging sequence of points such that $y_n \in \Gamma(x_n)$.*

Definition 27 *A non empty and compact-valued correspondence Γ is said to be **not** upper hemi-continuous (**not u.h.c.**) at x if for at least one converging sequence $x_n \to x$, it is possible to find a converging (sub)sequence $y_k \in \Gamma(x_{n_k})$, (with x_{n_k} a subsequence of x_n), whose limit point $y$is such that $y \notin \Gamma(x)$.*

We can have an idea of these concepts by drawing graphs. If we try to visualize when one of the two concepts fail we will immediately understand that l.h.c. does not allows for "discontinuities" that appear as "explosions" in the set of points, whereas the u.h.c. does not allows for "discontinuities" that appear as "implosions".[8]

Definition 28 *A Correspondence* Γ *which is both u.h.c. and l.h.c. at any* x *is said to be continuous.*

Note that a single valued correspondence is actually a function.

Exercise 28 *Show that a single valued correspondence is l.h.c. if and only if it is u.h.c.*

Exercise 29 *Is a continuous correspondence also a continuous function ? Is a continuous function also a continuous correspondence ? Try to justify formally your answer.*

Exercise 30 *Show the following useful result. Let* f_i, g_i $i = 1, 2, ...N$ *be continuous real valued functions such that* $f_i \geq g_i$*for all* i*. Define* $\Gamma(x) = \{y \in I\!R^N : \; g_i(x) \leq y_i \leq f_i(x),$ $i = 1, 2, ...N\}$*. Then* Γ*is a continuous correspondence.*

From Exercise 25 we saw that an elementary way of defining continuity of f at a point is to guarantee that if $x_n \to x$ then $f(x_n) \to f(x)$. Here below we provide some generalizations based on this definition:

Definition 29 *A function* $f : X \to I\!R$ *is upper (lower) semi-continuous at* x *if for all converging sequences* $x_n \to x$*,* $\limsup_{n\to\infty} f(x_n) \leq f(x)$ *(*$\liminf \; f(x_n) \geq f(x)$*).*

Upper semicontinuity is immediately extended to functions that possibly take values $-\infty$ and lower semicontinuity can be extended to functions that can possibly take the value $+\infty$. More in general, upper semicontinuos functions can be equivalently defined as those functions having closed upper level sets while lower semicontinuous functions those having close lower level sets for all $v \in \mathbb{R}$. Where the upper and lower level set at level v is defined, respectively, as:

$$U_v := \{x \in X | f(x) \geq v\}; \quad \text{and} \quad L_v := \{x \in X | f(x) \leq v\}.$$

We have the following extension to the Weierstrass theorem.

[8] In general, since the correspondence is (more or less implicitly) assumed to map into a compact Hausdorff space, u.h.c. is equivalent to the *graph* of Γ is closed, where the graph of a correspondence is defined as $Gr(\Gamma) = \{(x, y) \in X \times Y : \; y \in \Gamma(x)\}$ (see Theorem 14.12 in Aliprantis et al., 1994). While l.h.c. is *implied* by the fact that $Gr(\Gamma)$ is an open set (but the converse is not necessarily true: for example, a continuous correspondence is obviously l.h.c. but its graph is closed).

Lemma 0. Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be upper-semincontinuous and $C \subset \mathbb{R}^N$ be compact. If $f^* := \sup_{x \in C} f(x) > -\infty$ then there is a $x^* \in C$ such that $f(x^*) = f^* < \infty$.

Proof: Consider the function $\hat{f}$ defined pointwise as follows: $\hat{f}(x) = f(x)$ for $x \in C$ and $\hat{f}(x) = -\infty$. Clearly $f^* = \sup_{x \in \mathbb{R}^N} \hat{f}(x)$ and the set of maximisers of the two problems concide. The closeness of the set C and the upper-semicontinuity of f makes $\hat{f}$ upper-semicontinuous as well, that it has closed (possibly empty) upper level sets $X_r := \{x \in \mathbb{R}^N | \hat{f}(x) \geq r\}$ for all $r > -\infty$. Since C is bounded, X_r are bounded (again, possibly empty) for all $r > -\infty$. In other terms X_r are all compact sets. Now, the definition of sup implies that for each $r < f^*$, the set X_r is non-empty. x^* is hence one element of the the set generated by intersection $\cap_{r<f^*} X_r$. Since f never takes the value $+\infty$ and $\hat{f}$ is bounded above by f^*, it must be that $f^* = f(x^*) = \hat{f}(x^*) < \infty$. And we are done. **Q.E.D.**

There is a way of seeing continuity of a correspondence very similar to that for functions, as specified in the above definition. We just need to define the appropriate extension of Definition 13 for sets.

Definition 30 *Let $\{A_n\}_{n=1}^{\infty}$ a sequence of sets in $\mathbb{R}^n$. We say that $x \in \liminf A_n$ if every neighborhood I_x of x intersects all A_n for a n sufficiently large, i.e. for each I_x there is a N such that for all $n \geq N$ $I_x \cap A_n \neq \varnothing$. We say $x \in \limsup A_n$ if every neighborhood of x intersects infinitely many A_n. Clearly $\liminf A_n \subset \limsup A_n$. We say that $A_n \to A$ or $\lim A_n = A$ if limsup and liminf are the same set.*

In words, the set $\liminf \Gamma(x_n)$ is the set of all possible limit points of sequences $\{y_n\}_n$ such that $y_n \in \Gamma(x_n)$ for all n, while $\limsup \Gamma(x_n)$ is the set of all cluster points of such sequences.

Definition 31 *Let $\Gamma(\cdot)$ be a correspondence which maps points X into subsets of $\mathbb{R}^n$, and let $\{x_n\}_n$ be a sequence converging to x. We say that Γ is **u.h.c.** (resp. **l.h.c.**) [resp. continuous] if $\Gamma(x) \supset \limsup \Gamma(x_n)$ (resp. $\Gamma(x) \subset \liminf \Gamma(x_n)$), [resp. $\Gamma(x) = \lim \Gamma(x_n)$].*

The definition of u.h.c. in Definition 27 is hence more restrictive than that in Definition **??** in that the later does not require to work on compact spaces. In contrast, the correspondence between the two concepts of l.h.c. is perfect.

Exercise 31 *Show that when we consider compact valued correspondences the last definitions and those in Definitions 26 and 27 are equivalent.*

Note that it can be the case that a correspondence fails to be u.h.c. at a point x according to Defintion 27 while $\Gamma(x) = \lim \Gamma(x_n)$. That is, strictly speaking, we should have used a qualification such as *'weak'* for u.h.c. and continuity properties in Definition 31. Since in the theorems below we will assume compact valued correspondences this distinction will not matter.

Exercise 32 *Let $f : X \times Z$, and define the graph of f given z as*

$$Grf(z) = \{(x, y) \in X \times Y : \ -B \leq y \leq f(x, z)\},$$

where $B < \infty$ guarantees to have a compact valued correspondence for each given z. Show that if $Grf(z)$ is a continuous correspondence then f is jointly continuous in (x, z). What are the properties of f if $Grf(z)$ is a upper (lower) hemi-continuous correspondence?

The Maximum Theorem This is not a simple result, but it is one of the most important ones.

Theorem 10 *Let $X \subset I\!R^l$ and $Y \subset I\!R^m$, let $f : X \times Y \to I\!R$ be a continuous function, and let $\Gamma : X \twoheadrightarrow Y$ be a compact valued and continuous correspondence. Then (i) the function $h : X \to I\!R$ defined as*

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

is continuous, and the policy correspondence $G : X \to Y$ defined as

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

is (ii) nonempty, (iii) compact valued and (iv) upper hemi-continuous.

Proof (sketch): (i) Recall the 'implosions and explosions interpretation' for correspondences. It f is continuous it cannot have jumps. So the maximum value h can have jumps (i.e., it can be discontinuous) only if there are some implosions or explosions in the feasible set. In particular, with implosions we can have a sharp reduction of the sup, with explosions h can jump upward. The continuity of Γ guarantees that there are no such implosions and explosions in the feasible set, hence h must vary continuously with x.

Now, from the Weierstrass Theorem a maximum exists for any x, so (ii) is immediate. (iii) To show that G is compact valued, note first that $G(x) \subset \Gamma(x)$ hence G is bounded. To see that G maps closed sets for all x, note that for any convergent sequence y_n with

$y_n \in G(x)$ it must be that $f(x, y_n) = h(x)$ so from the continuity of f and Exercise 25, we have that

$$\lim_{n\to\infty} f(x, y_n) = f(x, \lim_{n\to\infty} y_n) = f(x, y) = h(x),$$

where the last equality comes for the fact that $f(x, y_n) = h(x)$ for all n, which implies $\lim_{n\to\infty} f(x, y_n) = h(x)$. Hence $y \in G(x)$ and, according to Definition 17, we have shown that $G(x)$ is closed for all x.

(iv) We now show that the policy correspondence cannot fail to be u.h.c.. Recall Definition 27. In order for a correspondence not to be u.h.c. at a point x one must be able to find two converging sequences $x_n \to x$ and $y_n \in G(x_n) \to y$ such that $y \notin G(x)$. However, if we recall the definition of the policy $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$, it is easy too see that as long as Γ is continuous this failure surely cannot happen because of $y \notin \Gamma(x)$. Moreover, because of continuity of both f and h for any couple of such converging sequences we must have that

$$f(x, y) = f(\lim x_n, \lim y_n) = \lim_n f(x_n, y_n) = \lim_n h(x_n) = h\left(\lim_n x_n\right) = h(x)$$

where in the first equality we used the definition of x an y as limit points, in the second we used the joint continuity of f; in the third we used the fact that $y_n \in G(x_n)$ for all n, and in the penultimate equality we used again the continuity of h. Hence, we have just seen that this cannot happen at all.[9] **Q.E.D.**

Notice that the policy correspondence is "only" upper hemi-continuous. Hence we can have explosions in the set that describes the optimal points, even tough the feasibility set cannot.

Exercise 33 *How can we have explosions in the policy correspondence if both f and Γ are continuous ?*

Exercise 34 *Solve exercise 3.16 of SLP.*

Theorem 10 is due to C. Berge (1959) and can be extended as follows:

Theorem 11 *Let $\Gamma : X \twoheadrightarrow Y$ a u.h.c. correspondence with non-empty and compact values. And let $f : X \times Y \to I\!R \cup \{-\infty\}$ be upper semi-continuous. Then the "value function" $h : X \to I\!R$ defined by*

$$h(x) = \max_{y\in\Gamma(x)} f(x, y)$$

is upper semi-continuous.

[9] Of course, the tricky part is to show that $G(x)$ is u.h.c. and that h is continuous simultaneously. In the proof of Theorem 3.6, SLP show that G is u.h.c. only using the continuity of f.

Proof: We want to show that the upper level sets of $h(x)$ are closed. When they are empty the result is trivial. So, suppose for some $v < \infty$ the upper level set of h - call it U_v is non-empty and take a converging sequence $x_n \to x$ with all $x_n \in U_v$. To be in U_v it must be that $h(x_n) > -\infty$ for all n. Note that given x_n, we are maximizing the upper semicontinuous function $f(x_n, \cdot)$ (that takes nowhere plus infinity) over the compact and non-empty set $\Gamma(x_n)$ and we get a value greater than $-\infty$. By Lemma 0 a maximum exists. In particular, for all n there is a $y_n \in \Gamma(x_n)$ such that $f(x_n, y_n) \geq v$. Since Γ is compact valued $\{y_n\}$ has a cluster point y [not immediate step]. Since Γ is $u.h.c$ we must have $y \in \Gamma(x)$. Now take a subsequence of $\{y_n\}$ converging to y. Since f is upper semicontinuous we have $f(x, y) \geq \limsup_k f(x_{n_k}, y_{n_k}) \geq v$. Since $y \in \Gamma(x)$ it must be that $h(x) \geq f(x, y)$, so $h(x) \geq v$ as required. **Q.E.D.**

As expected, if we allow both the feasibility set and the functions to jump upward, the value function may jump upward as well. Notice that downward jumps are not always preserved by the max operator. These are preserved by the min. Indeed, the same theorem guarantees that whenever $-f$ is upper semi-continuous (i.e., f is lower semi-continuous) then h is lower semi-continuous since $-h = -\min f$ must be upper semi-continuous. Do not get confused, Γ is always required to be u.h.c. [Can you explain intuitively why?].

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