Chapter 3

Deterministic Dynamic Programming

3.1 The Bellman Principle of Optimality

Richard Bellman (1957) states his Principle of Optimality in full generality as follows:

> *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."* Bellman (1957), page 83.

An *optimal policy* is a rule making decisions which yield an allowable sequence of decisions and maximize a preassigned function of the final state variables.

The Bellman Principle of Optimality (BPO) is essentially based on the following property of the real valued functions.

Lemma 1 *Let* $f : X \times Y \to I\!R \cup \{-\infty, +\infty\}$ *then*

$$V^* = \sup_{(x,y)\in X\times Y} f(x,y) = \sup_{x\in X}\left\{\sup_{y\in Y} f(x,y)\right\},$$

that is, if we define pointwise the value function $W : X \to I\!R \cup \{-\infty, +\infty\}$

$$\forall x \in X \qquad W(x) = \sup_{y\in Y}\ f(x,y)$$

then

$$V^* = \sup_{x\in X}\ W(x).$$

Proof. First notice that if we fix x, we have $V^* \geq f(y,x)$ for all y, hence $V^* \geq W(x)$ for all x. This is true even when $V^* = -\infty$ as in this case it must be that $W(x) = -\infty$ for all $x \in X$. As a consequence

$$V^* \geq \sup_{x \in X} W(x).$$

Now we need to show the inverse inequality. Since the case $V^* = -\infty$ has been analyzed above, we can have two cases.
(a) $V^* < \infty$. In this case, by the definition of sup in V^* we have that for every $\varepsilon > 0$ there exists a couple $(x', y') \in X \times Y$ such that $f(x', y') + \varepsilon > V^*$. In addition, we know that $W(x') = \sup_y f(x', y) \geq f(x', y')$, hence

$$\sup_x W(x) + \varepsilon \geq W(x') + \varepsilon > V^*.$$

Since the inequality $\sup_x W(x) + \varepsilon > V^*$ must be true for all $\varepsilon > 0$, it must be that $\sup_x W(x) \geq V^*$ (otherwise it is easy to see that one obtains a contradiction).
(b) If $V^* = \infty$ we have to show that for all real numbers $M < \infty$ there is a $x \in X$ such that $W(x) > M$. Assume it is not the case and let $\bar{M} < \infty$ such that $W(x) \leq \bar{M}$, for all x. Since for any x we have $W(x) \geq f(x,y)$ for all y, it must be that $\infty > \bar{M} \geq f(x,y)$ for all x, y, but this implies that $\bar{M}$ is an upper bound for f. Since V^* is the least upper bound we have a contradiction. **Q.E.D.**

Using the *infinite penalization approach*[1] the same result can be stated for the case where the choice (x, y) is restricted to a set $D \subset X \times Y$. In this case, one must be able to decompose the feasibility set in an appropriate way. In these notes, we will always

[1]The infinite penalization approach basically reduces a constrained maximization problem into a free maximization one. For example, consider the constrained problem

$$\sup_{(x,y) \in D \subset X \times Y} f(x,y).$$

If we define a new function f^* as follows

$$f^*(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \in D \\ -\infty \quad \text{otherwise} \end{cases}$$

then it should be easy to see that

$$\sup_{(x,y) \in D} f(x,y) = \sup_{(x,y) \in X \times Y} f^*(x,y).$$

analyze environments where this decomposition can be done.[2,3] Moreover, we will typically consider environments where the objective function f is a sum of terms. In this case, the following property of the supremum becomes quite useful.

Lemma 2 *Let a, and b two real numbers, if $b > 0$, then*

$$\sup_{x \in X} \; a + bf(x) = a + b \sup_{x \in X} \; f(x),$$

if $b < 0$, then

$$\sup_{x \in X} \; a + bf(x) = a + b \inf_{x \in X} \; f(x).$$

Proof. We will show the result when $b > 0$, assuming that the sup takes a finite value. [The case where the sup takes the value $-\infty$ or $+\infty$ is left as an exercise for the interested reader.] Let $f^* = \sup_{x \in X} \; f(x)$, and $V^* = \sup_{x \in X} \; a + bf(x)$. First, we show $V^* \leq a + bf^*$. Note that for all $x \in X$ we have $a + bf^* \geq a + bf(x)$, that is, $a + bf^*$ is an upper bound for the set

$$\{y : y = a + bf(x) \text{ for some } x \in X\}.$$

As a consequence, its least upper bound V^* must be such that $a + bf^* \geq V^* = \sup_{x \in X} a + bf(x)$. To show the converse, note that from the definition of f^* as a supremum, we have that for any $\varepsilon > 0$ there must exist a $\bar{x}_\varepsilon \in X$ such that $f(\bar{x}_\varepsilon) > f^* - \varepsilon$. Hence $a + bf(\bar{x}_\varepsilon) > a + bf^* - b\varepsilon$. Since $\bar{x}_\varepsilon \in X$, it is obvious that $V^* \geq a + bf(\bar{x}_\varepsilon)$. Hence $V^* \geq a + bf^* - b\varepsilon$. Since ε was arbitrarily chosen, we have our result: $V^* \geq a + bf^*$. **Q.E.D.**

Notice that in economics the use of the BPO is quite common. Consider for example the typical profit maximization problem

$$\begin{aligned} \pi^*(p, w) &= \max_{z,y} \; py - wz \\ s.t. \; y &\leq f(z), \end{aligned}$$

[2] Hence, $(x, y) \in D$ will always be equivalent to $x \in dom\Gamma$ and $y \in \Gamma(x)$ for some correspondence Γ, where the domain of a correspondence is the set of values for which it is non empty, i.e.

$$dom\Gamma = \{x : \Gamma(x) \neq \emptyset\}.$$

[3] The constrained maximization version of Lemma 1 is

$$\sup_{(x,y) \in D} \; f(x, y) = \sup_{x \in dom\Gamma} \left\{ \sup_{y \in \Gamma(x)} \; f(x, y) \right\}.$$

where y is output and z is the vector of inputs; and p and w are prices. Using Lemma 1 and 2, the problem can be decomposed as follows:

$$\pi^*(p, w) = \max_y \; py - C(y; w),$$

where C is the cost function, and for any given y is defined

$$\begin{aligned} C(y; w) &= \inf_z \; wz \\ s.t. \; y &\leq f(z). \end{aligned}$$

Let me now introduce some notation. To make easier the study of the notes I follow closely SLP, Chapter 4. Consider again the optimal growth problem. In the introductory section we defined the problem as follows

$$\begin{aligned} &\sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \\ s.t. \; k_0 &\geq 0 \\ 0 &\leq k_{t+1} \leq f(k_t) \quad \text{for all } t. \end{aligned}$$

In general, the class of dynamic problems we are going to consider is represented by

$$\begin{aligned} V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right) \\ s.t. \; x_0 &\in X \\ x_{t+1} &\in \Gamma(x_t) \quad \text{for all } t. \end{aligned} \tag{3.1}$$

One key ingredient is the time invariant function F of the present and the future states, whose discounted sum describes the objective function of the problem. The time invariant correspondence Γ describing feasibility, and $\beta \in (0, 1)$ is the discount factor. Finally, notice that we denoted the *true value function* V^* with an asterisk. This is done in order to make the distinction between the true value function V^* and a specific solution V to the Bellman functional equation implied by (3.1). We will indeed see that the two concepts are closely related but quite different.

Exercise 35 *Show that the general formulation in (3.1) can be specified to describe the optimal growth problem defined above.[Hint: very easy!]*

From these primitives, the problem can be rewritten in a more compact way. For any sequence $\mathbf{x} = \{x_{t+1}\}_{t=0}^{\infty}$ with initial value x_0 define the set

$$\Pi(x_0) = \left\{ \{x_{t+1}\}_{t=0}^{\infty} \text{ such that } x_{t+1} \in \Gamma(x_t) \quad \text{for all } t \right\}.$$

If $\mathbf{x} \in \Pi(x_0)$ we say $\mathbf{x}$ is a *feasible plan* (with respect to x_0).

We now make the following assumption on our primitive Γ.

Assumption 4.1 $\Gamma(x)$ is non-empty for any $x \in X$.

The relationship between the two definitions of feasibility is clarified by the following exercise.

Exercise 36 *Show that if Assumption 4.1 is true then $\Pi(x_0)$ is non-empty for each $x_0 \in X$.*

For any sequence $\mathbf{x}$ we define the intertemporal payoff function as follows

$$\mathbf{U}(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

so we can equivalently write the problem (3.1) in a compact way

$$V^*(x_0) = \sup_{\mathbf{x} \in \Pi(x_0)} \mathbf{U}(\mathbf{x}). \tag{3.2}$$

We allow the problem to have an unbounded value, so we write the infinite sum even when the series is divergent. What we will never do is to consider infinite sums when the series have *indeterminate* character.

Assumption 4.2 For all $x_0 \in X$ and $\mathbf{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \mathbf{U}_n(\mathbf{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists although it might be plus or minus infinity.

Exercise 37 *Show that if Assumption 4.2 is satisfied, we can write $\mathbf{U}(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ as follows*

$$\mathbf{U}(\mathbf{x}) = F(x_0, x_1) + \beta \mathbf{U}(\mathbf{x}')$$

for each feasible path $\mathbf{x} \in \Pi(x_0)$ with $x_0 \in X$.

We are now ready to state the BPO for our class of problems.

(Infinite) Bellman Principle If $\mathbf{x}$ is optimal then $\mathbf{x}'$ is optimal, where $\mathbf{x} = (x_0, x_1, ...)$ and $\mathbf{x}' = (x_1, x_2, ...)$ is the "one-step ahead" sequence.

The BPO principle is equivalent to the possibility of writing the value function V^* for our infinite dimensional problem in the form of a functional equation. Which is called the *Bellman Equation*[4]

$$V(x_0) = \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta V(x_1), \tag{3.3}$$

and this for any t. More formally, we have

[4] The corresponding finite horizon BPO is as follows

Theorem 12 *Let Assumptions **4.1** and **4.2** be satisfied by our problem. Then the function* $V^*(x_0) = \sup_{\mathbf{x}\in\Pi(x_0)} \mathbf{U}(\mathbf{x})$ *satisfies the functional equation (3.3) for all* $x_0 \in X$*. Moreover, if a feasible plan* $\mathbf{x}^* \in \Pi(x_0)$ *attains the supremum in (3.2) then the maximal plan* $\mathbf{x}^*$ *must satisfy (3.3) with* $V = V^*$*, i.e.*

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*),\ t = 0, 1, 2, ... \tag{3.4}$$

Proof What we have to show for the first part of the theorem is the following:

$$V^*(x_0) \equiv \sup_{\mathbf{x}\in\Pi(x_0)} \mathbf{U}(\mathbf{x}) = \sup_{x_1\in\Gamma(x_0)} F(x_0, x_1) + \beta V^*(x_1).$$

As a preliminary step, use Assumption **4.2** (and Exercise 37) to rewrite

$$\sup_{\mathbf{x}\in\Pi(x_0)} \mathbf{U}(\mathbf{x}) = \sup_{\mathbf{x}\in\Pi(x_0)} F(x_0, x_1) + \beta \mathbf{U}(\mathbf{x}').$$

By Lemma 1 (in its constrained version), we can decompose the sup operator as follows

$$\sup_{\mathbf{x}\in\Pi(x_0)} F(x_0, x_1) + \beta \mathbf{U}(\mathbf{x}') = \sup_{x_1\in\Gamma(x_0)} \sup_{\mathbf{x}'\in\Pi(x_1)} F(x_0, x_1) + \beta \mathbf{U}(\mathbf{x}').$$

The relationship between the correspondence Γ and the feasible set $\Pi(x_0)$ guarantees (by definition) that Assumption **4.1** (together with Exercise 36) suffices to allow us to do the decomposition.

The final step is to use Lemma 2 to pass through the second sup operator. That is, applying Lemma 2 to the second sup operator with $F(x_0, x_1) = a$ and $\beta = b > 0$ we have

$$\sup_{x_1\in\Gamma(x_0)} \sup_{\mathbf{x}'\in\Pi(x_1)} F(x_0, x_1) + \beta \mathbf{U}(\mathbf{x}') = \sup_{x_1\in\Gamma(x_0)} F(x_0, x_1) + \beta \sup_{\mathbf{x}'\in\Pi(x_1)} \mathbf{U}(\mathbf{x}').$$

One must keep in mind that this last step can only be done because of the specific characteristics of the sum. First of all, the discounted summation satisfies an obvious monotonicity assumption since $\beta > 0$. Second, it also satisfies an important property

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Finite Bellman Principle If a path $(x_0, x_1, ..., x_T)$ is optimal for the T horizon problem. Then the path $(x_1, x_2, ..., x_T)$ is optimal for the $T - 1$ horizon problem.

That is, we can write the value function $V_T^*(x_0)$ for the T-horizon problem in terms of the $T - 1$ horizon value function V_{T-1}^* as follows

$$V_T^*(x_0) = \sup_{x_1\in\Gamma(x_0)} F(x_0, x_1) + \beta V_{T-1}^*(x_1).$$

of continuity (as we saw a sum it is actually a linear mapping).[5] The last part of the proposition is easily derived by the fact that $\mathbf{x}^*$ reaches the supremum, i.e.

$$\mathbf{U}(\mathbf{x}^*) = V^*(x_0) = \max_{\mathbf{x}\in\Pi(x_0)} \mathbf{U}(\mathbf{x}),$$

and is left as an exercise. For an alternative proof see SLP Theorems 4.2 and 4.4. Finally, note that we used $\beta > 0$, however the case $\beta = 0$ is really easy, and again left as an exercise. **Q.E.D.**

Let's have another look at conditions (3.4). The key idea of the Bellman principle is that we can simply check for "one stage deviations". An optimal plan $\mathbf{x}^*$ has the property that for any t - once the past is given by x_t^*, and the effect of your choice x_{t+1} on the future returns is summarized by $V^*(x_{t+1})$ - there is no incentive to choose a different x_{t+1} from

[5]Monotonicity alone is not enough, as the following counterexample from Bertsekas et al. (1978) shows. In this example there are no states, and the problem is in two-periods. The agent must choose the control $c \in (-1, 0]$ and the continuation value is $J_0 = 0$. All the complications are due to the properties of the aggregator $H : U \times \mathcal{J} \to \mathcal{J}$, $\mathcal{J} \subset \mathbb{R}$. It is such that

$$H(u, J) = \begin{cases} u & \text{if} \quad J > -1 \\ u + J & \text{if} \quad J \leq -1. \end{cases}$$

One should easily see that the true optimal values are

$$J_1^* = \inf_u H(u, J_0) = -1$$

and

$$\begin{aligned} J_2^* &= \inf_{u_1,u_2} H(u_1, H(u_2, J_0)) = \\ &= \inf_{u_1,u_2} H(u_1, u_2) = -1. \end{aligned}$$

However, if one uses the recursive formulation and the Bellman operator gets something different. In particular

$$J_1 = T^1(J_0) = \inf_{u_2} H(u_2, J_0) = -1$$

and

$$\begin{aligned} T^2(J_0) &= T^1(J_1) = \inf_{u_1} H(u_1, J_1) \\ &= \inf_{u_1} H(u_1, -1) = -2. \end{aligned}$$

To have a sufficient condition we might require either continuity, or $\exists \alpha \geq 0 : \forall\, r \geq 0$, and J

$$H(u, J) \leq H(u, J + r) \leq H(u, J) + \alpha r.$$

that implied by the optimal plan x^*_{t+1}. That is, there is no incentive to revise the plan. That is why economists often call the paths generated by the BPO as *time consistent.*

The above theorem states a set of *necessary* conditions for an optimal path in terms of the functional equation. It is important to notice that regarding these properties the distinction between finite horizon and infinite horizon is merely technical. For example, for the finite periods version of the above theorem we do not need Assumption **4.2** since finite sums will always be well defined. In the infinite horizon, such assumptions are only needed to have a well defined continuation problem in each period.

If one looks for *sufficient* conditions the distinction between finite and infinite horizon becomes much more important. In particular, the finite version of Theorem 12 is also a sufficient condition for an optimum. That is, a path that satisfies the BPO is an optimal plan. The idea is simple. The BPO states that an optimal path is such that the agent does not have incentives to deviate for one period from his maximizing behavior and then reverting to an optimal behavior (summarized by $V^*(x)$). By induction on this principle, one can show that the agent has never incentive to deviate for finitely many periods either. In contrast, the BPO cannot say anything about infinite deviations. That is behaviors that never revert to the optimizing behavior any more. As a consequence, in order to use the BPO to detect optimal plans one must induce some additional structure on the problem so that the agent cannot gain from infinite deviations either. In term of the objective function $\mathbf{U}(\mathbf{x})$ one typically requires the so called *continuity at infinity* (see for example Fudenberg and Tirole, 1991, Chapter 5). Here below we follow SLP and state the additional condition in terms of the (true) value function V^*.

Theorem 13 *Assume **4.1** and **4.2**. Let a feasible path $\mathbf{x}^* \in \Pi(x_0)$ from x_0 satisfying (3.4), and such that*

$$\limsup_{t\to\infty} \beta^t V^*(x^*_t) \leq 0.$$

Then $\mathbf{x}^*$ attains the supremum in (3.2) when the initial state is x_0.

Proof. First notice that since $\mathbf{x}^*$ is feasible its value cannot be greater than the supremum, that is $V^*(x_0) \geq \mathbf{U}(\mathbf{x}^*)$. We have to show the inverse inequality. If we apply repeatedly (3.4) we get

$$V^*(x_0) = \mathbf{U}_n(\mathbf{x}^*) + \beta^{n+1} V^*(x^*_{n+1}) \text{ for } n = 1, 2, ...$$

now using $\limsup_{n\to\infty} \beta^{n+1} V^*(x^*_{n+1}) \leq 0$ we have

$$V^*(x_0) = \mathbf{U}_n(\mathbf{x}^*) + \beta^{n+1} V^*(x^*_{n+1}) \leq \lim_{n\to\infty} \mathbf{U}_n(\mathbf{x}^*) \equiv \mathbf{U}(\mathbf{x}^*).$$

Q.E.D.

The limit value condition imposed above can be interpreted as a transversality condition for the BPO. The same idea can be applied to the value function as in the first part of Theorem 12. In this case, we obtain a sufficient condition for a given function to be the true value function.

Theorem 14 *Assume 4.1 and 4.2. If V is a solution to the functional equation (3.3) and satisfies*

$$\lim_{t\to\infty} \beta^t V(x_t) = 0 \text{ for all } \mathbf{x} \in \Pi(x_0) \text{ and all } x_0 \in X,$$

Then V is the true value function, i.e. $V = V^*$.

Proof. The proof here is basically on the lines of the previous theorem. The only additional complication is that we are dealing with the sup instead of the Max. In particular, we are not sure of the existence of an optimal plan. However, notice that one key aspect of V^* to be verified for V is that

$$V^*(x_0) \geq U(\mathbf{x}) \text{ for all } \mathbf{x} \in \Pi(x_0) \text{ and all } x_0 \in X.$$

Now since V solves (3.3) for all t we have that

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta V(x_1) \geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2) \geq ... \\ &\geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T V(x_T) \text{ for all } \mathbf{x} \in \Pi(x_0). \end{aligned}$$

Hence, as long as $\beta^T V(x_T) \to 0$ we have the desired property for V. See also Theorem 4.3 in SLP. **Q.E.D.**

The above theorem also suggests that the "guess and verify" procedure we discussed in the introductory section simply provides *one* solution to the functional equation (3.3). However (3.3) might have multiple solutions, and we are obviously looking for the right value function V^*. Theorem 14 guarantees that a bounded solution V to (3.3) is actually the "right" value function.

3.2 The BPO under Bounded Returns: Continuity, Concavity and Differentiability of the Value Function

In this section we will specify the problem used to study the BPO by imposing additional restrictions on the primitives F and Γ so that to be able to show some properties for the value function V^*. Following SLP, we will heavily use the contraction mapping theorem (Theorem 7). To show that the Bellman operator is a contraction we will use the Blackwell sufficient conditions (Theorem 8), so we will work in the space of bounded functions with the sup norm.

Continuity To show continuity we will use the theorem of the maximum (Theorem 10). Here are the necessary assumptions to use it.

Assumption 4.3 $\Gamma(x)$ is a non-empty, compact valued and continuous correspondence, and $X \subseteq \mathbb{R}^l$.

Assumption 4.4 F is bounded and continuous and $\beta \in [0, 1)$.

Theorem 15 *Assume 4.3 and 4.4 and consider the metric space* $(\mathcal{C}(X), d_\infty)$ *of bounded and continuous functions with the sup norm. Then the Bellman operator* T *defined by*

$$(TV)(x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta V(x') \tag{3.5}$$

(i) maps $\mathcal{C}(X)$ *into itself; (ii) has a unique fixed point* $V \in \mathcal{C}(X)$; *for all* $V_0 \in \mathcal{C}(X)$ *we have*

$$d_\infty(T^n V_0, V) \leq \beta^n d_\infty(V_0, V), \text{ for any } n = 0, 1, ...$$

(iii) and the policy correspondence

$$G(x) = \{x' \in \Gamma(x) : V(x) = F(x, x') + \beta V(x')\}$$

is non empty, compact valued, and upper semi-continuous.

Proof. (i) If f is continuous and F is continuous the objective function of the maximization problem (3.5) is continuous. This, together with the properties of the correspondence Γ imply that we can directly apply Theorem 10 to show that the value function of this problem Tf is also continuous. The fact that F is bounded also implies that if f is bounded then Tf will be bounded too. So *(i)* is shown. To show *(ii)* and *(iii)* we use Theorem 7. We need to show that T describes a contraction and that the metric space $(\mathcal{C}(X), d_\infty)$ is complete. The completeness of $(\mathcal{C}(X), d_\infty)$ have been presented in Theorem 6. To show that T is a contraction we can use Theorem 8. In Exercise 22 we have precisely shown monotonicity and discounting for the optimal growth model, however, one can closely follow the same line of proof and prove the same statement in this more general case. So we have *(ii)*. Finally, since we have shown that the fixed point V is continuous, we can apply the maximum theorem again and show *(iii)*. **Q.E.D.**

Notice first, that when F is a bounded function also V^* must be bounded, and from Theorem 14 the unique bounded function V satisfying the Bellman equation must be the true value function, i.e. $V = V^*$. Moreover, from *(iii)* above we are guaranteed we can construct an optimal plan by taking a selection from the policy correspondence G as follows: start form x_0 an then for any $t \geq 1$ set $x_t \in G(x_{t-1})$. But then we have also shown existence without using possibly complicated extensions of the Weierstrass theorems in infinite dimensional spaces.

Concavity and Differentiability

Assumption 4.7 Γ has a convex graph, i.e. for each two $x_1, x_2 \in X$ and corresponding feasible $x_1' \in \Gamma(x_1)$, $x_2' \in \Gamma(x_2)$ we have

$$[\theta x_1' + (1-\theta)x_2'] \in \Gamma\left(\theta x_1 + (1-\theta)x_2\right) \text{ for any } \theta \in [0,1].$$

Assumption 4.8 F is concave and if $\theta \in (0,1)$ and $x_1 \neq x_2$ we have

$$F\left(\theta x_1 + (1-\theta)x_2,\ \theta x_1' + (1-\theta)x_2'\right) > \theta F\left(x_1, x_1'\right) + (1-\theta)F\left(x_2, x_2'\right).$$

Now we are ready to show our result.

Theorem 16 *Assume 4.3, 4.4, 4.7 and 4.8. (i) Then the fixed point V is strictly concave and the policy G is a continuous function g. (ii) Moreover, if F is differentiable then V is continuously differentiable and*

$$V'(x) = \frac{\partial F(x, g(x))}{\partial x} = F_1(x, g(x))$$

for all $x \in intX$ such that the policy is interior, i.e. $g(x) \in int\Gamma(x)$.

Proof. The proof of (i) uses the fact that under **4.7** and **4.8** the operator T maps continuous concave function into concave functions, and the space of continuous concave functions is a closed subset of the metric space $(\mathcal{C}(X), d_\infty)$. As a consequence we can apply Exercise 20 to be sure that the space of continuous and bounded functions in the sup norm is a complete metric space and apply the contraction mapping theorem. Under 4.8 it is easy to show that V is actually *strictly* concave. Since a strictly concave problem has a unique maximum, the policy correspondence must be single valued, hence g is a continuous function (see Exercises 28 and 12.3). Part *(ii)* will be shown graphically in class. The interested reader can see the proof in SLP, Theorem 4.11, page 85. **Q.E.D.**

The key element of the proof of part *(ii)* above is the following Lemma of Benveniste and Sheinkman (1979):

Lemma 3 *Let $x_0 \in intX$ and let D a neighborhood of x. If there exists a concave and differentiable function $W : D \to \Re$ such that for $x \in D$ $W(x) \leq V(x)$ and $V(x_0) = W(x_0)$ then V is differentiable at x_0 and $V'(x_0) = W'(x_0)$.*

Proof. If $p \in \partial V(x_0)$ then $p \in \partial W(x_0)$ since the subgradient inequality carries over. But W is differentiable, hence p is the unique subgradient of W, which implies that also V has only one subgradient at x_0. V is concave, and since any concave function with only one subgradient is differentiable V is differentiable. This last statement is not easy to show, see Rockafellar (1970). **Q.E.D.**

Monotonicity When $F(x, x')$ is monotone in x and the feasibility set $\Gamma(x)$ widens with x, it is easy to show that $V(x)$ is increasing.

Differentiability using Boldrin and Montrucchio (1998) Boldrin and Montrucchio (1998) use the properties of the contraction, the uniform convergence of the policy of a class of finite period problem to the policy of the infinite horizon problem, and a well known approximation theorem (Dieudonné, *Foundations of Mathematics* No. 8.6.3) to show differentiability.[6] Their result does not use concavity, hence their method can be used to study parameter sensitivity as well.

Differentiability of the policy under $\mathbf{C}^2$ Differentiability of the policy is strictly linked to the second order differentiability of the value function. Montrucchio (1997) shows that under some conditions the Bellman operator is a contraction also in the C^2 topology. Santos (1991) shown the same result using a different methodology. Recently, Santos (2003) shows how this result can be profitably used to compute bounds on the approximation errors in the numerical discretization procedures.

[6]The key Lemma they use guarantees that if $V_T'(k) \to \phi(k)$ uniformly as $T \to \infty$, and $V_T \to V$, with V_T differentiable for all T then $V'(k) = \phi(k)$.

Unbounded returns In most cases the complication of unbounded F is solved on an ad hoc basis. Alvarez and Stokey (1998) analyze the case where both F and Γ are homogeneous functions. Montrucchio and Privilegi (1998) and, more recently, Rincón-Zapatero and Rodriguez-Palmero (2003) generalize Boyd (1990) and use the contraction mapping theorem to show existence and uniqueness of V in a large class of Bellman equation problems with unbounded returns. Streufert (1990, 1998) analyze a large class of capital accumulation problems and uses the monotonicity property of the Bellman operator to show that V is upper hemi-continuous. He defines the notion of admissibility and uses biconvergence to show uniqueness and continuity.

3.3 The Euler's Variational Approach and the Bellman Principle of Optimality

To get the key idea of the Euler's Variational approach we should recall the problem we analyzed in Lemma 1, at the beginning of this section

$$\max_{(x,y)\in X\times Y} f(x,y).$$

To keep the analysis as simple as possible, assume the existence of a maximum couple (x^*, y^*), and define

$$V^* = f(x^*, y^*) \geq f(x,y) \text{ for all } (x,y) \in X \times Y. \tag{3.6}$$

According to the *Bellman's principle (*Lemma 1) the problem can be solved in two steps:

$$V^* = \max_{x\in X} W(x),$$

with

$$W(x) = \max_{y} f(x,y);$$

and vice versa: *any pair* that solves such a two steps problem is an optimal one. The *Euler's variational approach* starts by the observation that the optimal pair (x^*, y^*) satisfies (among other things)

$$f(x^*, y^*) \geq f(x, y^*) \text{ for all } x \in X, \text{ and} \tag{3.7}$$

$$f(x^*, y^*) \geq f(x^*, y) \text{ for all } y \in Y. \tag{3.8}$$

Notice the key difference from the Bellman principle: an optimal pair (x^*, y^*) has to be such that *along the optimal path* there is no incentive to deviate from it "unilaterally," that is only in one direction.[7] Notice that we can equivalently write the BPO two step procedure as

$$f(x^*, y^*) \geq f\left(x, y^*\left(x\right)\right) \text{ for all } x \in X$$

where (assuming there is only one optimum)

$$y^*\left(x\right) = \arg\max_{y \in Y} \ f\left(x, y\right).$$

That is, crucially $y^*\left(x\right)$ is not a fixed number, it is a function of x.

In our notation, the Euler variational approach translates in the observation that an optimal plan $\mathbf{x}^*$ for any t must satisfy

$$\begin{aligned} F(x_{t-1}^*, x_t^*) + \beta F(x_t^*, x_{t+1}^*) &\geq F(x_{t-1}^*, x_t) + \beta F(x_t, x_{t+1}^*) \\ \text{for all } x_t &\in \Gamma(x_{t-1}^*) \text{ such that } x_{t+1}^* \in \Gamma(x_t). \end{aligned} \tag{3.9}$$

That is, one property of the optimal plan is that the agent cannot gain by deviating (in a feasible fashion) from the optimal path in any period, taking the optimal path as given. This is again a one stage deviation principle. However, the key distinction with the BPO is that the deviation considered here does not take into account the future effects of such a deviation, but *takes as given both the past and the future.* Recall that the equivalent to condition (3.9) for the BPO is

$$\begin{aligned} F(x_{t-1}^*, x_t^*) + \beta V(x_t^*) &\geq F(x_{t-1}^*, x_t) + \beta V(x_t) \\ \text{for all } x_t &\in \Gamma(x_{t-1}^*). \end{aligned}$$

In other terms, the interpretation of the Euler condition under differentiability is that one-period *reversed arbitrage,* an arbitrage that immediately returns to the original path, is not profitable on an optimal path. This means that the cost calculated at $t = 0$ from acquiring an extra unit of capital at time t, $\beta^t u'(c_t^*) = \beta^t F_2(x_t^*, x_{t+1}^*)$, is at least as great as the benefit realized at time $t+1$ discounted back to period $t = 0$, from selling that additional unit of capital at $t+1$ for consumption. The extra unit of capital yields in utility terms $\beta^{t+1} F_1(x_{t+1}^*, x_{t+2}^*) = f'(k_{t+1})\beta^{t+1}u(c_{t+1}^*) : f'(k_{t+1})$ units of consumption good at time $t+1$, and each unit of that good is worth $\beta^{t+1}u(c_{t+1}^*)$ utils in period 0. Hence we have

$$u'(c_t^*) = \beta f'(k_{t+1}) u'(c_{t+1}^*).$$

The Variational approach uses one somehow weak property of optimality. As a consequence, in general the Euler's conditions are far from being sufficient for determining an

[7]Unilateral deviations of this sort are both in the concepts of saddle points and Nash equilibria.

optimal point. So the question is, suppose we have as a candidate the couple (x^{**}, y^{**}) such that satisfies (3.7) and (3.8). What are the additional conditions we need in order to guarantee that (x^{**}, y^{**}) is an optimal point? It turns out that a sufficient condition for an interior optimum is that at (x^{**}, y^{**}) f is subdifferentiable.[8] The idea is simple: if (x^{**}, y^{**}) satisfies (3.7) and (3.8) and at (x^{**}, y^{**}) f is subdifferentiable, then the vector $(0, 0)$ must belong to the subdifferential of f at (x^{**}, y^{**}). This property is at the core of the sufficiency of the first order conditions when f is concave.[9]

Example 1 *Let* $f(x, y) = (x-1)^2 (y-1)^2$ *with* $x, y \in [0, 3]$. *Obviously,*

$$\max_{(x,y)\in[0,3]\times[0,3]} f(x, y) = 4,$$

with solution $x^* = y^* = 3$, *however it is easy to see that the pair* $x^{**} = y^{**} = 1$ *satisfies (3.7) and (3.8) as well [check it!].*

Exercise 38 *Do the same exercise with* $f(x, y) = \sqrt{x}\sqrt{y}$ *and* $x, y \in [0, 3]$. *Notice that now we can use the first order conditions to find the 'unilateral' maximum for* x *given* $y = \bar{y}$, *since the problem is concave in* x *alone given* $\bar{y}$. *The same is true for* y *given* $x = \bar{x}$. *Is this form of concavity enough? Explain.*

Transversality So, Euler equations are necessary conditions for an optimum. We also said above that exactly following the same logic used in static maximization, when the problem is concave an interior optimal can be detected by the Euler's equations. However, this principle works only for finite horizon problems. When the time horizon is infinite Euler equations are not enough, and we need an additional restriction to detect optimal programs.

The reason is the same as the one for the BPO. The Euler conditions control for only one-stage deviations, which can be extended by induction to any finite stages deviation. But they cannot tell us anything about infinite period deviations.

[8] A real valued function $f : X \times Y \to \mathbb{R}$ is subdifferentiable (in the concave sense) at (x_0, y_0) if there exists a vector $p = (p_x, p_y)$ such that

$$f(x, y) \leq f(x_0, y_0) + p_x.(x - x_0) + p_y.(y - y_0) \text{ for any } x \in X, y \in Y,$$

and p is the subdifferential of f at (x_0, y_0).

[9] A function f defined in a convex X is concave if for any two $x, y \in X$ and any $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y).$$

By the separation theorem, any concave function is subdifferentiable.

Proposition 3 *Assume F is bounded, continuous, concave, and differentiable. Moreover assume Γ has a compact and convex graph. (i) If the (interior) sequence $\{x_t^*\}_{t=1}^{\infty}$ with $x_{t+1}^* \in int\Gamma(x_t^*)$ for any $t = 0, 1, 2, ...$ satisfies*

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0 \text{ for } t = 0, 1, ... \tag{3.10}$$

and for any other feasible sequence $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$ we have

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \geq 0, \tag{3.11}$$

then $\{x_t^*\}_{t=1}^{\infty}$ is an optimal sequence. (ii) If in addition $F_1(x, x') > 0$ for all $x, x' \in intX$ and $X \subseteq I\!R_+^l$, the condition (3.11) can be substituted by

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*) x_T^* \leq 0.$$

Proof. *(i)* We are done if we can show that for any feasible $\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)$ we have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*) \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}),$$

where both limit exist and the inequality has a meaning since F is bounded. Now, notice that from the concavity and differentiability of F we have that

$$F(x_t, x_{t+1}) \leq F(x_t^*, x_{t+1}^*) + F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_2(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*)$$

multiplying by β^t and summing up the first T terms one gets

$$\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \leq \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*) + D_T, \tag{3.12}$$

where

$$D_T = \sum_{t=0}^{T} \beta^t \left[F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_2(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*) \right].$$

Since $\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ converges for any sequence $\{x_t\}$, one can show that (3.12) implies that D_T must converge as well.[10] It then suffices to show that

$$\lim_{T\to\infty} D_T \leq 0$$

[10]See Mitchell (1990), page 715.

Notice that we can rearrange the terms in D_T and obtain[11]

$$D_T = \sum_{t=0}^{T-1} \beta^t \left[F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) \right] (x_{t+1} - x_{t+1}^*) - \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*).$$

Euler conditions (3.10) guarantee that the fist $T-1$ terms go to zero, hence

$$\lim_{T\to\infty} D_T = - \lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \leq 0$$

where the last inequality is implied by the transversality condition (3.11). In order to show *(ii)* notice that if $F_1 > 0$ and $x_T \geq 0$,

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \geq - \lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*) x_T^* \geq 0$$

and we are done. **Q.E.D.**

The convexity assumption on Γ is required only in order to define concavity of F. In fact, the theorem only uses the subdifferentiability property of F, so it will remain true as long as F is subdifferentiable along the optimal trajectory. Notice moreover that feasibility is used very little (only at $t = 0$). This is so since we assumed interiority, hence the sequence $\{x_{t+1}^*\}$ is in fact unconstrained optimal. That is, the optimal sequence dominates *all* other sequences. The sequence of subdifferentials are also called *supporting prices*. The reason will become clear in the next paragraph.

The Euler equations (3.10) are the usual first order conditions for x_{t+1}^* to be an interior maximum *given* x_t^* and x_{t+t}^* of the problem described in (3.9). The transversality condition (3.11) has the following interpretation. Notice first that F_1 is the marginal return of the state, for example, in the optimal growth model $F_1(x_t^*, x_{t+1}^*) = u'(c_t^*)f'(k_t^*)$ is the price of capital.[12] Since in the optimal growth model we have both $x_t \geq 0$ and $F_1 > 0$, the

[11] In more detail, consider the first few terms

$$\begin{aligned} & F_1(x_0^*, x_1^*)(x_0 - x_0^*) + F_2(x_0^*, x_1^*)(x_1 - x_1^*) \\ & + \beta \left[F_1(x_1^*, x_2^*)(x_1 - x_1^*) + F_2(x_1^*, x_2^*)(x_2 - x_2^*) \right] \\ = \; & \left[F_2(x_0^*, x_1^*) + \beta F_1(x_1^*, x_2^*) \right] (x_1 - x_1^*) \\ & + \beta \left[F_2(x_1^*, x_2^*) + \beta F_1(x_2^*, x_3^*) \right] (x_2 - x_2^*) \\ & - \beta^2 F_1(x_2^*, x_3^*)(x_2 - x_2^*) \end{aligned}$$

where the term $F_1(x_0^*, x_1^*)(x_0 - x_0^*)$ disappears since feasibility implies $x_0 = x_0^*$; and we have added and subtracted the term $\beta^2 F_1(x_2^*, x_3^*)(x_2 - x_2^*)$.

[12] The value of "one" unit of k_t^* is the price of the consumption goods $u'(c_t^*)$ times the amount of consumption goods that can be produced by one unit of capital.

transversality condition requires $\lim_{T\to\infty}\ \beta^T u'(c_T^*) f'(k_T^*) k_T^* \leq 0$. It is clear that for a finite horizon problems if $\beta^T u'(c_T^*) f'(k_T^*) > 0$ the agent will not maximize lifetime utility by ending the last period with a positive amount of capital k_T^*. The transversality condition states this intuitive argument in the limit. If $\lim_{T\to\infty}\ \beta^T u'(c_T^*) f'(k_T^*) k_T^* > 0$ the agent is holding valuable capital, and perhaps he can increase the present value of its utility by reducing it.

More in general, the transversality condition (3.11) requires any alternative trajectory $\{x_t\}$ satisfying

$$\lim_{t\to\infty} \beta^t F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) < 0$$

to be infeasible. That is, the transversality condition means that if given $\{x_t^*\}$ it is impossible to reduce the limit value of the optimal stock (considered in discounted terms) by choosing $x_t \neq x_t^*$ (except perhaps for incurring in an infinite loss because $\{x\}$ is not feasible) then the value of the capital has been exhausted along the trajectory, and $\{x_t^*\}$ must be optimal as long there are no finite period gains (the Euler condition).

Exercise 39 *Reproduce the proof of the sufficiency of the Euler plus transversality conditions for the optimal growth model. That is, show the following statement. Assume that a consumption path* $\mathbf{c}^*$ *solves* $u'(c_t^*) = \beta f'\left(k_{t+1}^*\right) u'\left(c_{t+1}^*\right)$ *for all* t*, that*

$$\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* = 0,$$

and that both u *and* f *are concave functions (with the usual interpretations). Then* $\mathbf{c}^*$ *is an optimal path for the optimal growth problem.*

Necessity of the Transversality Condition. A typical situation where the transversality is a necessary condition is when the capital stocks are bounded in the optimal growth model. One can of course derive a general proof of it. We will just provide the intuition behind it in a special case. Recall that in the optimal growth model the transversality condition is

$$\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* = 0.$$

In 'finance' terms, the Euler conditions state the unprofitability of reversed arbitrages; while the transversality condition defines a no-arbitrage condition for *unreversed arbitrages:* arbitrages which never return to the original path (Gray and Salant, 1983). Suppose (c, k) is an optimal path and suppose the agent decides to increase consumption in period 0, this is possible if he/she foregoes one unit of capital to be used in next period production. The marginal gain in period zero is $u'(c_0^*)$. Now let T any natural number and define a T*-period reversed arbitrage* the case where the planner reacquire the unit of capital foregone in period 0 only at time $T+1$. After period $T+1$ we are back to the original path.

This deviation from the optimal path $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ generates two costs. First, the direct cost of reacquiring the capital in period $T+1$, which in period zero utility terms is $\beta^{T+1}u'(c_{T+1}^*)$. The indirect cost arises since the net marginal product of capital that unit of capital is lost every period between $t=1$ and $t=T+1$; this is a foregone shadow interest loss. The indirect cost at time t in period zero utils is

$$\beta^t u'(c_t^*)\left(f'(k_t^*)-1\right).$$

Adding to those losses the direct cost and equating to the marginal benefit yields the following zero marginal profit condition:

$$u'(c_0^*) = \sum_{t=1}^{T+1} \beta^t u'(c_t^*)\left(f'(k_t^*)-1\right) + \beta^{T+1}u'(c_{T+1}^*). \tag{3.13}$$

Notice that for $T=0$ the expression reduces to the Euler equation. It is also clear from this condition, that the unprofitability of one-period reversed arbitrage expressed via the Euler equations implies the unprofitability of any $T-$period reversed arbitrage [just rearrange terms to obtain a sum of Euler equations].

However, this is not the end of the story. The infinite horizon implies that the agent should contemplate also the possibility of an unreversed arbitrage, in which a the unit of capital is permanently sacrificed at $t=0$. Of course, there are not repurchase costs associated with this deviation. Hence the zero marginal profit condition for the unreversed arbitrage is

$$u'(c_0^*) = \sum_{t=1}^{\infty} \beta^t u'(c_t^*)\left(f'(k_t^*)-1\right)$$

but this equation is compatible with (3.13) as $T\to\infty$ only if

$$\lim_{T\to\infty} \beta^T u'(c_T^*) = 0,$$

which, in the bounded k_{T+1}^* case, implies the transversality condition. Thus, the transversality condition expresses the zero marginal profit condition for the open-ended arbitrages which are only admissible in the infinite horizon context. Hence both Euler equations and transversality are necessary for optimality. We of course know, that when the problem is concave, the Euler equation together with the transversality are sufficient conditions for optimality.

3.4 Optimal Control and the Maximization Principle of Pontryagin

Under the name "optimal control" one typically refer to an approach strictly linked with the Euler's approach and developed by Pontryagin et al. (1962).

In fact, one can study an optimal control problem with both the recursive and the variational techniques. The basic discrete-time optimal control problem consists in maximizing and an objective function of the form

$$\mathbf{U}_T = \phi(x_{T+1}) + \sum_{t=0}^{T} \beta^t u\left(x_t, c_t\right).$$

subject to a dynamic system

$$x_{t+1} = f(x_t, c_t), \tag{3.14}$$

where f describes how today's *control* $c_t \in C$ affects future state x_{t+1}, given today's state x_t, and $\phi(x_{T+1})$ summarizes the final effect of the state. The set C is any direct restriction on controls other than the law of motion. As usual, we have the initial condition on state $x_0 = \overline{x}$. One can use the natural extension of the Bellman approach to study such a problem. For example, the Bellman equation for this problem is

$$V_{t+1}(x) = \sup_{c \in C}\; u(x,c) + \beta V_t(f(x,c))$$

with $V_0(x_{T+1}) = \phi(x_{T+1})$.

One can see the problem in the joint space of control and state plans and apply a generalized version (for infinite dimensional spaces) of the Lagrange multiplier theorem to it. Another approach is, however, to note that (3.14) uniquely determines the path of states $\mathbf{x} = \{x_t\}$ once the path of controls $\mathbf{c} = \{c_t\}$ is specified and hence we really have to select $\mathbf{c}$ with objective

$$\mathbf{U}_T = \mathbf{U}_T(\mathbf{c}) = \phi(x_{T+1}(\mathbf{c})) + \sum_{t=0}^{T} \beta^t u\left(x_t(\mathbf{c}), c_t\right).$$

Still another approach is to view the problem in the space of states by considering the implicitly defined set of all trajectories that can be obtained by application of admissible controls. Each of these approaches has theoretical advantages for the purpose of deriving necessary conditions and practical advantages for the purpose of developing computational procedures for obtaining solutions.

The Maximum Principle of Pontryagin The Maximum Pontryagin's principle is typically associated to optimal control problems since it emphasizes the role of controls alone. It gives a set of *necessary* conditions for optimality. It is assumed that f is such that given x_0, a given path $\mathbf{x}$ is uniquely determined by $\mathbf{c}$, hence the objective functional can be considered to be dependent only on $\mathbf{c}$. It is typically assumed that both u and f have partial derivatives with respect to x which are jointly continuous in (c, x). We will further assume that the optimal trajectory is interior with respect to C.

Theorem 17 *Let $\{x_{t+1}, c_t\}_{t=0}^{T}$ the optimal control and state trajectory for the above optimal control problem given x_0. Then under some regularity conditions there is an adjoint trajectory $\{\lambda_{t+1}\}_{t=0}^{T}$, such that, given x_0, $\{x_{t+1}, c_t, \lambda_{t+1}\}_{t=0}^{T}$ satisfy:*

$$x_{t+1} = f(x_t, c_t) \quad \textit{system equation} \tag{3.15}$$

$$\lambda_t = H_x(\lambda_{t+1}, x_t, c_t) \quad \textit{adjoint equation} \tag{3.16}$$

$$\lambda_{T+1} = \phi_x(x_{T+1}) \quad \textit{adjoint final equation} \tag{3.17}$$

$$0 = H_c(\lambda_{t+1}, x_t, c_t) \quad \textit{variational condition} \tag{3.18}$$

where H is the Hamiltonian function

$$H(\lambda_{t+1}, x_t, c_t) = u\,(x_t, c_t) + \beta \lambda_{t+1} f(x_t, c_t).$$

Proof (Sketch). We do not investigate what are the regularity conditions that guarantee the existence of the multipliers $(\lambda_{t+1})_{t=0}^{T}$. Under weaker conditions one can show that a similar result can be shown by using an extended Hamiltonian of the form $\hat{H} = p_t u\,(x_t, c_t) + \beta \lambda_{t+1} f(x_t, c_t)$ for some sequence $\{p_t\}$. The core part of the proof has two main ingredients. The adjoint equations and the adjoint final condition on one hand, together with the fact that we can approximate arbitrarily well the value of a deviation by a first order Taylor's expansion when the deviation is *small.* Notice indeed that the proof considers only deviations around the optimal path, and that the statement has a local nature. Specifically, let us write the Lagrangian associated with the above problem

$$L(\lambda, \mathbf{x}, \mathbf{c}) = \sum_{t=0}^{T} \beta^t u\,(x_t, c_t) + \phi(x_{T+1}) + \sum_{t=0}^{T} \beta^{t+1} \lambda_{t+1} \left[f(x_t, c_t) - x_{t+1} \right].$$

The key point of the approach is that once we take into account the adjoint equation, and if we consider local variations, we can check only deviations of the controls. More precisely, from the regularity properties of the problem one can show that along a plan that satisfies the adjoint equation the total value of the program can be approximated in terms of first order by a variation of the Lagrangian in only the controls, keeping states as constant. That is, suppose $(\mathbf{x}^*, \mathbf{c}^*, \lambda^*) = (x_{t+1}, c_t, \lambda_{t+1})_{t=0}^T$ is a triplet with $\mathbf{c} \in C$, and satisfying the adjoint and the final adjoint equations, given x_0, we have:

$$\mathbf{U}_T(\mathbf{c}^*) - \mathbf{U}_T(\mathbf{c}) = L(\lambda^*, \mathbf{x}^*, \mathbf{c}^*) - L(\lambda^*, \mathbf{x}^*, \mathbf{c}) + o\left(\|\mathbf{c}^* - \mathbf{c}\|\right),$$

with $\lim_{\mathbf{c}\to\mathbf{c}^*} \frac{o(\|\mathbf{c}^*-\mathbf{c}\|)}{\|\mathbf{c}^*-\mathbf{c}\|} = 0$. The last step is to realize that if states have second order effects (along the plan satisfying the adjoint equation) deriving deviations of the Lagrangian we can ignore any component that does not depend explicitly from the controls $\mathbf{c}$. This implies that it is enough to consider deviations in the Hamiltonian alone. Indeed we can rewrite the Lagrangian emphasizing the Hamiltonian as follows

$$\begin{aligned} L(\lambda, \mathbf{x}, \mathbf{c}) &= \phi(x_{T+1}) + \sum_{t=0}^{T} \beta^t \left[u\left(x_t, c_t\right) + \beta\lambda_{t+1} f(x_t, c_t) - \beta\lambda_{t+1} x_{t+1}\right] \\ &= \phi(x_{T+1}) + \sum_{t=0}^{T} \beta^t \left[H(\lambda_{t+1}, x_t, c_t) - \beta\lambda_{t+1} x_{t+1}\right], \end{aligned}$$

and notice that the remaining addends $\beta\lambda_{t+1}x_{t+1}$ can be ignored if $\mathbf{x}$ and λ satisfy the adjoint equations. See also Luenberger (1969), page 262. **Q.E.D.**

Notice that the variational condition in the above theorem is expressed in terms of stationary point for H. In continuous time this condition requires c^* to maximize the Hamiltonian each period. In fact, this distinction represents an important difference between the continuous time and the discrete time versions of the Maximum Principle. The idea behind this fact is simple. In continuous time one can construct "small" deviation in controls by varying a lot the path, but for a very short period of time: $\int_0^T |c(t) - u(t)|\, dt < \varepsilon$. This is not possible in discrete time, where to have a "small" deviation one must remain close to the optimal path for any t. As a consequence, the Pontryagin's Maximum Principle in more powerful in continuous time than when the time is discrete.

In continuous time, Mangasarian (1966) and Arrow and Kurz (1970) derived sufficient conditions for optimality. Mangasarian showed that if H is concave in (x, c) (and C convex), the necessary conditions of the Pontryagin's maximum theorem become also sufficient for an optimum. The discrete time version of the sufficiency theorem would be as follows[13]

Proposition 4 *Let $\left\{x_{t+1}^*, c_t^*, \lambda_{t+1}^*\right\}_{t=0}^T$ a sequence satisfying all the conditions of Theorem 17 above. Moreover assume that $\lambda_{t+1}^* \geq 0$ for all t, and that both u, f and ϕ are concave in (x_t, c_t), then the sequence $\left(x_{t+1}^*, c_t^*\right)$ is a global optimum for the problem.*

[13] See Takayama (1985), especially pages 660-666, for a didactical review in continuous time.

Proof. The proof uses the fact that a concave function is subdifferentiable to show a sequence of key inequalities using a similar derivation to that in Proposition 3. The condition $\lambda_{t+1}^* \geq 0$ only guarantees that when both u and f are concave then H is concave in (x_t, c_t). When f is linear it can be dispensed. **Q.E.D.**

Static Maximization and Pontryagin We have already mentioned that the maximum principle is basically an extension of the Lagrangian theorem. It improves at least in two direction. First, the theorem is particularly suitable for infinite dimensional spaces. The infinite dimensional version of the Lagrange theorem uses the same line of proof of the usual Lagrange theorem in finite dimensional spaces. However, the generalized Inverse Function Theorem of Liusternik is by no mean a simple result.[14] In addition, in continuous time, the theorem is not stated in terms of derivatives with respect to c, hence it allow for non differentiable cases. For example, the method allows for both corner and bang-bang solutions.

Consider the following exercise.

Exercise 40 *(i) Write the neoclassical growth model in terms of an optimal control problem. That is, distinguish states x from controls c, and specify f, u and C for this problem. [Hint: you might want to write the feasible set for controls C as a function of the state]. (ii) Next, derive the Euler equations from the Pontryagin maximum principle, and interpret economically the adjoint variables λ_t.*

Using the Pontryagin maximum principle one can deal perhaps with a larger class of problems than the one covered by the Euler's variational approach. In both cases however, when the horizon is infinite, one need to derive appropriate transversality condition.

[14] See Chapter 9 of Luenberger (1969).

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