1

Dynamic Consumption Theory

Optimizing models of intertemporal choices are widely used by theoretical and empirical studies of consumption. This chapter outlines their basic analytical structure, along with some extensions. The technical tools introduced here aim at familiarizing the reader with recent applied work on consumption and saving, but they will also prove useful in the rest of the book, when we shall study investment and other topics in economic dynamics.

The chapter is organized as follows. Section 1.1 illustrates and solves the basic version of the intertemporal consumption choice model, deriving theoretical relationships between the dynamics of permanent income, current income, consumption, and saving. Section 1.2 discusses problems raised by empirical tests of the theory, focusing on the excess sensitivity of consumption to expected income changes and on the excess smoothness of consumption following unexpected income variations. Explanations of the empirical evidence are offered by Section 1.3, which extends the basic model by introducing a precautionary saving motive. Section 1.4 derives the implications of optimal portfolio allocation for joint determination of optimal consumption when risky financial assets are available. The Appendix briefly introduces dynamic programming techniques applied to the optimal consumption choice. Bibliographic references and suggestions for further reading bring the chapter to a close.

1.1 Permanent Income and Optimal Consumption

The basic model used in the modern literature on consumption and saving choices is based on two main assumptions:

1. Identical economic agents maximize an *intertemporal utility function*, defined on the consumption levels in each period of the optimization horizon, subject to the constraint given by overall available resources.

2. Under uncertainty, the maximization is based on *expectations* of future relevant variables (for example, income and the rate of interest) formed *rationally* by agents, who use optimally all information at their disposal.

We will therefore study the optimal behavior of a *representative agent* who lives in an uncertain environment and has rational expectations. Implications of the theoretical model will then be used to interpret aggregate data. The representative consumer faces an infinite horizon (like any aggregate economy), and solves at time t an intertemporal choice problem of the following general form:

$$\max_{\{c_{t+i};i=0,1,\dots\}} U(c_t, c_{t+1}, \dots) \equiv U_t,$$

subject to the constraint (for $i = 0, \dots, \infty$)

$$A_{t+i+1} = (1 + r_{t+i})A_{t+i} + y_{t+i} - c_{t+i},$$

where A_{t+i} is the stock of financial wealth at the beginning of period $t + i$; r_{t+i} is the real rate of return on financial assets in period $t + i$; y_{t+i} is *labor* income earned at the end of period $t + i$, and c_{t+i} is consumption, also assumed to take place at the end of the period. The constraint therefore accounts for the evolution of the consumer's financial wealth from one period to the next.

Several assumptions are often made in order easily to derive empirically testable implications from the basic model. The main assumptions (some of which will be relaxed later) are as follows.

- *Intertemporal separability (or additivity over time)* The generic utility function $U_t(\cdot)$ is specified as

  $$U_t(c_t, c_{t+1}, \dots) = v_t(c_t) + v_{t+1}(c_{t+1}) + \dots$$

  (with $v'_{t+i} > 0$ and $v''_{t+i} < 0$ for any $i \geq 0$), where $v_{t+i}(c_{t+i})$ is the valuation at t of the utility accruing to the agent from consumption c_{t+i} at $t + i$. Since v_{t+i} depends only on consumption at $t + i$, the ratio of marginal utilities of consumption in any two periods is independent of consumption in any other period. This rules out goods whose effects on utility last for more than one period, either because the goods themselves are durable, or because their consumption creates long-lasting habits. (*Habit formation phenomena* will be discussed at the end of this chapter.)
- *A way of discounting utility in future periods that guarantees intertemporally consistent choices*. Dynamic inconsistencies arise when the valuation at time t of the relative utility of consumption in any two *future* periods, $t+k_1$ and $t+k_2$ (with $t < t+k_1 < t+k_2$), differs from the valuation of the same relative

utility at a different time $t+i$. In this case the optimal levels of consumption for $t+k_1$ and $t+k_2$ originally chosen at t may not be considered optimal at some later date: the consumer would then wish to reconsider his original choices simply because time has passed, even if no new information has become available. To rule out this phenomenon, it is necessary that the ratios of discounted marginal utilities of consumption in $t+k_1$ and $t+k_2$ depend, in addition to c_{t+k_1} and c_{t+k_2}, only on the distance $k_2 - k_1$, and not also on the moment in time when the optimization problem is solved. With a discount factor for the utility of consumption in $t+k$ of the form $(1+\rho)^{-k}$ (called "*exponential discounting*"), we can write

$$v_{t+k}(c_{t+k}) = \left(\frac{1}{1+\rho}\right)^k u(c_{t+k}),$$

and dynamic consistency of preferences is ensured: under certainty, the agent may choose the optimal consumption plan once and for all at the beginning of his planning horizon.[1]

- *The adoption of expected utility as the objective function under uncertainty (additivity over states of nature)* In discrete time, a *stochastic process* specifies a random variable for each date t, that is a real number associated to the realization of a *state of nature*. If it is possible to give a probability to different states of nature, it is also possible to construct an expectation of future income, weighting each possible level of income with the probability of the associated state of nature. In general, the probabilities used depend on available information, and therefore change over time when new information is made available. Given her information set at t, I_t, the consumer maximizes expected utility conditional on I_t: $U_t = E\left(\sum_{i=0}^{\infty} v_{t+i}(c_{t+i}) \mid I_t\right)$. Together with the assumption of intertemporal separability (additivity over periods of time), the adoption of expected utility entails an inverse relationship between the degree of intertemporal substitutability, measuring the agent's propensity to substitute current consumption with future consumption *under certainty*, and risk aversion, determining the agent's choices among different consumption levels *under uncertainty* over the state of nature: the latter, and the inverse of the former, are both measured in absolute terms by $-v_t''(c)/v_t'(c)$ at time t and for consumption level c. (We will expand on this point on page 6.)

[1] A strand of the recent literature (see the last section of this chapter for references) has explored the implications of a different discount function: a "*hyperbolic*"discount factor declines at a relatively higher rate in the short run (consumers are relatively "impatient" at short horizons) than in the long run (consumers are "patient" at long horizons, implying dynamic inconsistent preferences).

- Finally, we make the simplifying assumption that *there exists only one financial asset with certain and constant rate of return* r. Financial wealth A is the stock of the safe asset allowing the agent to transfer resources through time in a perfectly forecastable way; the only uncertainty is on the (exogenously given) future labor incomes y. Stochastic rates of return on n financial assets are introduced in Section 1.4 below.

Under the set of hypotheses above, the consumer's problem may be specified as follows:

$$\max_{\{c_{t+i}, i=0,1,\dots\}} U_t = E_t \left[\sum_{i=0}^{\infty} \left(\frac{1}{1+\rho} \right)^i u(c_{t+i}) \right] \tag{1.1}$$

subject to the constraint (for $i = 0, \dots, \infty$):[2]

$$A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i}, \qquad A_t \text{ given.} \tag{1.2}$$

In (1.1) ρ is the consumer's intertemporal rate of time preference and $E_t[\cdot]$ is the (rational) expectation formed using information available at t: for a generic variable x_{t+i} we have $E_t x_{t+i} = E(x_{t+i} \mid I_t)$. The hypothesis of rational expectations implies that the forecast error $x_{t+i} - E(x_{t+i} \mid I_t)$ is uncorrelated with the variables in the information set $I_t : E_t(x_{t+i} - E(x_{t+i} \mid I_t)) = 0$ (we will often use this property below). The value of current income y_t in included in I_t.

In the constraint (1.2) financial wealth A may be negative (the agent is not liquidity-constrained); however, we impose the restriction that the consumer's debt cannot grow at a rate greater than the financial return r by means of the following condition (known as the *no-Ponzi-game condition*):

$$\lim_{j \to \infty} \left(\frac{1}{1+r} \right)^j A_{t+j} \geq 0. \tag{1.3}$$

The condition in (1.3) is equivalent, in the infinite-horizon case, to the non-negativity constraint $A_{T+1} \geq 0$ for an agent with a life lasting until period T: in the absence of such a constraint, the consumer would borrow to finance infinitely large consumption levels. Although in its general formulation (1.3) is an inequality, if marginal utility of consumption is always positive this condition will be satisfied as an equality. Equation (1.3) with strict equality is called *transversality condition* and can be directly used in the problem's solution.

Similarly, without imposing (1.3), interests on debt could be paid for by further borrowing on an infinite horizon. Formally, from the budget constraint (1.2) at time t, repeatedly substituting A_{t+i} up to period $t+j$, we get the

[2] In addition, a non-negativity constraint on consumption must be imposed: $c_{t+i} \geq 0$. We assume that this constraint is always fulfilled.

following equation:

$$\frac{1}{1+r}\sum_{i=0}^{j-1}\left(\frac{1}{1+r}\right)^{i}c_{t+i}+\left(\frac{1}{1+r}\right)^{j}A_{t+j}=\frac{1}{1+r}\sum_{i=0}^{j-1}\left(\frac{1}{1+r}\right)^{i}y_{t+i}+A_{t}.$$

The present value of consumption flows from t up to $t+j-1$ can exceed the consumer's total available resources, given by the sum of the initial financial wealth A_t and the present value of future labor incomes from t up to $t+j-1$. In this case $A_{t+j}<0$ and the consumer will have a stock of debt at the beginning of period $t+j$. When the horizon is extended to infinity, the constraint (1.3) stops the agent from consuming more than his lifetime resources, using further borrowing to pay the interests on the existing debt in any period up to infinity. Assuming an infinite horizon and using (1.3) with equality, we get the consumer's *intertemporal budget constraint* at the beginning of period t (in the absence of liquidity constraints that would rule out, or limit, borrowing):

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}c_{t+i}=\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}y_{t+i}+A_{t}. \tag{1.4}$$

1.1.1 Optimal consumption dynamics

Substituting the consumption level derived from the budget constraint (1.2) into the utility function, we can write the consumer's problem as

$$\max U_t=E_t\sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{i}u\big((1+r)A_{t+i}-A_{t+i+1}+y_{t+i}\big)$$

with respect to wealth A_{t+i} for $i=1,2,\ldots,$ given initial wealth A_t and subject to the transversality condition derived from (1.3). The first-order conditions

$$E_t\,u'(c_{t+i})=\frac{1+r}{1+\rho}E_t\,u'(c_{t+i+1})$$

are necessary and sufficient if utility $u(c)$ is an increasing and concave function of consumption (i.e. if $u'(c)>0$ and $u''(c)<0$). For the consumer's choice in the first period (when $i=0$), noting that $u'(c_t)$ is known at time t, we get the so-called *Euler equation*:

$$u'(c_t)=\frac{1+r}{1+\rho}E_tu'(c_{t+1}). \tag{1.5}$$

At the optimum the agent is indifferent between consuming immediately one unit of the good, with marginal utility $u'(c_t)$, and saving in order to consume $1+r$ units in the next period, $t+1$. The same reasoning applies to any period t

in which the optimization problem is solved: the Euler equation gives the dynamics of marginal utility in any two successive periods.[3]

The evolution over time of marginal utility and consumption is governed by the difference between the rate of return r and the intertemporal rate of time preference ρ. Since $u''(c_t) < 0$, lower consumption yields higher marginal utility: if $r > \rho$, the consumer will find it optimal to increase consumption over time, exploiting a return on saving higher than the utility discount rate; when $r = \rho$, optimal consumption is constant, and when $r < \rho$ it is decreasing. The shape of marginal utility as a function of c (i.e. the concavity of the utility function) determines the magnitude of the effect of $r - \rho$ on the time path of consumption: if $|u''(c)|$ is large relative to $u'(c)$, large variations of marginal utility are associated with relatively small fluctuations in consumption, and then optimal consumption shows little changes over time even when the rate of return differs substantially from the utility discount rate.

Also, the agent's degree of risk aversion is determined by the concavity of the utility function. It has been already mentioned that our assumptions on preferences imply a negative relationship between risk aversion and intertemporal substitutability (where the latter measures the change in consumption between two successive periods owing to the difference between r and ρ or, if r is not constant, to changes in the rate of return). It is easy to find such relationship for the case of a *CRRA* (*constant relative risk aversion*) utility function, namely:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \qquad \gamma > 0,$$

with $u'(c) = c^{-\gamma}$. The degree of relative risk aversion—whose general measure is the absolute value of the elasticity of marginal utility, $-u''(c)\,c/u'(c)$—is in

[3] An equivalent solution of the problem is found by maximizing the Lagrangian function:

$$\mathcal{L}_t = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^i u(c_{t+i})$$
$$- \lambda \left[\sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i E_t c_{t+i} - (1+r)A_t - \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i E_t y_{t+i} \right],$$

where λ is the Lagrange multiplier associated with the intertemporal budget constraint (here evaluated at the end of period t). From the first-order conditions for c_t and c_{t+1}, we derive the Euler equation (1.5). In addition, we get $u'(c_t) = \lambda$. The shadow value of the budget constraint, measuring the increase of maximized utility that is due to an infinitesimal increase of the resources available at the end of period t, is equal to the marginal utility of consumption at t. At the optimum, the Euler equation holds: the agent is indifferent between consumption in the current period and consumption in any future period, since both alternatives provide additional utility given by $u'(c_t)$. In the Appendix to this chapter, the problem's solution is derived by means of dynamic programming techniques.

this case independent of the consumption level, and is equal to the parameter γ.[4] The measure of intertemporal substitutability is obtained by solving the consumer's optimization problem under certainty. The Euler equation corresponding to (1.5) is

$$c_t^{-\gamma} = \frac{1+r}{1+\rho} c_{t+1}^{-\gamma} \quad \Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \frac{1+r}{1+\rho}.$$

Taking logarithms, and using the approximations $\log(1+r) \simeq r$ and $\log(1+\rho) \simeq \rho$, we get

$$\Delta \log c_{t+1} = \frac{1}{\gamma}(r - \rho).$$

The elasticity of intertemporal substitution, which is the effect of changes in the interest rate on the growth rate of consumption $\Delta \log c$, is constant and is measured by the reciprocal of the coefficient of relative risk aversion γ.

1.1.2 Consumption level and dynamics

Under uncertainty, the expected value of utility may well differ from its realization. Letting

$$u'(c_{t+1}) - E_t u'(c_{t+1}) \equiv \eta_{t+1},$$

we have by definition that $E_t \eta_{t+1} = 0$ under the hypothesis of rational expectations. Then, from (1.5), we get

$$u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t) + \eta_{t+1}. \tag{1.6}$$

If we assume also that $r = \rho$, the stochastic process describing the evolution over time of marginal utility is

$$u'(c_{t+1}) = u'(c_t) + \eta_{t+1}, \tag{1.7}$$

and the change of marginal utility from t to $t+1$ is given by a stochastic term unforecastable at time t ($E_t \eta_{t+1} = 0$).

In order to derive the implications of the above result for the dynamics of consumption, it is necessary to specify a functional form for $u(c)$. To obtain a linear relation like (1.7), directly involving the level of consumption, we can assume a quadratic utility function $u(c) = c - (b/2)c^2$, with linear marginal utility $u'(c) = 1 - bc$ (positive only for $c < 1/b$). This simple and somewhat

[4] The denominator of the CRRA utility function is zero if $\gamma = 1$, but marginal utility can nevertheless have unitary elasticity: in fact, $u'(c) = c^{-\gamma} = 1/c$ if $u(c) = \log(c)$. The presence of the constant term "-1" in the numerator makes utility well defined also when $\gamma \to 1$. This limit can be computed, by l'Hôpital's rule, as the ratio of the limits of the numerator's derivative, $dc^{1-\gamma}/d\gamma = -\log(c)c^{1-\gamma}$, and the denominator's derivative, which is -1.

restrictive assumption lets us rewrite equation (1.7) as

$$c_{t+1} = c_t + u_{t+1}, \tag{1.8}$$

where $u_{t+1} \equiv -(1/b)\eta_{t+1}$ is such that $E_t u_{t+1} = 0$. If marginal utility is linear in consumption, as is the case when the utility function is quadratic, the process (1.8) followed by the level of consumption is a martingale, or a random walk, with the property:[5]

$$E_t c_{t+1} = c_t. \tag{1.9}$$

This is the main implication of the intertemporal choice model with rational expectations and quadratic utility: the best forecast of next period's consumption is current consumption. The consumption change from t to $t+1$ cannot be forecast on the basis of information available at t: formally, u_{t+1} is *orthogonal* to the information set used to form the expectation E_t, including all variables known to the consumer and dated $t, t-1, \ldots$ This implication has been widely tested empirically. Such *orthogonality tests* will be discussed below.

The solution of the consumer's intertemporal choice problem given by (1.8) cannot be interpreted as a consumption function. Indeed, that equation does not link consumption in each period to its determinants (income, wealth, rate of interest), but only describes the dynamics of consumption from one period to the next. The assumptions listed above, however, make it possible to derive the *consumption function*, combining what we know about the dynamics of optimal consumption and the intertemporal budget constraint (1.4). Since the realizations of income and consumption must fulfill the constraint, (1.4) holds also with expected values:

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t c_{t+i} = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} + A_t. \tag{1.10}$$

Linearity of the marginal utility function, and a discount rate equal to the interest rate, imply that the level of consumption expected for any future period is equal to current consumption. Substituting $E_t c_{t+i}$ with c_t on the left-hand side of (1.10), we get

$$\frac{1}{r}c_t = A_t + \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} \equiv A_t + H_t. \tag{1.11}$$

The last term in (1.11), the present value at t of future expected labor incomes, is the consumer's "human wealth" H_t. The consumption function can then be

[5] A *martingale* is a stochastic process x_t with the property $E_t x_{t+1} = x_t$. With $r = \rho$, marginal utility and, under the additional hypothesis of quadratic utility, the level of consumption have this property. No assumptions have been made about the distribution of the process $x_{t+1} - x_t$, for example concerning time-invariance, which is a feature of a random walk process.

written as

$$c_t = r(A_t + H_t) \equiv y_t^P \tag{1.12}$$

Consumption in t is now related to its determinants, the levels of financial wealth A_t and human wealth H_t. The consumer's overall wealth at the beginning of period t is given by $A_t + H_t$. Consumption in t is then the annuity value of total wealth, that is the return on wealth in each period: $r(A_t + H_t)$. That return, that we define as *permanent income* (y_t^P), is the flow that could be earned for ever on the stock of total wealth. The conclusion is that the agent chooses to consume in each period exactly his *permanent income*, computed on the basis of expectations of future labor incomes.

1.1.3 Dynamics of income, consumption, and saving

Given the consumption function (1.12), we note that the evolution through time of consumption and permanent income coincide. Leading (1.12) one period, we have

$$y_{t+1}^P = r(A_{t+1} + H_{t+1}). \tag{1.13}$$

Taking the expectation at time t of y_{t+1}^P, subtracting the resulting expression from (1.13), and noting that $E_t A_{t+1} = A_{t+1}$ from (1.2), since realized income y_t is included in the consumer's information set at t, we get

$$y_{t+1}^P - E_t y_{t+1}^P = r(H_{t+1} - E_t H_{t+1}). \tag{1.14}$$

Permanent income calculated at time $t + 1$, conditional on information available at that time, differs from the expectation formed one period earlier, conditional on information at t, only if there is a "surprise" in the agent's human wealth at time $t+1$. In other words, the "surprise" in permanent income at $t + 1$ is equal to the annuity value of the "surprise" in human wealth arising from new information on future labor incomes, available only at $t + 1$.

Since $c_t = y_t^P$, from (1.9) we have

$$E_t y_{t+1}^P = y_t^P.$$

All information available at t is used to calculate permanent income y_t^P, which is also the best forecast of the next period's permanent income. Using this result, the evolution over time of permanent income can be written as

$$y_{t+1}^P = y_t^P + r\left[\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (E_{t+1} - E_t) y_{t+1+i}\right],$$

where the "surprise" in human wealth in $t + 1$ is expressed as the revision in expectations on future incomes: y^P can change over time only if those expectations change, that is if, when additional information accrues to the agent in

$t+1$, $(E_{t+1}-E_t)y_{t+1+i} \equiv E_{t+1}y_{t+1+i} - E_t y_{t+1+i}$ is not zero for all i. The evolution over time of consumption follows that of permanent income, so that we can write

$$\begin{aligned} c_{t+1} &= c_t + r\left[\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (E_{t+1}-E_t)y_{t+1+i}\right] \\ &= c_t + u_{t+1}. \end{aligned} \tag{1.15}$$

It can be easily verified that the change of consumption between t and $t+1$ cannot be foreseen as of time t (since it depends only on information available in $t+1$): $E_t u_{t+1} = 0$. Thus, equation (1.15) enables us to attach a well defined economic meaning and a precise measure to the error term u_{t+1} in the Euler equation (1.8).

Intuitively, permanent income theory has important implications not only for the optimal consumption path, but also for the behavior of the agent's *saving*, governing the accumulation of her financial wealth. To discover these implications, we start from the definition of *disposable income* y^D, the sum of labor income, and the return on the financial wealth:

$$y_t^D = rA_t + y_t.$$

Saving s_t (the difference between disposable income and consumption) is easily derived by means of the main implication of permanent income theory ($c_t = y_t^P$):

$$s_t \equiv y_t^D - c_t = y_t^D - y_t^P = y_t - rH_t. \tag{1.16}$$

The level of saving in period t is then equal to the difference between current (labor) income y_t and the annuity value of human wealth rH_t. Such a difference, being *transitory* income, does not affect consumption: if it is positive it is entirely saved, whereas, if it is negative it determines a decumulation of financial assets of an equal amount. Thus, the consumer, faced with a variable labor income, changes the stock of financial assets so that the return earned on it (rA) allows her to keep consumption equal to permanent income.

Unfolding the definition of human wealth H_t in (1.16), we can write saving at t as

$$\begin{aligned} s_t &= y_t - \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} \\ &= \frac{1}{1+r}y_t - \left[\frac{1}{1+r} - \left(\frac{1}{1+r}\right)^2\right]E_t y_{t+1} \\ &\quad - \left[\left(\frac{1}{1+r}\right)^2 - \left(\frac{1}{1+r}\right)^3\right]E_t y_{t+2} + \ldots \\ &= -\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i E_t \Delta y_{t+i}, \end{aligned} \tag{1.17}$$

where $\Delta y_{t+i} = y_{t+i} - y_{t+i-1}$. Equation (1.17) sheds further light on the motivation for saving in this model: the consumer saves, accumulating financial assets, to face expected future declines of labor income (a *"saving for a rainy day"* behavior). Equation (1.17) has been extensively used in the empirical literature, and its role will be discussed in depth in Section 1.2.

1.1.4 Consumption, saving, and current income

Under certainty on future labor incomes, permanent income does not change over time. As a consequence, with $r = \rho$, consumption is constant and unrelated to current income y_t. On the contrary, when future incomes are uncertain, permanent income changes when new information causes a revision in expectations. Moreover, there is a link between current income and consumption if changes in income cause revisions in the consumer's expected permanent income. To explore the relation between current and permanent income, we assume a simple first-order autoregressive process generating income y:

$$y_{t+1} = \lambda y_t + (1-\lambda)\bar{y} + \varepsilon_{t+1}, \qquad E_t \varepsilon_{t+1} = 0, \tag{1.18}$$

where $0 \leq \lambda \leq 1$ is a parameter and $\bar{y}$ denotes the unconditional mean of income. The stochastic term ε_{t+1} is the component of income at $t+1$ that cannot be forecast on the basis of information available at t (i.e. the income innovation). Suppose that the stochastic process for income is in the consumer's information set. From (1.18) we can compute the revision, between t and $t+1$, of expectations of future incomes caused by a given realization of the stochastic term ε_{t+1}. The result of this calculation will then be substituted into (1.15) to obtain the effect on consumption c_{t+1}.

The revision in expectations of future incomes is given by

$$E_{t+1} y_{t+1+i} - E_t y_{t+1+i} = \lambda^i \varepsilon_{t+1}, \qquad \forall i \geq 0.$$

Substituting this expression into (1.15) for each period $t+1+i$, we have

$$r\left[\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i \lambda^i \varepsilon_{t+1}\right] = \left[\varepsilon_{t+1}\frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{\lambda}{1+r}\right)^i\right], \tag{1.19}$$

and solving the summation, we get[6]

$$c_{t+1} = c_t + \left(\frac{r}{1+r-\lambda}\right)\varepsilon_{t+1}, \tag{1.20}$$

which directly links current income innovation ε_{t+1} to current consumption

[6] The right-hand side expression in (1.19) can be written $\varepsilon_{t+1}(r/1+r)S_\infty(\lambda/1+r)$ if we denote by $S_N(\alpha)$ a *geometric series* with parameter α, of order N. Since $S_N(\alpha) - \alpha S_N(\alpha) = (1+\alpha+\alpha^2+\ldots+\alpha^N) - (\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{N+1}) = 1+\alpha^{N+1}$, such a series takes values $S_N(\alpha) = (1+\alpha^{N+1})/(1-\alpha)$ and, as long as $\alpha < 1$, converges to $S_\infty(\alpha) = (1-\alpha)^{-1}$ as N tends to infinity. Using this formula in (1.19) yields the result.

c_{t+1}. Like equation (1.8), (1.20) is a Euler equation; the error term is the innovation in permanent income, here expressed in terms of the current income innovation. Given an unexpected increase of income in period $t+1$ equal to ε_{t+1}, the consumer increases consumption in $t+1$ and expected consumption in all future periods by the annuity value of the increase in human wealth, $r\varepsilon_{t+1}/(1+r-\lambda)$. The portion of additional income that is not consumed, i.e.

$$\varepsilon_{t+1} - \frac{r}{1+r-\lambda}\varepsilon_{t+1} = \frac{1-\lambda}{1+r-\lambda}\varepsilon_{t+1},$$

is saved and added to the outstanding stock of financial assets. Starting from the next period, the return on this saving will add to disposable income, enabling the consumer to keep the higher level of consumption over the whole infinite future horizon.

It is important to notice that the magnitude of the consumption change between t and $t+1$ resulting from an innovation in current income ε_{t+1} depends, for a given interest rate r, on the parameter λ, capturing the degree of persistence of an innovation in $t+1$ on future incomes. To see the role of this parameter, it is useful to consider two polar cases.

1. $\lambda = 0$. In this case $y_{t+1} = \bar{y} + \varepsilon_{t+1}$. The innovation in current income is purely *transitory* and does not affect the level of income in future periods. Given an innovation ε_{t+1}, the consumer's human wealth, calculated at the beginning of period $t+1$, changes by $\varepsilon_{t+1}/(1+r)$. This change in H_{t+1} determines a variation of permanent income—and consumption—equal to $r\varepsilon_{t+1}/(1+r)$. In fact, from (1.20) with $\lambda = 0$, we have

$$c_{t+1} = c_t + \left(\frac{r}{1+r}\right)\varepsilon_{t+1}. \tag{1.21}$$

2. $\lambda = 1$. In this case $y_{t+1} = y_t + \varepsilon_{t+1}$. The innovation in current income is *permanent*, causing an equal change of all future incomes. The change in human wealth is then ε_{t+1}/r and the variation in permanent income and consumption is simply ε_{t+1}. From (1.20), with $\lambda = 1$, we get

$$c_{t+1} = c_t + \varepsilon_{t+1}.$$

Exercise 1 *In the two polar cases $\lambda = 0$ and $\lambda = 1$, find the effect of ε_{t+1} on saving in $t+1$ and on saving and disposable income in the following periods.*

Exercise 2 *Using the stochastic process for labor income in (1.18), prove that the consumption function that holds in this case (linking c_t to its determinants A_t, y_t, and $\bar{y}$) has the following form:*

$$c_t = rA_t + \frac{r}{1+r-\lambda}\, y_t + \frac{1-\lambda}{1+r-\lambda}\, \bar{y}.$$

What happens if $\lambda = 1$ and if $\lambda = 0$?

1.2 Empirical Issues

The dynamic implications of the permanent income model of consumption illustrated above motivated many recent empirical studies on consumption. Similarly, the life-cycle theory of consumption developed mainly by F. Modigliani has been subjected to empirical scrutiny. The partial-equilibrium perspective of this chapter makes it difficult to discuss the relationship between long-run saving and growth rates at the aggregate level: as we shall see in Chapter 4, the link between income growth and saving depends also on the interest rate, and becomes more complicated when the assumption of an exogenously given income process is abandoned. But even empirical studies based on *cross-sectional* individual data show that saving, if any, occurs only in the middle and old stages of the agent's life: consumption tracks income too closely to explain wealth accumulation only on the basis of a life-cycle motive.

As regards aggregate short-run dynamics, the first empirical test of the fundamental implication of the permanent income/rational expectations model of consumption is due to R. E. Hall (1978), who tests the orthogonality of the error term in the Euler equation with respect to past information. If the theory is correct, no variable known at time $t-1$ can explain changes in consumption between $t-1$ and t. Formally, the test is carried out by evaluating the statistical significance of variables dated $t-1$ in the Euler equation for time t. For example, augmenting the Euler equation with the income change that occurred between $t-2$ and $t-1$, we get

$$\Delta c_t = \alpha \Delta y_{t-1} + e_t, \tag{1.22}$$

where $\alpha = 0$ if the permanent income theory holds. Hall's results for the USA show that the null hypothesis cannot be rejected for several past aggregate variables, including income. However, some lagged variables (such as a stock index) are significant when added to the Euler equation, casting some doubt on the validity of the model's basic version.

Since Hall's contribution, the empirical literature has further investigated the dynamic implications of the theory, focusing mainly on two empirical regularities apparently at variance with the model: the consumption's *excess sensitivity* to current income changes, and its *excess smoothness* to income innovations. The remainder of this section illustrates these problems and shows how they are related.

1.2.1 Excess sensitivity of consumption to current income

A different test of the permanent income model has been originally proposed by M. Flavin (1981). Flavin's test is based on (1.15) and an additional equation for the stochastic process for income y_t. Consider the following stochastic process for income (AR(1) in first differences):

$$\Delta y_t = \mu + \lambda \Delta y_{t-1} + \varepsilon_t, \tag{1.23}$$

where ε_t is the change in current income, Δy_t, that is unforecastable using past income realizations. According to the model, the change in consumption between $t-1$ and t is due to the revision of expectations of future incomes caused by ε_t. Letting θ denote the intensity of this effect, the behavior of consumption is then

$$\Delta c_t = \theta \varepsilon_t. \tag{1.24}$$

Consumption is *excessively sensitive* to current income if c_t reacts to changes of y_t by more than is justified by the change in permanent income, measured by $\theta \varepsilon_t$.

Empirically, the Excess Sensitivity Hypothesis is formalized by augmenting (1.24) with the change in current income,

$$\Delta c_t = \beta \Delta y_t + \theta \varepsilon_t + v_t, \tag{1.25}$$

where β (if positive) measures the overreaction of consumption to a change in current income, and v_t captures the effect on consumption of information about permanent income, available to agents at t but unrelated to current income changes.

According to the permanent income model, an increase in current income causes a change in consumption only by the amount warranted by the revision of permanent income. Only *innovations* (that is, unpredictable changes) in income cause consumption changes: the term $\theta \varepsilon_t$ in (1.25) captures precisely this effect. An estimated value for β greater than zero is then interpreted as signaling an overreaction of consumption to *anticipated* changes in income.

The test on β in (1.25) is equivalent to Hall's orthogonality test in (1.22). In fact, substituting the stochastic process for income (1.23) into (1.25), we get

$$\Delta c_t = \beta \mu + \beta \lambda \, \Delta y_{t-1} + (\theta + \beta) \varepsilon_t + v_t. \tag{1.26}$$

From this expression for the consumption change, we note that the hypothesis $\beta = 0$ in (1.25) implies that $\alpha = 0$ in (1.22): if consumption is excessively sensitive to income, then the orthogonality property of the error term in the equation for Δc_t does not hold. Equation (1.26) highlights a potential difficulty with the orthogonality test. Indeed, Δc_t may be found to be uncorrelated with Δy_{t-1} if the latter does not forecast future income changes. In this case $\lambda = 0$ and the orthogonality test fails to reject the theory, even though consumption is excessively sensitive to predictable changes in income. Thus, differently from Hall's test, the approach of Flavin provides an estimate of the excess sensitivity of consumption, measured by β, which is around 0.36 on US quarterly data over the 1949–79 period.[7]

Among the potential explanations for the excess sensitivity of consumption, a strand of the empirical literature focused on the existence of *liquidity constraints*, which limit the consumer's borrowing capability, thus preventing the realization of the optimal consumption plan. With binding liquidity constraints, an increase in income, though perfectly anticipated, affects consumption only when it actually occurs.[8] A different rationale for excess sensitivity, based on the precautionary saving motive, will be analyzed in Section 1.3.[9]

1.2.2 Relative variability of income and consumption

One of the most appealing features of the permanent income theory, since the original formulation due to M. Friedman, is a potential explanation of why consumption typically is *less volatile* than current income: even in simple textbook Keynesian models, a marginal propensity to consume $c < 1$ in aggregate consumption functions of the form $C = \bar{c} + cY$ is crucial in obtaining the basic concept of multiplier of autonomous expenditure. By relating consumption not to current but to permanent, presumably less volatile, income, the limited reaction of consumption to changes in current income is theoretically motivated. The model developed thus far, adopting the framework of intertemporal optimization under rational expectations, derived the implications of this original intuition, and formalized the relationship between current income, consumption, and saving. (We shall discuss in the next chapter formalizations of simple textbook insights regarding investment dynamics: investment, like changes in consumption, is largely driven by revision of expectations regarding future variables.)

In particular, according to theory, the agent chooses current consumption on the basis of all available information on future incomes and changes optimal consumption over time only in response to unanticipated changes

[7] However, Flavin's test cannot provide an estimate of the change in permanent income resulting from a current income innovation θ, if ε and v in (1.26) have a non-zero covariance. Using aggregate data, any change in consumption due to v_t is also reflected in innovations in current income ε_t, since consumption is a component of aggregate income. Thus, the covariance between ε and v tends to be positive.

[8] Applying instrumental variables techniques to (1.25), Campbell and Mankiw (1989, 1991) directly interpret the estimated β as the fraction of liquidity-constrained consumers, who simply spend their current income.

[9] While we do not focus in this chapter on aggregate equilibrium considerations, it is worth mentioning that binding liquidity constraints and precautionary savings both tend to increase the aggregate saving rate: see Aiyagari (1994), Jappelli and Pagano (1994).

(innovations) in current income, causing revisions in permanent income. Therefore, on the empirical level, it is important to analyze the relationship between current income innovations and changes in permanent income, taking into account the degree of persistence over time of such innovations.

The empirical research on the properties of the stochastic process generating income has shown that income y is *non-stationary*: an innovation at time t does not cause a temporary deviation of income from trend, but has permanent effects on the level of y, which does not display any tendency to revert to a deterministic trend. (For example, in the USA the estimated long-run change in income is around 1.6 times the original income innovation.[10]) The implication of this result is that consumption, being determined by permanent income, should be *more* volatile than current income.

To clarify this point, consider again the following process for income:

$$\Delta y_{t+1} = \mu + \lambda \Delta y_t + \varepsilon_{t+1}, \tag{1.27}$$

where μ is a constant, $0 < \lambda < 1$, and $E_t \varepsilon_{t+1} = 0$. The income *change* between t and $t+1$ follows a stationary autoregressive process; the income *level* is permanently affected by innovations ε.[11] To obtain the effect on permanent income and consumption of an innovation ε_{t+1} when income is governed by (1.27), we can apply the following property of ARMA stochastic processes, which holds whether or not income is stationary (Deaton, 1992). For a given stochastic process for y of the form

$$a(L) y_t = \mu + b(L) \varepsilon_t,$$

where $a(L) = a_0 + a_1 L + a_2 L^2 + \ldots$ and $b(L) = b_0 + b_1 L + b_2 L^2 + \ldots$ are two polynomials in the lag operator L (such that, for a generic variable x, we have $L^i x_t = x_{t-i}$), we derive the following expression for the variance of the change in permanent income (and consequently in consumption):[12]

$$\frac{r}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i (E_{t+1} - E_t) y_{t+1+i} = \frac{r}{1+r} \frac{\sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i b_i}{\sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i a_i} \varepsilon_{t+1}. \tag{1.28}$$

In the case of (1.27), we can write

$$y_t = \mu + (1+\lambda) y_{t-1} - \lambda y_{t-2} + \varepsilon_t;$$

[10] The feature of non-stationarity of income (in the USA and in other countries as well) is still an open issue. Indeed, some authors argue that, given the low power of the statistical tests used to assess the non-stationarity of macroeconomic time series, it is impossible to distinguish between non-stationarity and the existence of a deterministic time trend on the basis of available data.

[11] A stochastic process of this form, with $\lambda = 0.44$, is a fairly good statistical description of the (aggregate) income dynamics for the USA, as shown by Campbell and Deaton (1989) using quarterly data for the period 1953–84.

[12] The following formula can also be obtained by computing the revisions in expectations of future incomes, as has already been done in Section 1.1.

hence we have $a(L) = 1 - (1+\lambda)L + \lambda L^2$ and $b(L) = 1$. Applying the general formula (1.28) to this process, we get

$$\Delta c_{t+1} = \frac{r}{1+r}\left(\frac{r(1+r-\lambda)}{(1+r)^2}\right)^{-1}\varepsilon_{t+1} = \frac{1+r}{1+r-\lambda}\varepsilon_{t+1}.$$

This is formally quite similar to (1.20), but, because the income process is stationary only in first differences, features a different numerator on the right-hand side: the relation between the innovation ε_{t+1} and the change in consumption Δc_{t+1} is linear, but the slope is greater than 1 if $\lambda > 0$ (that is if, as is realistic in business-cycle fluctuations, above-average growth tends to be followed by still fast—if mean-reverting—growth in the following period). The same coefficient measures the ratio of the variability of consumption (given by the standard deviation of the consumption change) and the variability of income (given by the standard deviation of the innovation in the income process):

$$\frac{\sigma_{\Delta c}}{\sigma_{\varepsilon}} = \frac{1+r}{1+r-\lambda}.$$

For example, $\lambda = 0.44$ and a (quarterly) interest rate of 1% yield a coefficient of 1.77. The implied variability of the (quarterly) change of consumption would be 1.77 times that of the income innovation. For non-durable goods and services, Campbell and Deaton (1989) estimate a coefficient of only 0.64. Then, the response of consumption to income innovations seems to be at variance with the implications of the permanent income theory: the reaction of consumption to unanticipated changes in income is too smooth (this phenomenon is called *excess smoothness*). This conclusion could be questioned by considering that the estimate of the income innovation, ε, depends on the variables included in the econometric specification of the income process. In particular, if a *univariate* process like (1.27) is specified, the information set used to form expectations of future incomes and to derive innovations is limited to past income values only. If agents form their expectations using additional information, not available to the econometrician, then the "true" income innovation, which is perceived by agents and determines changes in consumption, will display a smaller variance than the innovation estimated by the econometrician on the basis of a limited information set. Thus, the observed *smoothness* of consumption could be made consistent with theory if it were possible to measure the income innovations perceived by agents.[13]

A possible solution to this problem exploits the essential feature of the permanent income theory under rational expectations: agents choose optimal consumption (and saving) using all available information on future incomes.

[13] Relevant research includes Pischke (1995) and Jappelli and Pistaferri (2000).

It is the very behavior of consumers that reveals their available information. If such behavior is observed by the econometrician, it is possible to use it to construct expected future incomes and the associated innovations. This approach has been applied to saving, which, as shown by (1.17), depends on expected future changes in income.

To formalize this point, we start from the definition of saving and make explicit the information set used by agents at time t to forecast future incomes, I_t:

$$s_t = -\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i E(\Delta y_{t+i} \mid I_t). \tag{1.29}$$

The information set available to the econometrician is Ω_t, with $\Omega_t \subseteq I_t$ (agents know everything the econometrician knows but the reverse is not necessarily true). Moreover, we assume that saving is observed by the econometrician: $s_t \in \Omega_t$. Then, taking the expected value of both sides of (1.29) with respect to the information set Ω_t and applying the "law of iterated expectations," we get

$$\begin{aligned} E(s_t \mid \Omega_t) &= -\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i E\left[E(\Delta y_{t+i} \mid I_t) \mid \Omega_t\right] \\ \Longrightarrow s_t &= -\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i E(\Delta y_{t+i} \mid \Omega_t), \end{aligned} \tag{1.30}$$

where we use the assumption that saving is included in Ω_t. According to theory, then, saving is determined by the discounted future changes in labor incomes, even if they are forecast on the basis of the smaller information set Ω_t.

Since saving choices, according to (1.29), are made on the basis of all information available to agents, it is possible to obtain predictions on future incomes that do not suffer from the limited information problem typical of the univariate models widely used in the empirical literature. Indeed, predictions can be conditioned on past saving behavior, thus using the larger information set available to agents. This is equivalent to forming predictions of income changes Δy_t by using not only past changes, Δy_{t-1}, but also past saving, s_{t-1}.

In principle, this extension of the forecasting model for income could reduce the magnitude of the estimated innovation variance σ_ε. In practice, as is shown in some detail below, the evidence of excess smoothness of consumption remains unchanged after this extension.

1.2.3 Joint dynamics of income and saving

Studying the implications derived from theory on the joint behavior of income and saving usefully highlights the connection between the two empirical puzzles mentioned above (*excess sensitivity* and *excess smoothness*). Even though the two phenomena focus on the response of consumption to income changes of a different nature (consumption is excessively sensitive to *anticipated* income changes, and excessively smooth in response to *unanticipated* income variations), it is possible to show that the excess smoothness and excess sensitivity phenomena are different manifestations of the same empirical anomaly.

To outline the connection between the two, we proceed in three successive steps.

1. First, we assume a stochastic process *jointly* governing the evolution of income and saving over time and derive its implications for equations like (1.22), used to test the orthogonality property of the consumption change with respect to lagged variables. (Recall that the violation of the orthogonality condition entails excess sensitivity of consumption to predicted income changes.)
2. Then, given the expectations of future incomes based on the assumed stochastic process, we derive the behavior of saving implied by theory according to (1.17), and obtain the restrictions that must be imposed on the estimated parameters of the process for income and saving to test the validity of the theory.
3. Finally, we compare such restrictions with those required for the orthogonality property of the consumption change to hold.

We start with a simplified representation of the bivariate stochastic process governing income—expressed in first differences as in (1.27) to allow for non-stationarity, and imposing $\mu = 0$ for simplicity—and saving:

$$\Delta y_t = a_{11}\Delta y_{t-1} + a_{12}s_{t-1} + u_{1t}, \tag{1.31}$$

$$s_t = a_{21}\Delta y_{t-1} + a_{22}s_{t-1} + u_{2t}. \tag{1.32}$$

With s_{t-1} in the model, it is now possible to generate forecasts on future income changes by exploiting the additional informational value of past saving. Inserting the definition of saving ($s_t = rA_t + y_t - c_t$) into the accumulation constraint (1.2), we get

$$A_{t+1} = A_t + (rA_t + y_t - c_t) \Rightarrow s_t = A_{t+1} - A_t. \tag{1.33}$$

Obviously, the flow of saving is the change of the stock of financial assets from one period to the next, and this makes it possible to write the change in consumption by taking the first difference of the definition of saving used above:

$$\begin{aligned}\Delta c_t &= \Delta y_t + r\Delta A_t - \Delta s_t \\ &= \Delta y_t + rs_{t-1} - s_t + s_{t-1} \\ &= \Delta y_t + (1+r)s_{t-1} - s_t.\end{aligned} \tag{1.34}$$

Finally, substituting for Δy_t and s_t from (1.31) and (1.32), we obtain the following expression for the consumption change Δc_t:

$$\Delta c_t = \gamma_1 \Delta y_{t-1} + \gamma_2 s_{t-1} + v_t, \tag{1.35}$$

where

$$\gamma_1 = a_{11} - a_{21}, \qquad \gamma_2 = a_{12} - a_{22} + (1+r), \qquad v_t = u_{1t} - u_{2t}.$$

The implication of the permanent income theory is that the consumption change between $t-1$ and t cannot be predicted on the basis of information available at time $t-1$. This entails the *orthogonality* restriction $\gamma_1 = \gamma_2 = 0$, which in turn imposes the following restrictions on the coefficients of the joint process generating income and savings:

$$a_{11} = a_{21}, \qquad a_{22} = a_{12} + (1+r). \tag{1.36}$$

If these restrictions are fulfilled, the consumption change $\Delta c_t = u_{1t} - u_{2t}$ is unpredictable using lagged variables: the change in consumption (and in permanent income) is equal to the current income innovation (u_{1t}) less the innovation in saving (u_{2t}), which reflects the revision in expectations of future incomes calculated by the agent on the basis of all available information. Now, from the definition of savings (1.17), using the expectations of future income changes derived from the model in (1.31) and (1.32), it is possible to obtain the restrictions imposed by the theory on the stochastic process governing income and savings. Letting

$$\mathbf{x}_t \equiv \begin{pmatrix} \Delta y_t \\ s_t \end{pmatrix}, \qquad \mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathbf{u}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix},$$

we can rewrite the process in (1.31)–(1.32) as

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{u}_t. \tag{1.37}$$

From (1.37), the expected values of Δy_{t+i} can be easily derived:

$$E_t \mathbf{x}_{t+i} = \mathbf{A}^i \mathbf{x}_t, \qquad i \geq 0;$$

hence (using a matrix algebra version of the geometric series formula)

$$\begin{aligned} -\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i E_t \mathbf{x}_{t+i} &= -\sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \mathbf{A}^i \mathbf{x}_t \\ &= -\left[\left(\mathbf{I} - \frac{1}{1+r}\mathbf{A}\right)^{-1} - \mathbf{I}\right] \mathbf{x}_t. \end{aligned} \tag{1.38}$$

The element of vector x we are interested in (saving s) can be "extracted" by applying to $\mathbf{x}$ a vector $e_2 \equiv (0 \quad 1)'$, which simply selects the second element of $\mathbf{x}$. Similarly, to apply the definition in (1.17), we have to select the first element of the vector in (1.38) using $e_1 \equiv (1 \quad 0)'$. Then we get

$$\mathbf{e}_2'\mathbf{x}_t = -\mathbf{e}_1'\left[\left(\mathbf{I} - \frac{1}{1+r}\mathbf{A}\right)^{-1} - \mathbf{I}\right]\mathbf{x}_t \Rightarrow \mathbf{e}_2' = -\mathbf{e}_1'\left[\left(\mathbf{I} - \frac{1}{1+r}\mathbf{A}\right)^{-1} - \mathbf{I}\right],$$

yielding the relation

$$\mathbf{e}_2' = (\mathbf{e}_2' - \mathbf{e}_1')\frac{1}{1+r}\mathbf{A}. \tag{1.39}$$

Therefore, the restrictions imposed by theory on the coefficients of matrix $\mathbf{A}$ are

$$a_{11} = a_{21}, \qquad a_{22} = a_{12} + (1+r). \tag{1.40}$$

These restrictions on the joint process for income and saving, which rule out the excess smoothness phenomenon, are exactly the same as those—in equation (1.35)—that must be fulfilled for the orthogonality property to hold, and therefore also ensure elimination of excess sensitivity.[14] Summarizing, the phenomena of excess sensitivity and excess smoothness, though related to income changes of a different nature (anticipated and unanticipated, respectively), signal the same deviation from the implications of the permanent income theory. If agents excessively react to expected income changes, they must necessarily display a lack of reaction to unanticipated income changes. In fact, any variation in income is made up of a predicted component and a (unpredictable) innovation: if the consumer has an "excessive" reaction to the former component, the intertemporal budget constraint forces him to react in an "excessively smooth" way to the latter component of the change in current income.

1.3 The Role of Precautionary Saving

Recent developments in consumption theory have been aimed mainly at solving the empirical problems illustrated above. The basic model has been extended in various directions, by relaxing some of its most restrictive assumptions. On the one hand, as already mentioned, liquidity constraints can prevent

[14] The coincidence of the restrictions necessary for orthogonality and for ruling out excess smoothness is obtained only in the special case of a first-order stochastic process for income and saving. In the more general case analyzed by Flavin (1993), the orthogonality restrictions are nested in those necessary to rule out excess smoothness. Then, in general, orthogonality conditions analogous to (1.36) imply—but are not implied by—those analogous to (1.40).

the consumer from borrowing as much as required by the optimal consumption plan. On the other hand, it has been recognized that in the basic model saving is motivated only by a rate of interest higher than the rate-of-time preference and/or by the need for redistributing income over time, when current incomes are unbalanced between periods. Additional motivations for saving may be relevant in practice, and may contribute to the explanation of, for example, the apparently insufficient decumulation of wealth by older generations, the high correlation between income and consumption of younger agents, and the excess smoothness of consumption in reaction to income innovations. This section deals with the latter strand of literature, studying the role of a precautionary saving motive in shaping consumers' behavior.

First, we will spell out the microeconomic foundations of precautionary saving, pointing out which assumption of the basic model must be relaxed to allow for a precautionary saving motive. Then, under the new assumptions, we shall derive the dynamics of consumption and the consumption function, and compare them with the implications of the basic version of the permanent income model previously illustrated.

1.3.1 Microeconomic foundations

Thus far, with a quadratic utility function, *uncertainty* has played only a limited role. Indeed, only the expected value of income y affects consumption choices—other characteristics of the income distribution (e.g. the variance) do not play any role.

With quadratic utility, marginal utility is linear and the expected value of the marginal utility of consumption coincides with the marginal utility of expected consumption. An increase in uncertainty on future consumption, with an unchanged expected value, does not cause any reaction by the consumer.[15] As we shall see, if marginal utility is a *convex* function of consumption, then the consumer displays a *prudent* behavior, and reacts to an increase in uncertainty by saving more: such saving is called *precautionary*, since it depends on the uncertainty about future consumption.

Convexity of the marginal utility function $u'(c)$ implies a positive sign of its second derivative, corresponding to the third derivative of the utility function: $u'''(c) > 0$. A precautionary saving motive, which does not arise with quadratic utility ($u'''(c) = 0$), requires the use of different functional forms, such

[15] In the basic version of the model, the consumer is interested only in the *certainty equivalent* value of future consumption.

as exponential utility.[16] With risk aversion ($u''(c) < 0$) and convex marginal utility ($u'''(c) > 0$), under uncertainty about future incomes (and consumption), unfavorable events determine a loss of utility greater than the gain in utility obtained from favorable events of the same magnitude. The consumer fears low-income states and adopts a prudent behavior, saving in the current period in order to increase expected future consumption.

An example can make this point clearer. Consider a consumer living for two periods, t and $t+1$, with no financial wealth at the beginning of period t. In the first period labor income is $\bar{y}$ with certainty, whereas in the second period it can take one of two values—y_{t+1}^A or $y_{t+1}^B < y_{t+1}^A$—with equal probability. To focus on the precautionary motive, we rule out any other motivation for saving by assuming that $E_t(y_{t+1}) = \bar{y}$ and $r = \rho = 0$. In equilibrium the following relation holds: $E_t u'(c_{t+1}) = u'(c_t)$. At time t the consumer chooses saving s_t (equal to $\bar{y} - c_t$) and his consumption at time $t+1$ will be equal to saving s_t plus realized income. Considering actual realizations of income, we can write the budget constraint as

$$\left.\begin{array}{l} c_{t+1}^A \\ c_{t+1}^B \end{array}\right\} = \bar{y} - c_t + \left\{\begin{array}{l} y_{t+1}^A \\ y_{t+1}^B \end{array}\right.$$

$$= s_t + \left\{\begin{array}{l} y_{t+1}^A \\ y_{t+1}^B \end{array}\right. .$$

Using the definition of saving, $s_t \equiv \bar{y} - c_t$, the Euler equation becomes

$$E_t(u'(y_{t+1} + s_t)) = u'(\bar{y} - s_t). \tag{1.41}$$

Now, let us see how the consumer chooses saving in two different cases, beginning with that of *linear* marginal utility ($u'''(c) = 0$). In this case we have $E_t u'(\cdot) = u'(E_t(\cdot))$. Recalling that $E_t(y_{t+1}) = \bar{y}$, condition (1.41) becomes

$$u'(\bar{y} + s_t) = u'(\bar{y} - s_t), \tag{1.42}$$

and is fulfilled by $s_t = 0$. The consumer does not save in the first period, and his second-period consumption will coincide with current income. The uncertainty on income in $t+1$ reduces overall utility but does not induce the consumer to modify his choice: there is no precautionary saving. On the contrary, if, as in Figure 1.1, marginal utility is *convex* ($u'''(c) > 0$), then, from "Jensen's inequality," $E_t u'(c_{t+1}) > u'(E_t(c_{t+1}))$.[17] If the consumer were to choose

[16] A quadratic utility function has another undesirable property: it displays increasing absolute risk aversion. Formally, $-u''(c)/u'(c)$ is an increasing function of c. This implies that, to avoid uncertainty, the agent is willing to pay more the higher is his wealth, which is not plausible.

[17] Jensen's inequality states that, given a strictly convex function $f(x)$ of a random variable x, then $E(f(x)) > f(Ex)$.

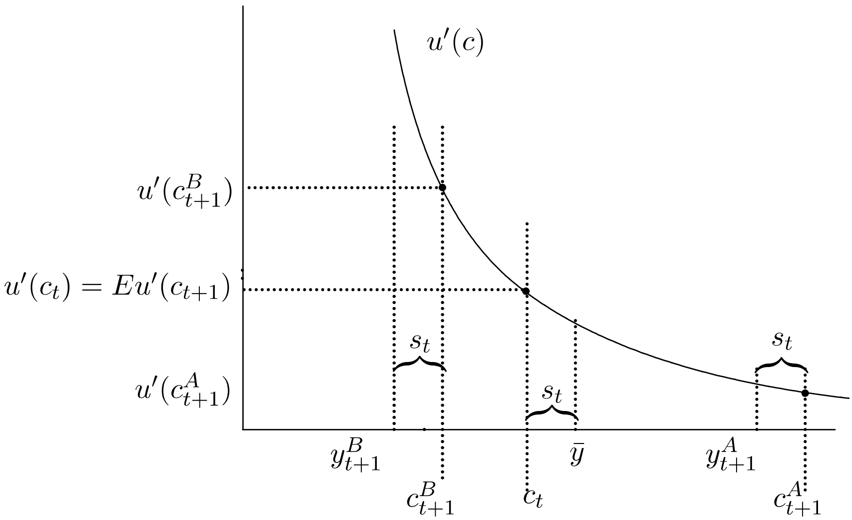

Figure 1.1. Precautionary savings

zero saving, as was optimal under a linear marginal utility, we would have (for $s_t = 0$, and using Jensen's inequality)

$$E_t(u'(c_{t+1})) > u'(c_t). \tag{1.43}$$

The optimality condition would be violated, and expected utility would not be maximized. To re-establish equality in the problem's first-order condition, marginal utility must decrease in $t+1$ and increase in t: as shown in the figure, this may be achieved by shifting an amount of resources s_t from the first to the second period. As the consumer saves more, decreasing current consumption c_t and increasing c_{t+1} in both states (good and bad), marginal utility in t increases and expected marginal utility in $t+1$ decreases, until the optimality condition is satisfied. Thus, with convex marginal utility, uncertainty on future incomes (and consumption levels) entails a positive amount of saving in the first period and determines a consumption path trending upwards over time ($E_t c_{t+1} > c_t$), even though the interest rate is equal to the utility discount rate. Formally, the relation between uncertainty and the upward consumption path depends on the degree of consumer's *prudence*, which we now define rigorously. Approximating (by means of a second-order Taylor expansion) around c_t the left-hand side of the Euler equation $E_t u'(c_{t+1}) = u'(c_t)$, we get

$$E_t(c_{t+1} - c_t) = -\frac{1}{2}\frac{u'''(c_t)}{u''(c_t)}E_t(c_{t+1} - c_t)^2 \equiv \frac{1}{2}aE_t(c_{t+1} - c_t)^2, \tag{1.44}$$

where $a \equiv -u'''(c)/u''(c)$ is the coefficient of *absolute prudence*. Greater uncertainty, increasing $E_t((c_{t+1}-c_t)^2)$, induces a larger increase in consumption between t and $t+1$. The definition of the coefficient measuring prudence is formally similar to that of risk-aversion coefficients: however, the latter is related to the curvature of the utility function, whereas prudence is determined by the curvature of marginal utility. It is also possible to define the coefficient of *relative prudence*, $-u'''(c)c/u''(c)$. Dividing both sides of (1.44) by c_t, we get

$$E_t\left(\frac{c_{t+1}-c_t}{c_t}\right) = -\frac{1}{2}\frac{u'''(c_t)\cdot c_t}{u''(c_t)}E_t\left(\frac{c_{t+1}-c_t}{c_t}\right)^2 = \frac{1}{2}pE_t\left(\frac{c_{t+1}-c_t}{c_t}\right)^2,$$

where $p \equiv -(u'''(c)\cdot c/u''(c))$ is the coefficient of relative prudence. Readers can check that this is constant for a CRRA function, and determine its relationship to the coefficient of relative risk aversion.

Exercise 3 *Suppose that a consumer maximizes*

$$\log(c_1) + E[\log(c_2)]$$

under the constraint $c_1+c_2 = w_1+w_2$ *(i.e., the discount rate of period 2 utility and the rate of return on saving* $w_1 - c_1$ *are both zero). When* c_1 *is chosen, there is uncertainty about* w_2*: the consumer will earn* $w_2 = x$ *or* $w_2 = y$ *with equal probability. What is the optimal level of* c_1*?*

1.3.2 Implications for the consumption function

We now solve the consumer's optimization problem in the case of a non-quadratic utility function, which motivates precautionary saving. The setup of the problem is still given by (1.1) and (1.2), but the utility function in each period is now of the exponential form:

$$u(c_{t+i}) = -\frac{1}{\gamma}e^{-\gamma c_{t+i}}, \tag{1.45}$$

where $\gamma > 0$ is the coefficient of absolute prudence (and also, for such a *constant absolute risk aversion*—CARA—utility function, the coefficient of absolute risk aversion).[18] Assume that labor income follows the AR(1) stochastic process:

$$y_{t+i} = \lambda y_{t+i-1} + (1-\lambda)\bar{y} + \varepsilon_{t+i}, \tag{1.46}$$

where ε_{t+i} are independent and identically distributed (i.i.d.) random variables, with zero mean and variance σ_ε^2. We keep the simplifying hypothesis that $r = \rho$.

[18] Since for the exponential utility function $u'(0) = 1 < \infty$, in order to rule out negative values for consumption it would be necessary to explicitly impose a non-negativity constraint; however, a closed-form solution to the problem would not be available if that constraint were binding.

The problem's first-order condition, for $i = 0$, is given by

$$e^{-\gamma c_t} = E_t(e^{-\gamma c_{t+1}}). \tag{1.47}$$

To proceed, we guess that the stochastic process followed by consumption over time has the form

$$c_{t+i} = c_{t+i-1} + K_{t+i-1} + v_{t+i}, \tag{1.48}$$

where K_{t+i-1} is a deterministic term (which may however depend on the period's timing within the individual's life cycle) and v_{t+i} is the innovation in consumption ($E_{t+i-1}v_{t+i} = 0$). Both the sequence of K_t terms and the features of the distribution of v must be determined so as to satisfy the Euler equation (1.47) and the intertemporal budget constraint (1.4). Using (1.48), from the Euler equation, after eliminating the terms in c_t, we get

$$e^{\gamma K_t} = E_t(e^{-\gamma v_{t+1}}) \Rightarrow K_t = \frac{1}{\gamma} \log E_t(e^{-\gamma v_{t+1}}). \tag{1.49}$$

The value of K depends on the characteristics of the distribution of v, yet to be determined. Using the fact that $\log E(\cdot) > E(\log(\cdot))$ by Jensen's inequality and the property of consumption innovations $E_t v_{t+1} = 0$, we can however already write

$$K_t = \frac{1}{\gamma} \log E_t(e^{-\gamma v_{t+1}}) > \frac{1}{\gamma} E_t(\log(e^{-\gamma v_{t+1}})) = \frac{1}{\gamma} E_t(-\gamma v_{t+1}) = 0 \Rightarrow K_t > 0. \tag{1.50}$$

The first result is that the consumption path is *increasing* over time: the consumption change between t and $t+1$ is expected to equal $K_t > 0$, whereas with quadratic utility (maintaining the assumption $\rho = r$) consumption changes would have zero mean. Moreover, from (1.49) we interpret $-K_t$ as the "certainty equivalent" of the consumption innovation v_{t+1}, defined as the (negative) certain change of consumption from t to $t+1$ that the consumer would accept to avoid the uncertainty on the marginal utility of consumption in $t+1$.

To obtain the consumption function (and then to determine the effect of the precautionary saving motive on the *level* of consumption) we use the intertemporal budget constraint (1.10) computing the expected values $E_t c_{t+i}$ from (1.48). Knowing that $E_t v_{t+i} = 0$, we have

$$\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i c_t + \frac{1}{1+r} \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \sum_{j=1}^{i} K_{t+j-1} = A_t + H_t. \tag{1.51}$$

Solving for c_t, we finally get

$$c_t = r(A_t + H_t) - \frac{r}{1+r} \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \sum_{j=1}^{i} K_{t+j-1}. \tag{1.52}$$

The level of consumption is made up of a component analogous to the definition of permanent income, $r(A_t + H_t)$, less a term that depends on the constants K and captures the effect of the precautionary saving motive: since the individual behaves prudently, her consumption increases over time, but (consistently with the intertemporal budget constraint) the level of consumption in t is lower than in the case of quadratic utility.

As the final step of the solution, we derive the form of the stochastic term v_{t+i}, and its relationship to the income innovation ε_{t+i}. To this end we use the budget constraint (1.4), where c_{t+i} and y_{t+i} are realizations and not expected values, and write future realized incomes as the sum of the expected value at time t and the associated "surprise": $y_{t+i} = E_t y_{t+i} + (y_{t+i} - E_t y_{t+i})$. The budget constraint becomes

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i c_{t+i} = A_t + H_t + \frac{1}{1+r}\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i (y_{t+i} - E_t y_{t+i}).$$

Substituting for c_{t+i} (with $i > 0$) from (1.48) and for c_t from the consumption function (1.52), we get

$$\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i \sum_{j=1}^{i} v_{t+j} = \sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i (y_{t+i} - E_t y_{t+i}).$$

Given the stochastic process for income (1.46) we can compute the income "surprises,"

$$y_{t+i} - E_t y_{t+i} = \sum_{k=0}^{i-1} \lambda^k \varepsilon_{t+i-k},$$

and insert them into the previous equation, to obtain

$$\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i \sum_{j=1}^{i} v_{t+j} = \sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i \sum_{k=0}^{i-1} \lambda^k \varepsilon_{t+i-k}. \tag{1.53}$$

Developing the summations, collecting terms containing v and ε with the same time subscript, and using the fact that v and ε are serially uncorrelated processes, we find the following condition that allows us to determine the form of v_{t+i}:

$$\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i (v_{t+h} - \lambda^{i-1}\varepsilon_{t+h}) = 0, \qquad \forall h \geq 1. \tag{1.54}$$

Solving the summation in (1.54), we arrive at the final form of the stochastic terms of the Euler equation guessed in (1.48): at all times $t + h$,

$$v_{t+h} = \frac{r}{1+r-\lambda}\varepsilon_{t+h}. \tag{1.55}$$

As in the quadratic utility case (1.20), the innovation in the Euler equation can be interpreted as the *annuity value* of the revision of the consumer's human wealth arising from an innovation in income for the assumed stochastic process.

Expression (1.55) for v_{t+1} can be substituted in the equation for K_t (1.49). The fact that the innovations ε are i.i.d. random variables implies that K_t does not change over time: $K_{t+i-1} = K$ in (1.48). The evolution of consumption over time is then given by

$$c_{t+1} = c_t + K + \frac{r}{1+r-\lambda}\varepsilon_{t+1}. \tag{1.56}$$

Substituting the constant value for K into (1.52), we get a closed-form consumption function:[19]

$$\begin{aligned} c_t &= r(A_t + H_t) - \frac{r}{1+r}\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i i\cdot K \\ &= r(A_t + H_t) - \frac{r}{1+r}K\frac{1+r}{r^2} \\ &= r(A_t + H_t) - \frac{K}{r}. \end{aligned}$$

Finally, to determine the constant K and its relationship with the uncertainty about future labor incomes, some assumptions on the distribution of ε have to be made. If ε is *normally* distributed, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, then, letting $\theta \equiv r/(1+r-\lambda)$, we have[20]

$$K_t = \frac{1}{\gamma}\log E_t(e^{-\gamma\theta\varepsilon_{t+1}}) = \frac{1}{\gamma}\log e^{\frac{\gamma^2\theta^2\sigma_\varepsilon^2}{2}} = \frac{\gamma\theta^2\sigma_\varepsilon^2}{2}. \tag{1.57}$$

The dynamics of consumption over time and its level in each period are then given by

$$\begin{aligned} c_{t+1} &= c_t + \frac{\gamma\theta^2\sigma_\varepsilon^2}{2} + \theta\varepsilon_{t+1}, \\ c_t &= r(A_t + H_t) - \frac{1}{r}\frac{\gamma\theta^2\sigma_\varepsilon^2}{2}. \end{aligned}$$

[19] To verify this result, note that

$$\begin{aligned} \sum_{i=1}^{\infty}\alpha^i i &= \sum_{i=1}^{\infty}\alpha^i + \sum_{i=2}^{\infty}\alpha^i + \sum_{i=3}^{\infty}\alpha^i + \ldots \\ &= \sum_{i=1}^{\infty}\alpha^i + \alpha\sum_{i=1}^{\infty}\alpha^i + \alpha^2\sum_{i=1}^{\infty}\alpha^i + \ldots \\ &= (1 + \alpha + \alpha^2 + \ldots)\sum_{i=1}^{\infty}\alpha^i \\ &= \sum_{i=0}^{\infty}\alpha^i\left(\sum_{i=0}^{\infty}\alpha^i - 1\right), \end{aligned}$$

which equals $\frac{1}{1-\alpha}\frac{\alpha}{1-\alpha} = \alpha/(1-\alpha)^2$ as long as $\alpha < 1$, which holds true in the relevant $\alpha = 1/(1+r)$ case with $r > 0$.

[20] To derive (1.57) we used the following statistical fact: if $x \sim N(E(x), \sigma^2)$, then e^x is a *lognormal* random variable with mean $E(e^x) = e^{E(x)+\sigma^2/2}$.

The innovation variance σ_ε^2 has a positive effect on the change in consumption between t and $t+1$, and a negative effect on the level of consumption in t. Increases in the uncertainty about future incomes (captured by the variance of the innovations in the process for y) generate larger changes of consumption from one period to the next and drops in the level of current consumption. Thus, allowing for a precautionary saving motive can rationalize the slow decumulation of wealth by old individuals, and can explain why (increasing) income and consumption paths are closer to each other than would be implied by the basic permanent income model. Moreover, if positive innovations in current income are associated with higher uncertainty about future income, the *excess smoothness* phenomenon may be explained, since greater uncertainty induces consumers to save more and may then reduce the response of consumption to income innovations.

Exercise 4 *Assuming $u(c) = c^{1-\gamma}/(1-\gamma)$ and $r \neq \rho$, derive the first-order condition of the consumer's problem under uncertainty. If c_{t+1}/c_t has a lognormal distribution (i.e. if the rate of change of consumption $\Delta \log c_{t+1}$ is normally distributed with constant variance σ^2), write the Euler equation in terms of the expected rate of change of consumption $E_t(\Delta \log c_{t+1})$. How does the variance σ^2 affect the behavior of the rate of change of c over time? (*Hint: *make use of the fact mentioned in note 20, recall that $c_{t+1}/c_t = e^{\Delta \log c_{t+1}}$, and express the Euler equations in logarithmic terms.)*

1.4 Consumption and Financial Returns

In the model studied so far, the consumer uses a single financial asset with a certain return to implement the optimal consumption path. A precautionary saving motive has been introduced by abandoning the hypothesis of quadratic utility. However, there is still no choice on the allocation of saving. If we assume that the consumer can invest his savings in n financial assets with uncertain returns, we generate a more complicated choice of the composition of financial wealth, which interacts with the determination of the optimal consumption path. The chosen portfolio allocation will depend on the characteristics of the consumer's utility function (in particular the degree of risk aversion) and of the distribution of asset returns. Thereby extended, the model yields testable implications on the *joint* dynamics of consumption and asset returns, and becomes the basic version of the *consumption-based capital asset pricing model* (CCAPM).

With the new hypothesis of n financial assets with uncertain returns, the consumer's budget constraint must be reformulated accordingly. The beginning-of-period stock of the jth asset, measured in units of consumption, is given by A^j_{t+i}. Therefore, total financial wealth is $A_{t+i} = \sum_{j=1}^n A^j_{t+i} \cdot r^j_{t+i+1}$ denotes the real rate of return of asset j in period $t+i$, so that $A^j_{t+i+1} = (1+r^j_{t+i+1})A^j_{t+i}$. This return is not known by the agent at the beginning of period $t+i$. (This explains the time subscript $t+i+1$, whereas labor income—observed by the agent at the beginning of the period—has subscript $t+i$.) The accumulation constraint from one period to the next takes the form

$$\sum_{j=1}^{n} A^j_{t+i+1} = \sum_{j=1}^{n} (1+r^j_{t+i+1})A^j_{t+i} + y_{t+i} - c_{t+i}, \qquad i = 0, \ldots, \infty. \tag{1.58}$$

The solution at t of the maximization problem yields the levels of consumption and of the stocks of the n assets from t to infinity. Like in the solution of the consumer's problem analyzed in Section 1.1 (but now with uncertain asset returns), we have a set of n Euler equations,

$$u'(c_t) = \frac{1}{1+\rho} E_t\left[(1+r^j_{t+1})\, u'(c_{t+1})\right] \qquad \text{for} \quad j = 1, \ldots, n. \tag{1.59}$$

Since $u'(c_t)$ is not stochastic at time t, we can write the first-order conditions as

$$\begin{aligned} 1 &= E_t\left[(1+r^j_{t+1})\frac{1}{1+\rho}\frac{u'(c_{t+1})}{u'(c_t)}\right] \\ &\equiv E_t\left[(1+r^j_{t+1})\, M_{t+1}\right], \end{aligned} \tag{1.60}$$

where M_{t+1} is the "stochastic discount factor" applied at t to consumption in the following period. Such a factor is the intertemporal marginal rate of substitution, i.e. the discounted ratio of marginal utilities of consumption in any two subsequent periods. From equation (1.60) we derive the fundamental result of the CCAPM, using the following property:

$$E_t\left[\left(1+r^j_{t+1}\right) M_{t+1}\right] = E_t(1+r^j_{t+1})\, E_t(M_{t+1}) + \text{cov}_t(r^j_{t+1}, M_{t+1}). \tag{1.61}$$

Inserting (1.61) into (1.60) and rearranging terms, we get

$$E_t(1+r^j_{t+1}) = \frac{1}{E_t(M_{t+1})}\left[1 - \text{cov}_t\left(r^j_{t+1}, M_{t+1}\right)\right]. \tag{1.62}$$

In the case of the safe asset (with certain return r^0) considered in the previous sections,[21] (1.62) reduces to

$$1 + r^0_{t+1} = \frac{1}{E_t(M_{t+1})}. \tag{1.63}$$

Substituting (1.63) into (1.62), we can write the expected return of each asset j in excess of the safe asset as

$$E_t(r^j_{t+1}) - r^0_{t+1} = -(1+r^0_{t+1})\, \text{cov}_t(r^j_{t+1}, M_{t+1}). \tag{1.64}$$

[21] The following results hold also if the safe return rate r^0 is random, as long as it has zero covariance with the stochastic discount factor M.

Equation (1.64) is the main result from the model with risky financial assets: in equilibrium, an asset j whose return has a negative covariance with the stochastic discount factor yields an expected return higher than r^0. In fact, such an asset is "risky" for the consumer, since it yields lower returns when the marginal utility of consumption is relatively high (owing to a relatively low level of consumption). The agent willingly holds the stock of this asset in equilibrium only if such risk is appropriately compensated by a "premium," given by an expected return higher than the risk-free rate r^0.

1.4.1 Empirical implications of the CCAPM

In order to derive testable implications from the model, we consider a CRRA utility function,

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma},$$

where $\gamma > 0$ is the coefficient of relative risk aversion. In this case, (1.60) becomes

$$1 = E_t\left[(1 + r^j_{t+1})\frac{1}{1+\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \qquad \text{for} \quad j = 1, \ldots, n. \tag{1.65}$$

Moreover, let us assume that the rate of growth of consumption and the rates of return of the n assets have a lognormal joint conditional distribution.[22] Taking logs of (1.65) (with the usual approximation $\log(1+\rho) \simeq \rho$), we get

$$0 = -\rho + \log E_t\left[(1 + r^j_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right],$$

and by the property mentioned in the preceding footnote we obtain

$$\log E_t\left[(1 + r^j_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = E_t(r^j_{t+1} - \gamma\,\Delta \log c_{t+1}) + \frac{1}{2}\Sigma_j, \tag{1.66}$$

where

$$\Sigma_j = E\left\{\left[(r^j_{t+1} - \gamma\,\Delta\log c_{t+1}) - E_t(r^j_{t+1} - \gamma\,\Delta\log c_{t+1})\right]^2\right\}.$$

Note that the unconditional expectation $E[\cdot]$ in the definition of Σ_j may be used under the hypothesis that the innovations in the joint process for returns

[22] In general, when two random variables x and y have a lognormal joint conditional probability distribution, then $\log E_t(x_{t+1} y_{t+1}) = E_t(\log(x_{t+1} y_{t+1})) + \frac{1}{2} var_t(\log(x_{t+1} y_{t+1}))$, where $var_t(\log(x_{t+1} y_{t+1})) = E_t\left\{\left[\log(x_{t+1} y_{t+1}) - E_t\left(\log(x_{t+1} y_{t+1})\right)\right]^2\right\}$.

and the consumption growth rate have constant variance (homoskedasticity). Finally, from (1.66) we can derive the expected return on the jth asset:

$$E_t\, r^j_{t+1} = \gamma E_t\,(\Delta \log c_{t+1}) + \rho - \frac{1}{2}\Sigma_j. \tag{1.67}$$

Several features of equation (1.67) can be noticed. In the first place, (1.67) can be immediately interpreted as the Euler equation that holds for each asset j. This interpretation can be seen more clearly if (1.67) is rewritten with the expected rate of change of consumption on the left-hand side. (See the solution to exercise 4 for the simpler case of only one safe asset.)

Second, the most important implication of (1.67) is the existence of a precise relationship between the forecastable component of (the growth rate of) consumption and asset returns. A high growth rate of consumption is associated with a high rate of return, so as to enhance saving, for a given intertemporal discount rate ρ. The degree of risk aversion γ is a measure of this effect, which is the same for all assets. At the empirical level, (1.67) suggests the following methodology to test the validity of the model.

1. A forecasting model for $\Delta \log c_{t+1}$ is specified; vector $\mathbf{x}_t$ contains only those variables, from the wider information set available to agents at time t, which are relevant for forecasting consumption growth.
2. The following system for $\Delta \log c_{t+1}$ and r^j_{t+1} is estimated:

$$\Delta \log c_{t+1} = \boldsymbol{\delta}'\,\mathbf{x}_t + u_{t+1},$$

$$r^j_{t+1} = \boldsymbol{\pi}'_j\mathbf{x}_t + k_j + v^j_{t+1}, \qquad j = 1, \ldots, n,$$

   where k_j is a constant and u and v are random errors uncorrelated with the elements of $\mathbf{x}$.
3. The following restrictions on the estimated parameters are tested:

$$\boldsymbol{\pi}_j = \gamma\, \boldsymbol{\delta}, \qquad j = 1, \ldots, n.$$

Finally, the value of Σ_j differs from one asset return to another, because of differences in the variability of return innovations and differences in the covariances between such innovations and the innovation of the consumption change. In fact, by the definition of Σ_j and the lognormality assumption, we have

$$\begin{aligned}\Sigma_j = {} & E\left[(r^j_{t+1} - E_t(r^j_{t+1}))^2\right] + \gamma^2 E\left[(\Delta \log c_{t+1} - E_t(\Delta \log c_{t+1}))^2\right] \\ & - 2\gamma E\left[(r^j_{t+1} - E_t(r^j_{t+1}))(\Delta \log c_{t+1} - E_t(\Delta \log c_{t+1}))\right] \\ \equiv {} & \sigma_j^2 + \gamma^2\sigma_c^2 - 2\gamma\sigma_{jc}. \end{aligned} \tag{1.68}$$

The expected return of an asset is negatively affected by the variance of the return itself and is positively affected by its covariance with the rate of change in consumption. Thus, using (1.67) and (1.68), we obtain, for any asset j,

$$E_t\, r^j_{t+1} = \gamma\, E_t(\,\Delta \log c_{t+1}) + \rho - \frac{\gamma^2 \sigma_c^2}{2} - \frac{\sigma_j^2}{2} + \gamma \sigma_{jc}. \tag{1.69}$$

This equation specializes the general result given in (1.62), and it is interesting to interpret each of the terms on its right-hand side. Faster expected consumption growth implies that the rate of return should be higher than the rate of time preference ρ, to an extent that depends on intertemporal substitutability as indexed by γ. "Precaution," also indexed by γ, implies that the rate of return consistent with optimal consumption choices is lower when consumption is more volatile (a higher σ_c^2). The variance of returns has a somewhat counterintuitive negative effect on the required rate or return: however, this term appears only because of Jensen's inequality, owing to the approximation that replaced $\log E_t(1 + r^j_{t+1})$ with $E_t r^j_{t+1}$ in equation (1.69). But it is again interesting and intuitive to see that the return's covariance with consumption growth implies a higher required rate of return. In fact, the consumer will be satisfied by a lower expected return if an asset yields more when consumption is decreasing and marginal utility is increasing; this asset provides a valuable hedge against declines in consumption to risk-averse consumers. Hence an asset with positive covariance between the own return innovations and the innovations in the rate of change of consumption is not attractive, unless (as must be the case in equilibrium) it offers a high expected return.

When there is also an asset with a safe return r^0, the model yields the following relationship between r^0 and the stochastic properties of $\Delta \log c_{t+1}$ (see again the solution of exercise 4):

$$r^0_{t+1} = \gamma\, E_t\,(\Delta \log c_{t+1}) + \rho - \frac{\gamma^2 \sigma_c^2}{2}. \tag{1.70}$$

(The return variance and covariance with consumption are both zero in this case.) Equations (1.69) and (1.70) show the determinants of the returns on different assets in equilibrium. All returns depend positively on the intertemporal rate of time preference ρ, since, for a given growth rate of consumption, a higher discount rate of future utility induces agents to borrow in order to finance current consumption: higher interest rates are then required to offset this incentive and leave the growth rate of consumption unchanged. Similarly, given ρ, a higher growth rate of consumption requires higher rates of return to offset the incentive to shift resources to the present, reducing the difference between the current and the future consumption levels. (The strength of this effect is inversely related to the intertemporal elasticity of substitution, given by $1/\gamma$ in the case of a CRRA utility function.) Finally, the uncertainty about the rate of change of consumption captured by σ_c^2 generates a precautionary saving motive, inducing the consumer to accumulate financial assets with a depressing effect on their rates of return. According to (1.69), the expected rate of return on the jth risky asset is also determined by σ_j^2 (as a result of the approximation) and by the covariance between rates of return and consumption changes. The strength of the latter effect is directly related to the degree of the consumer's risk aversion.

For any asset j, the "risk premium," i.e. the difference between the expected return $E_t\, r_{t+1}^j$ and the safe return r_{t+1}^0, is

$$E_t\, r_{t+1}^j - r_{t+1}^0 = -\frac{\sigma_j^2}{2} + \gamma\sigma_{jc}. \tag{1.71}$$

An important strand of literature, originated by Mehra and Prescott (1985), has tested this implication of the model. Many studies have shown that the observed premium on stocks (amounting to around 6% per year in the USA), given the observed covariance σ_{jc}, can be explained by (1.71) only by values of γ too large to yield a plausible description of consumers' attitudes towards risk. Moreover, when the observed values of $\Delta \log c$ and σ_c^2 are plugged into (1.70), with plausible values for ρ and γ, the resulting safe rate of return is much higher than the observed rate. Only the (implausible) assumption of a negative ρ could make equation (1.70) consistent with the data.

These difficulties in the model's empirical implementation are known as the *equity premium puzzle* and the *risk-free rate puzzle*, respectively, and have motivated various extensions of the basic model. For example, a more general specification of the consumer's preferences may yield a measure of risk aversion that is independent of the intertemporal elasticity of substitution. It is therefore possible that consumers at the same time display a strong aversion toward risk, which is consistent with (1.71), and a high propensity to intertemporally substitute consumption, which solves the *risk-free rate puzzle*.

A different way of making the above model more flexible, recently put forward by Campbell and Cochrane (1999), relaxes the hypothesis of intertemporal separability of utility. The next section develops a simple version of their model.

1.4.2 Extension: the habit formation hypothesis

As a general hypothesis on preferences, we now assume that what provides utility to the consumer in each period is not the whole level of consumption by itself, but only the amount of consumption in excess of a "habit" level. An individual's habit level changes over time, depending on the individual's own past consumption, or on the history of aggregate consumption.

In each period t, the consumer's utility function is now

$$u(c_t, x_t) = \frac{(c_t - x_t)^{1-\gamma}}{1-\gamma} \equiv \frac{(z_t\, c_t)^{1-\gamma} - 1}{1-\gamma},$$

where $z_t \equiv (c_t - x_t)/c_t$ is the *surplus consumption* ratio, and x_t (with $c_t > x_t$) is the level of *habit*. The evolution of x over time is here determined by aggregate (per capita) consumption and is not affected by the consumption choices of the individual consumer. Then, marginal utility is simply

$$u_c(c_t, x_t) = (c_t - x_t)^{-\gamma} \equiv (z_t\, c_t)^{-\gamma}.$$

The first-order conditions of the problem—see equation (1.65)—now have the following form:

$$1 = E_t\left[(1 + r^j_{t+1})\frac{1}{1+\rho}\left(\frac{z_{t+1}}{z_t}\right)^{-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right], \qquad \text{for} \quad j = 1, \dots, n. \quad (1.72)$$

The evolution over time of habit and aggregate consumption, denoted by $\bar{c}$, are modeled as

$$\Delta \log z_{t+1} = \phi \varepsilon_{t+1}, \quad (1.73)$$

$$\Delta \log \bar{c}_{t+1} = g + \varepsilon_{t+1}. \quad (1.74)$$

Aggregate consumption grows at the constant average rate g, with innovations $\varepsilon \sim N(0, \sigma_c^2)$. Such innovations affect the consumption habit,[23] with the parameter ϕ capturing the sensitivity of z to ε. Under the maintained hypothesis of lognormal joint distribution of asset returns and the consumption growth rate (and using the fact that, with identical individuals, in equilibrium $c = \bar{c}$), taking logarithms of (1.72), we get

$$0 = -\rho + E_t r^j_{t+1} - \gamma E_t(\Delta \log z_{t+1}) - \gamma E_t(\Delta \log c_{t+1})$$
$$+ \tfrac{1}{2}\,\mathrm{var}_t(r^j_{t+1} - \gamma \Delta \log z_{t+1} - \gamma \Delta \log c_{t+1}).$$

Using the stochastic processes specified in (1.73) and (1.74), we finally obtain the risk premium on asset j and the risk-free rate of return:

$$E_t r^j_{t+1} - r^0_{t+1} = -\frac{\sigma_j^2}{2} + \gamma\,(1+\phi)\sigma_{jc}, \quad (1.75)$$

$$r^0_{t+1} = \gamma\, g + \rho - \frac{\gamma^2(1+\phi)^2\,\sigma_c^2}{2}. \quad (1.76)$$

Comparing (1.75) and (1.76) with the analogous equations (1.71) and (1.70), we note that the magnitude of ϕ has a twofold effect on returns. On the one

[23] The assumed stochastic process for the logarithm of s satisfies the condition $c > x$ ($s > 0$): consumption is never below habit.

hand, a high sensitivity of habit to innovations in c enhances the precautionary motive for saving, determining a stronger incentive to asset accumulation and consequently a decrease in returns, as already shown by the last term in (1.70).[24] On the other hand, a high ϕ magnifies the effect of the covariance between risky returns and consumption (σ_{jc}) on the premium required to hold risky assets in equilibrium.

Therefore, the introduction of habit formation can (at least partly) solve the two problems raised by empirical tests of the basic version of the CCAPM: for given values of other parameters, a sufficiently large value of ϕ can bring the risk-free rate implied by the model closer to the lower level observed on the markets, at the same time yielding a relatively high risk premium.

Appendix A1: Dynamic Programming

This appendix outlines the dynamic programming methods widely used in the macroeconomic literature and in particular in consumption theory. We deal first with the representative agent's intertemporal choice under certainty on future income flows; the extension to the case of uncertainty follows.

A1.1 Certainty

Let's go back to the basic model of Section 1.1, assuming that future labor incomes are known to the consumer and that the safe asset has a constant return. The maximization problem then becomes

$$\max_{c_{t+i}} \left[U_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho} \right)^i u(c_{t+i}) \right],$$

subject to the accumulation constraint (for all $i \geq 0$),

$$A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i}.$$

Under certainty, we can write the constraint using the following definition of total wealth, including the stock of financial assets A and human capital H: $W_t = (1+r)(A_t + H_t)$. W_t measures the stock of total wealth at the end of period t but before consumption c_t occurs, whereas A_t and H_t measure financial and human wealth at the beginning of the period. In terms of total wealth W, the accumulation constraint for

[24] A constant ϕ is assumed here for simplicity. Campbell and Cochrane (1999) assume that ϕ decreases with s: the variability of consumption has a stronger effect on returns when the level of consumption is closer to habit.

period t becomes

$$
\begin{aligned}
W_{t+1} &= (1+r)\left[A_{t+1} + \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} y_{t+1+i}\right] \\
&= (1+r)\left[(1+r)A_t + y_t - c_t + \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} y_{t+1+i}\right] \\
&= (1+r)\left[(1+r)(A_t + H_t) - c_t\right] \\
&= (1+r)(W_t - c_t).
\end{aligned}
$$

The evolution over time of total wealth is then (for all $i \geq 0$)

$$W_{t+i+1} = (1+r)(W_{t+i} - c_{t+i}).$$

Formally, W_{t+i} is the *state variable*, giving, in each period $t+i$, the total amount of resources available to the consumer; and c_{t+i} is the *control variable*, whose level, optimally chosen by the utility-maximizing consumer, affects the amount of resources available in the next period, $t+i+1$. The intertemporal separability of the objective function and the accumulation constraints allow us to use dynamic programming methods to solve the above problem, which can be decomposed into a sequence of two-period optimization problems. To clarify matters, suppose that the consumer's horizon ends in period T, and impose a non-negativity constraint on final wealth: $W_{T+1} \geq 0$. Now consider the optimization problem at the beginning of the final period T, given the stock of total wealth W_T. We maximize $u(c_T)$ with respect to c_T, subject to the constraints $W_{T+1} = (1+r)(W_T - c_T)$ and $W_{T+1} \geq 0$. The solution yields the optimal level of consumption in period T as a function of wealth: $c_T = c_T(W_T)$. Also, the maximum value of utility in period $T(V)$ depends, through the optimal consumption choice, on wealth. The resulting *value function* $V_T(W_T)$ summarizes the solution of the problem for the final period T.

Now consider the consumer's problem in the previous period, $T-1$, for a given value of W_{T-1}. Formally, the problem is

$$\max_{c_{T-1}}\left(u(c_{T-1}) + \frac{1}{1+\rho}V_T(W_T)\right),$$

subject to the constraint $W_T = (1+r)(W_{T-1} - c_{T-1})$. As in the case above, the problem's solution has the following form: $c_{T-1} = c_{T-1}(W_{T-1})$, with an associated maximized value of utility (now over periods $T-1$ and T) given by $V_{T-1}(W_{T-1})$. The same procedure can be applied to earlier periods recursively (*backward recursion*). In general, the problem can be written in terms of the *Bellman equation*:

$$V_t(W_t) = \max_{c_t}\left(u(c_t) + \frac{1}{1+\rho}V_{t+1}(W_{t+1})\right), \tag{1.A1}$$

subject to $W_{t+1} = (1+r)(W_t - c_t)$. Substituting for W_{t+1} into the objective function and differentiating with respect to c_t, we get the following first-order condition:

$$u'(c_t) = \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}). \tag{1.A2}$$

Using the Bellman equation at time t and differentiating with respect to W_t, we obtain $V'_{t+1}(W_{t+1})$:

$$
\begin{aligned}
V'_t(W_t) &= u'(c_t)\frac{\partial c_t}{\partial W_t} + \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}) - \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1})\frac{\partial c_t}{\partial W_t} \\
&= \left(u'(c_t) - \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1})\right)\frac{\partial c_t}{\partial W_t} + \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}) \\
&= \frac{1+r}{1+\rho}V'_{t+1}(W_{t+1}),
\end{aligned}
$$

where we use the fact that the term in square brackets in the second line equals zero by (1.A2). Finally, using again the first-order condition, we find

$$
V'_t(W_t) = u'(c_t). \tag{1.A3}
$$

The effect on utility V_t of an increase in wealth W_t is equal to the marginal utility from immediately consuming the additional wealth. Along the optimal consumption path, the agent is indifferent between immediate consumption and saving. (The term in square brackets is zero.) The additional wealth can then be consumed in any period with the same effect on utility, measured by $u'(c_t)$ in (1.A2): this is an application of the *envelope theorem*.

Inserting condition (1.A3) in period $t + 1$ into (1.A2), we get the Euler equation,

$$
u'(c_t) = \frac{1+r}{1+\rho}u'(c_{t+1}),
$$

which is the solution of the problem (here under certainty) already discussed in Section 1.1.

The recursive structure of the problem and the backward solution procedure provide the optimal consumption path with the property of *time consistency*. Maximization of (1.A1) at time t takes into account $V_{t+1}(W_{t+1})$, which is the optimal solution of the same problem at time $t+1$, obtained considering also $V_{t+2}(W_{t+2})$, and so forth. As time goes on, then, consumption proceeds optimally along the path originally chosen at time t. (This time consistency property of the solution is known as Bellman's *optimality principle*.)

Under regularity conditions, the iteration of Bellman equation starting from a (bounded and continuous) value function $V_T(\cdot)$ leads to a limit function $V(\cdot)$, which is unique and invariant over time. Such a function $V = \lim_{j\to\infty} V_{T-j}$ solves the consumer's problem over an infinite horizon. In this case also, the function that gives the agent's consumption $c(W)$ is invariant over time. Operationally, if the problem involves (1) a quadratic utility function, or (2) a logarithmic utility function and Cobb–Douglas constraints, it can be solved by first guessing a functional form for $V(\cdot)$ and then checking that such function satisfies Bellman equation (1.A1).

As an example, consider the case of the CRRA utility function[25]

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

The Bellman equation is

$$V(W_t) = \max_{c_t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} V(W_{t+1}) \right),$$

subject to the constraint $W_{t+1} = (1+r)(W_t - c_t)$. Let us assume (to be proved later on) that the value function has the same functional form as utility:

$$V(W_t) = K \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{1.A4}$$

with K being a positive constant to be determined. Using (1.A4), we can write the Bellman equation as

$$K \frac{W_t^{1-\gamma}}{1-\gamma} = \max_{c_t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} K \frac{W_{t+1}^{1-\gamma}}{1-\gamma} \right). \tag{1.A5}$$

From this equation, using the constraint and differentiating with respect to c_t, we get the first-order condition

$$c_t^{-\gamma} = \frac{1+r}{1+\rho} K \left[(1+r)(W_t - c_t)\right]^{-\gamma},$$

and solving for c_t we obtain the consumption function $c_t(W_t)$:

$$c_t = \frac{1}{1 + (1+r)^{\frac{1-\gamma}{\gamma}} (1+\rho)^{-\frac{1}{\gamma}} K^{\frac{1}{\gamma}}} \, W_t, \tag{1.A6}$$

where K is still to be determined.

To complete the solution, we combine the Bellman equation (1.A5) with the consumption function (1.A6) and define

$$B \equiv (1+r)^{1-\gamma/\gamma} (1+\rho)^{-1/\gamma}$$

to simplify notation. We can then write

$$K \frac{W_t^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \left[\frac{W_t}{1 + BK^{\frac{1}{\gamma}}} \right]^{1-\gamma} + \frac{1}{1+\rho} \frac{K}{1-\gamma} \left[(1+r) \left(\frac{BK^{\frac{1}{\gamma}}}{1 + BK^{\frac{1}{\gamma}}} \right) W_t \right]^{1-\gamma}, \tag{1.A7}$$

where the terms in square brackets are, respectively, C_t and W_{t+1}. The value of K that satisfies (1.A7) is found by equating the coefficient of $W_t^{1-\gamma}$ on the two sides of the

[25] The following solution procedure can be applied also when $\gamma > 1$ and the utility function is unbounded. To guarantee this result an additional condition will be imposed below; see Stokey, Lucas, and Prescott (1989) for further details.

equation, noting that $(1+r)^{1-\gamma}(1+\rho)^{-1} \equiv B^{\gamma}$, and solving for K :

$$K = \left(\frac{1}{1-B}\right)^{\gamma}. \tag{1.A8}$$

Under the condition that $B < 1$, the complete solution of the problem is

$$V(W_t) = \left(\frac{1}{1-(1+r)^{\frac{1-\gamma}{\gamma}}(1+\rho)^{-\frac{1}{\gamma}}}\right)^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma},$$

$$c(W_t) = \left[1-(1+r)^{\frac{1-\gamma}{\gamma}}(1+\rho)^{-\frac{1}{\gamma}}\right] W_t.$$

A1.2 Uncertainty

The recursive structure of the problem ensures that, even under uncertainty, the solution procedure illustrated above is still appropriate. The consumer's objective function to be maximized now becomes

$$U_t = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^{i} u(c_{t+1}),$$

subject to the usual budget constraint (1.2). Now we assume that future labor incomes y_{t+i} $(i > 0)$ are uncertain at time t, whereas the interest rate r is known and constant. The state variable at time t is the consumer's certain amount of resources at the end of period t: $(1+r)A_t + y_t$. The value function is then $V_t((1+r)A_t + y_t)$, where subscript t means that the value of available resources depends on the information set at time t. Under uncertainty, the Bellman equation becomes

$$V_t[(1+r)A_t + y_t] = \max_{c_t} \left\{ u(c_t) + \frac{1}{1+\rho} E_t V_{t+1}[(1+r)A_{t+1} + y_{t+1}] \right\}. \tag{1.A9}$$

The value of $V_{t+1}(\cdot)$ is stochastic, since future income are uncertain, and enters (1.A9) as an expected value.

Differentiating with respect to c_t and using the budget constraint, we get the following first-order condition:

$$u'(c_t) = \frac{1+r}{1+\rho} E_t V'_{t+1}[(1+r)A_{t+1} + y_{t+1}].$$

As in the certainty case, by applying the envelope theorem and using the condition obtained above, we have

$$V'_t(\cdot) = \frac{1+r}{1+\rho} E_t V'_{t+1}(\cdot)$$
$$= u'(c_t).$$

Combining the last two equations, we finally get the stochastic Euler equation

$$u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}),$$

already derived in Section 1.1 as the first-order condition of the problem.

Review Exercises

Exercise 5 *Using the basic version of the rational expectations/permanent income model (with quadratic utility and $r = \rho$), assume that labor income is generated by the following stochastic process:*

$$y_{t+1} = \bar{y} + \varepsilon_{t+1} - \delta\varepsilon_t, \qquad \delta > 0,$$

where $\bar{y}$ is the mean value of income and ε is an innovation with $E_t\varepsilon_{t+1} = 0$.

(a) Discuss the impact of an increase of $\bar{y}$ ($\Delta\bar{y} > 0$) on the agent's permanent income, consumption and saving.
(b) Now suppose that, in period $t+1$ only, a positive innovation in income occurs: $\varepsilon_{t+1} > 0$. In all past periods income has been equal to its mean level: $y_{t-i} = \bar{y}$ for $i = 0, \ldots, \infty$. Find the change in consumption between t and $t+1$ (Δc_{t+1}) as a function of ε_{t+1}, providing the economic intuition for your result.
(c) With reference to question (b), discuss what happens to saving in periods $t+1$ and $t+2$.

Exercise 6 *Suppose the consumer has the following utility function:*

$$U_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^i u(c_{t+i}, S_{t+i}),$$

where S_{t+i} is the stock of durable goods at the beginning of period $t+i$. There is no uncertainty. The constraints on the optimal consumption choice are:

$$S_{t+i+1} = (1-\delta)S_{t+i} + d_{t+i},$$
$$A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i} - p_{t+i}\, d_{t+i},$$

where δ is the physical depreciation rate of durable goods, d is the expenditure on durable goods, p is the price of durable goods relative to non-durables, and S_t and A_t are given. Note that the durable goods purchased at time $t+i$ start to provide utility to the consumer only from the following period, as part of the stock at the beginning of period $t+i+1$ (S_{t+i+1}). Set up the consumer's utility maximization problem and obtain the first-order conditions, providing the economic intuition for your result.

Exercise 7 *The representative consumer maximizes the following intertemporal utility function:*

$$U_t = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^i u(c_{t+i}, c_{t+i-1}),$$

where

$$u(c_{t+i}, c_{t+i-1}) = (c_{t+i} - \gamma c_{t+i-1}) - \frac{b}{2}(c_{t+i} - \gamma c_{t+i-1})^2, \qquad \gamma > 0.$$

In each period $t+i$, utility depends not only on current consumption, but also on consumption in the preceding period, $t+i-1$. All other assumptions made in the chapter are maintained (in particular $\rho = r$).

(a) Give an interpretation of the above utility function in terms of habit formation.
(b) From the first-order condition of the maximization problem, derive the dynamic equation for c_{t+1}, and check that this formulation of utility violates the property of orthogonality of Δc_{t+1} with respect to variables dated t.

Exercise 8 *Suppose that labor income y is generated by the following stochastic process:*

$$y_t = \lambda y_{t-1} + x_{t-1} + \varepsilon_{1t},$$

$$x_t = \varepsilon_{2t},$$

where x_t $(= \varepsilon_{2t})$ does not depend on its own past values $(x_{t-1}, x_{t-2}, \ldots)$ and $E(\varepsilon_{1t} \cdot \varepsilon_{2t}) = 0$. x_{t-1} is the only additional variable (realized at time $t-1$) which affects income in period t besides past income y_{t-1}. Moreover, suppose that the information set used by agents to calculate their permanent income y_t^P is $I_{t-1} = \{y_{t-1}, x_{t-1}\}$, whereas the information set used by the econometrician to estimate the agents' permanent income is $\Omega_{t-1} = \{y_{t-1}\}$. Therefore, the additional information in x_{t-1} is used by agents in forecasting income but is ignored by the econometrician.

(a) Using equation (1.7) in the text (lagged one period), find the changes in permanent income computed by the agents (Δy_t^P) and by the econometrician ($\Delta \tilde{y}_t^P$), considering the different information set used (I_{t-1} or Ω_{t-1}).
(b) Compare the variance of Δy_t^P e $\Delta \tilde{y}_t^P$, and show that the variability of permanent income according to agents' forecast is lower than the variability obtained by the econometrician with limited information. What does this imply for the interpretation of the excess smoothness phenomenon?

Exercise 9 *Consider the consumption choice of an individual who lives for two periods only, with consumption c_1 and c_2 and incomes y_1 and y_2. Suppose that the utility function in each period is*

$$u(c) = \begin{cases} ac - (b/2)c^2 & \text{for} \quad c < a/b; \\ (a^2/2b) & \text{for} \quad c \geq a/b. \end{cases}$$

(Even though the above utility function is quadratic, we rule out the possibility that a higher consumption level reduces utility.)

(a) *Plot marginal utility as a function of consumption.*
(b) *Suppose that* $r = \rho = 0$, $y_1 = a/b$, *and* y_2 *is uncertain:*

$$y_2 = \begin{cases} a/b + \sigma, & \text{with probability } 0.5; \\ a/b - \sigma, & \text{with probability } 0.5. \end{cases}$$

Write the first-order condition relating c_1 *to* c_2 *(random variable) if the consumer maximizes expected utility. Find the optimal consumption when* $\sigma = 0$, *and discuss the effect of a higher* σ *on* c_1.

Further Reading

The consumption theory based on the intertemporal smoothing of optimal consumption paths builds on the work of Friedman (1957) and Modigliani and Brumberg (1954). A critical assessment of the life-cycle theory of consumption (not explicitly mentioned in this chapter) is provided by Modigliani (1986). Abel (1990, part 1), Blanchard and Fischer (1989, para. 6.2), Hall (1989), and Romer (2001, ch. 7) present consumption theory at a technical level similar to ours. Thorough overviews of the theoretical and empirical literature on consumption can be found in Deaton (1992) and, more recently, in Browning and Lusardi (1997) and Attanasio (1999), with a particular focus on the evidence from microeconometric studies. When confronting theory and microeconomic data, it is of course very important (and far from straightforward) to account for heterogeneous objective functions across individuals or households. In particular, empirical work has found that theoretical implications are typically not rejected when the marginal utility function is allowed to depend flexibly on the number of children in the household, on the household head's age, and on other observable characteristics. Information may also be heterogeneous: the information set of individual agents need not be more refined than the econometrician's (Pischke, 1995), and survey measures of expectations formed on its basis can be used to test theoretical implications (Jappelli and Pistaferri, 2000).

The seminal paper by Hall (1978) provides the formal framework for much later work on consumption, including the present chapter. Flavin (1981) tests the empirical implications of Hall's model, and finds evidence of *excess sensitivity* of consumption to expected income. Campbell (1987) and Campbell and Deaton (1989) derive theoretical implication for saving behavior and address the problem of *excess smoothness* of consumption to income innovations. Campbell and Deaton (1989) and Flavin (1993) also provide the joint interpretation of "excess sensitivity" and "excess smoothness" outlined in Section 1.2.

Empirical tests of the role of liquidity constraints, also with a cross-country perspective, are provided by Jappelli and Pagano (1989, 1994), Campbell and Mankiw (1989, 1991) and Attanasio (1995, 1999). Blanchard and Mankiw (1988) stress the importance of the precautionary saving motive, and Caballero (1990) solves analytically the optimization problem with precautionary saving assuming an exponential utility function, as in Section 1.3. Weil (1993) solves the same problem in the case of constant but unrelated intertemporal elasticity of substitution and relative risk aversion parameters. A precautionary saving motive arises also in the models of Deaton (1991) and Carroll (1992), where liquidity constraints force consumption to closely track current income and induce agents to accumulate a limited stock of financial assets to support consumption in the event of sharp reductions in income (*buffer-stock saving*). Carroll (1997, 2001) argues that the empirical evidence on consumers' behavior can be well explained by incorporating in the life-cycle model both a precautionary saving motive and a moderate degree of impatience. Sizeable responses of consumption to predictable income changes are also generated by models of dynamic inconsistent preferences arising from hyperbolic discounting of future utility; Angeletos *et al.* (2001) and Frederick, Loewenstein, and O'Donoghue (2002) provide surveys of this strand of literature.

The general setup of the CCAPM used in Section 1.4 is analyzed in detail by Campbell, Lo, and MacKinley (1997, ch. 8) and Cochrane (2001). The model's empirical implications with a CRRA utility function and a lognormal distribution of returns and consumption are derived by Hansen and Singleton (1983) and extended by, among others, Campbell (1996). Campbell, Lo, and MacKinley (1997) also provide a complete survey of the empirical literature. Campbell (1999) has documented the international relevance of the *equity premium* and the *risk-free rate puzzles*, originally formulated by Mehra and Prescott (1985) and Weil (1989). Aiyagari (1993), Kocherlakota (1996), and Cochrane (2001, ch. 21) survey the theoretical and empirical literature on this topic. Costantinides, Donaldson, and Mehra (2002) provide an explanation of those puzzles by combining a life-cycle perspective and borrowing constraints. Campbell and Cochrane (1999) develop the CCAPM with *habit formation* behavior outlined in Section 1.4 and test it on US data. An exhaustive survey of the theory and the empirical evidence on consumption, asset returns, and macroeconomic fluctuations is found in Campbell (1999).

Dynamic programming methods with applications to economics can be found in Dixit (1990), Sargent (1987, ch. 1) and Stokey, Lucas, and Prescott (1989), at an increasing level of difficulty and analytical rigor.

References

Abel, A. (1990) "Consumption and Investment," in B. Friedman and F. Hahn (ed.), *Handbook of Monetary Economics*, Amsterdam: North-Holland.

Aiyagari, S. R. (1993) "Explaining Financial Market Facts: the Importance of Incomplete Markets and Transaction Costs," *Federal Reserve Bank of Minneapolis Quarterly Review*, 17, 17–31.

—— (1994) "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109, 659–684.

Angeletos, G.-M., D. Laibson, A. Repetto, J. Tobacman, and S. Winberg (2001) “The Hyperbolic Consumption Model: Calibration, Simulation and Empirical Evaluation,” *Journal of Economic Perspectives*, 15(3), 47–68.

Attanasio, O. P. (1995) “The Intertemporal Allocation of Consumption: Theory and Evidence,” *Carnegie–Rochester Conference Series on Public Policy*, 42, 39–89.

—— (1999) “Consumption,” in J. B. Taylor and M. Woodford (ed.), *Handbook of Macroeconomics*, vol. 1B, Amsterdam: North-Holland, 741–812.

Blanchard, O. J. and S. Fischer (1989) *Lectures on Macroeconomics*, Cambridge, Mass.: MIT Press.

—— and N. G. Mankiw (1988) “Consumption: Beyond Certainty Equivalence,” *American Economic Review (Papers and Proceedings)*, 78, 173–177.

Browning, M. and A. Lusardi (1997) “Household Saving: Micro Theories and Micro Facts,” *Journal of Economic Literature*, 34, 1797–1855.

Caballero, R. J. (1990) “Consumption Puzzles and Precautionary Savings,” *Journal of Monetary Economics*, 25, 113–136.

Campbell, J. Y. (1987) “Does Saving Anticipate Labour Income? An Alternative Test of the Permanent Income Hypothesis,” *Econometrica*, 55, 1249–1273.

—— (1996) “Understanding Risk and Return,” *Journal of Political Economy*, 104, 298–345.

—— (1999) “Asset Prices, Consumption and the Business Cycle,” in J. B. Taylor and M. Woodford (ed.), *Handbook of Macroeconomics*, vol. 1C, Amsterdam: North-Holland.

—— and J. H. Cochrane (1999) “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior,” *Journal of Political Economy*, 2, 205–251.

—— and A. Deaton (1989) “Why is Consumption So Smooth?” *Review of Economic Studies*, 56, 357–374.

—— and N. G. Mankiw (1989) “Consumption, Income and Interest Rates: Reinterpreting the Time-Series Evidence,” *NBER Macroeconomics Annual*, 4, 185–216.

—— —— (1991) “The Response of Consumption to Income: a Cross-Country Investigation,” *European Economic Review*, 35, 715–721.

—— A. W. Lo, and A. C. MacKinley (1997) *The Econometrics of Financial Markets*, Princeton: Princeton University Press.

Carroll, C. D. (1992) “The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence,” *Brookings Papers on Economic Activity*, 2, 61–156.

—— (1997) “Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis,” *Quarterly Journal of Economics* , 102, 1–55.

—— (2001) “A Theory of the Consumption Function, With and Without Liquidity Constraints,” *Journal of Economic Perspectives*, 15 (3), 23–45.

Cochrane, J. H. (2001) *Asset Pricing*, Princeton: Princeton University Press.

Costantinides G. M., J. B. Donaldson, and R. Mehra (2002) “Junior Can’t Borrow: A New Perspective on the Equity Premium Puzzle,” *Quarterly Journal of Economics*, 117, 269–298.

Deaton, A. (1991) “Saving and Liquidity Constraints,” *Econometrica*, 59, 1221–1248.

—— (1992) *Understanding Consumption*, Oxford: Oxford University Press.

Dixit, A. K. (1990) *Optimization in Economic Theory*, 2nd edn, Oxford: Oxford University Press.

Flavin, M. (1981) "The Adjustment of Consumption to Changing Expectations about Future Income," *Journal of Political Economy* , 89, 974–1009.

—— (1993) "The Excess Smoothness of Consumption: Identification and Interpretation," *Review of Economic Studies*, 60, 651–666.

Frederick S., G. Loewenstein, and T. O'Donoghue (2002) "Time Discounting and Time Preference: A Critical Review," *Journal of Economic Literature*, 40, 351–401.

Friedman, M. (1957) *A Theory of the Consumption Function*, Princeton: Princeton University Press.

Hall, R. E. (1978) "Stochastic Implications of the Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy*, 96, 971–987.

—— (1989) "Consumption," in R. Barro (ed.), *Handbook of Modern Business Cycle Theory*, Oxford: Basil Blackwell.

Hansen, L. P. and K. J. Singleton (1983) "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns," *Journal of Political Economy*, 91, 249–265.

Jappelli, T. and M. Pagano (1989) "Consumption and Capital Market Imperfections: An International Comparison," *American Economic Review*, 79, 1099–1105.

—— —— (1994) "Saving, Growth and Liquidity Constraints," *Quarterly Journal of Economics*, 108, 83–109.

—— and L. Pistaferri (2000), "Using Subjective Income Expectations to Test for Excess Sensitivity of Consumption to Predicted Income Growth," *European Economic Review* 44, 337–358.

Kocherlakota, N. R. (1996) "The Equity Premium: It's Still a Puzzle," *Journal of Economic Literature*, 34(1), 42–71.

Mehra, R. and E. C. Prescott (1985) "The Equity Premium: A Puzzle," *Journal of Monetary Economics* , 15(2), 145–161.

Modigliani, F. (1986) "Life Cycle, Individual Thrift, and the Wealth of Nations," *American Economic Review*, 76, 297–313.

—— and R. Brumberg (1954) "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in K. K. Kurihara (ed.), *Post-Keynesian Economics*, New Brunswick, NJ: Rutgers University Press.

Pischke, J.-S. (1995) "Individual Income, Incomplete Information, and Aggregate Consumption," *Econometrica*, 63, 805–840.

Romer, D. (2001) *Advanced Macroeconomics*, 2nd edn, New York: McGraw-Hill.

Sargent, T. J. (1987) *Dynamic Macroeconomic Theory*, Cambridge, Mass.: Harvard University Press.

Stokey, N., R. J. Lucas, and E. C. Prescott (1989) *Recursive Methods in Economic Dynamics*, Cambridge, Mass.: Harvard University Press.

Weil, P. (1989) "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics*, 24, 401–421.

—— (1993) "Precautionary Savings and the Permanent Income Hypothesis," *Review of Economic Studies*, 60, 367–383.