Chapter 8

Theory of Fixed Investment and Employment Dynamics

Investment is expenditures by firms on equipment and structures. Business (fixed) investment is commonly held to be an important determinant of an economy's long-run growth. While the significance of short-term changes in business investment is less widely recognized, the importance of such changes for the business cycle has been known to economists since the beginning of the last century. For example, many believe that the US record expansion in the 90s had been driven, at least in part, by strong investment in computers and related equipment.

For individual plants, investment is simply the expenditure required to adjust its stock of capital. Capital includes all equipment and structures the plant uses. The plant combines capital with other inputs, such as labor and energy, to produce goods or services. When an extraction company acquires diesel engines, it is investing in equipment. When an automobile manufacturer builds a new warehouse, it is investing in structures.

Since investment spending raises future capital and thus the quantity of goods and services that may be produced in the future, plants will tend to adjust their investment levels in response to forecasted changes in the market's demand for their own output. Changes in productivity will also tend to increase investment. For example, if the efficiency with which inputs may be combined to produce output increases, the firm may be able to sell more of its product, since it can offer it at a more attractive price. The firm may then expand and more workers may be hired. These workers will need equipment, and, as a result, investment will rise.

8.1 The Value of the Firm

We denote by V_t^* the value of the firm. If he stock market works efficiently, V_t^* should correspond to the expected discounted present value (DPV) of all future profits π_{t+j}^* , $j = 1, ..., \infty$ from period t onward. But at what rate should firm discount cash flows? Recall the Lucas's tree model, where the consumer trades a risk free bond and and a risky asset (the trees). If we interpret the risky asset as shares of our firm, the first order conditions of the consumer are

$$\begin{aligned} u'(c_t^*) &= \mathbf{E}_t\left[\beta(1+r_t)u'\left(c_{t+1}^*\right)\right] \\ V_t^* u'(c_t^*) &= \mathbf{E}_t\left[\beta\left(V_{t+1}^* + \pi_{t+1}^*\right)u'\left(c_{t+1}^*\right)\right], \end{aligned}$$

where $SDF_{t+1} = \frac{\beta u'\left(c_{t+1}^*\right)}{u'(c_t^*)}$ is the stochastic discount factor, and $\frac{V_{t+1}^*+\pi_{t+1}^*}{V_t^*}$ represents the rate of return from holding the firm. When $r_t = r$ for all t, the law of iterated expectations implies $\frac{1}{1+r} = \mathbf{E}_t\left[SDF_{t+j+1}\right]$ for all $j \geq 0$.

If we focus on equilibria with no-bubbles, unraveling the second conditions and using the law of iterated expectations, yields

$$V_t^* = \mathbf{E}_t\left[\sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j}^*)}{u'(c_t^*)} \pi_{t+j}^*\right].$$

Rearranged, this condition implies:

$$V_t^* = \mathbf{E}_t\left[\sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^j \pi_{t+j}^*\right].$$

Cash flow should hence be discounted by at the real risk free rate.[1]

The above evaluation of the firm also applies to the case where agents were heterogeneous, facing idiosyncratic shocks, as long as the asset market is *complete,* that is, the existing securities would span all states of nature. In a world with incomplete asset markets things can get very complicated. The bottom line is that prices are not longer uniquely determinated. There are however cases where we can still use the above definition to evaluate the value of a firm or of a project. This is the situation where the profits of the firms are zero in all states of nature not spanned by the securities. In this case, the existing securities will still span all states of nature which are relevant for the firm.

[1] It is easy to see that when r_t changes with time, we have $\frac{1}{1+r_{t+n}} = \mathbf{E}_t\left[SDF_{t+n+1}\right]$ and

$$V_t^* = \mathbf{E}_t\left[\sum_{j=1}^{\infty} \frac{\pi_{t+j}^*}{\Pi_n^j\left(1+r_{t+n}\right)}\right].$$

8.2 Problem of the Representative Firm

Consider the problem of an infinitely lived firm that in every period chooses how much to invest, i.e. how much to add to its stock of productive capital.

Since we aim at studying the behavior of *aggregate* investment, we assume that the firm owns capital. We could have assumed that the firm hires capital from consumers or from a firm who produces it. This requires a second agent and the distinction between internal and external adjustment costs. At the aggregate level, internal and external adjustment costs have equivalent implications (see Sala-i-Martin, 2005).[2]

This firm has hence a dynamic choice. Because it takes time to manufacture, deliver, and install new capital goods, investment expenditures today do not immediately raise the level of a plant's capital. So investment involves a dynamic trade-off: by investing today, the firm foregoes current profits to spend resources in order to increase its stock of future capital and raise future production and future profits.

Clearly, every period the firm will also choose labor input n_t, but we abstract from this static choice, and assume the firm has already maximized with respect to n_t when is called to make the optimal investment decision.[3]

The law of capital is the usual one

$$k_{t+1} = (1 - \delta) k_t + i_t.$$

In each period, the firm produces with the stock of capital k_t, which hence partially depreciates (δ), it then makes the (gross) investment decision i_t that will determine the capital stock in place for next period's production k_{t+1}.

[2] We will assumed that the interest rate and prices are exogenously given to the firm. This seems a reasonable assumption if we want to think about the behavior of individual firms. *Aggregate* investment however, both depends upon and affects the interest rate of the economy. That is, in the aggregate the interest rate is endogenous. One can endogenize the real interest rate by embodying the individual neoclassical firms we will describe in a general equilibrium model where there are also consumers and the interest rate is determined by the equalization between the desired investment by firms and desired savings by households. This will give raise to the Neoclassical model of economic growth we saw in Chapter 1.

[3] Recalling the analysis of Chapter 1, an alternative possibility would be to assume that n_t is supplied inelastically by individuals. Hence if we normalize the aggregate labor supply to one, market clearing always requires $n_t = 1$ for all levels of capital. In other terms, for each k the market wage fully adjust - since there is a vertical labor supply - to

$$w(k) = F_n(k, 1).$$

.

For notational reasons, in what follow we abuse a bit in notation and denote by V_t^* the values of the firm *including* period t dividends π_t^*. Given k_0, and a sequence of profit functions $\{\Pi_t\}_{t=0}^{\infty}$ the sequential problem the firm is facing in period $t = 0$ can be formulated as follows

$$\begin{aligned} V_0^* &= \max_{\{i_t, k_{t+1}\}} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \Pi_t \left(k_t, k_{t+1}, i_t \right) \right] \\ & \quad s.t. \\ k_{t+1} &= (1-\delta) k_t + i_t \\ k_{t+1} &\geq 0, \ , \text{ for all } t; \ k_0 \text{ given.} \end{aligned}$$

8.3 The Neoclassical Theory of Investment

We did not specify the cash flow (or profit) functions Π_t yet. The traditional neoclassical theory of investment uses a very simple formulation of the problem: Denote the production function of the firm by $f(k_t)$, the level of technology of the firm at time t by z_t, and the price of a unit of investment good or the unit price of capital goods as p_t;[4] we have

$$\Pi_t \left(k_t, k_{t+1}, i_t \right) = \Pi \left(k_t, k_{t+1}, i_t; p_t, z_t \right) = z_t f \left(k_t \right) - p_t i_t,$$

hence the optimal profit in each period is $\pi_t^* = z_t f \left(k_t^* \right) - p_t i_t^*$.

Consider now the deterministic version of the model. Given k_0 and the sequential of prices and shocks $\{p_t, z_t\}_{t=0}^{\infty}$ problem specializes to

$$\begin{aligned} V_0^* &= \max_{\{i_t, k_{t+1}\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[z_t f \left(k_t \right) - p_t i_t \right] \\ & \quad s.t. \\ k_{t+1} &= (1-\delta) k_t + i_t; \\ k_{t+1} &\geq 0, \text{ for all } t; \ k_0 \text{ given.} \end{aligned} \tag{8.1}$$

We now derive the Euler equation for the problem. We are hence looking for a feasible deviation from the optimal *interior* program $\{i_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, where interiority simple requires $k_{t+1}^* > 0$ for all t. In the spirit of the Euler variational approach, the perturbation is aimed at changing k_{t+1}^* (and i_t^*, i_{t+1}^*), while keeping unchanged all k_s^* for $s \neq t+1$, in particular both k_t^* and k_{t+2}^*.

[4]Notice that this is a price relative to the price of the final good, which is normalized to one as usual.

Let ε any real number (positive and negative) in an open neighborhood O of zero. Such neighborhood is obviously constructed to maintain feasibility. For each ε, the perturbed plan $\{\hat{\imath}_t^{\varepsilon}, \hat{k}_{t+1}^{\varepsilon}\}_{t=0}^{\infty}$ is constructed from $\{i_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ as follows: $\hat{k}_{t+1}^{\varepsilon} = k_{t+1}^* + \varepsilon$, and $\hat{k}_s^{\varepsilon} = k_s^*$ for $s \neq t+1$. It is easy to check from the law of motions that such perturbation to the optimal plan is achieved by modifying the investment plan as follows: $\hat{\imath}_t^{\varepsilon} = i_t^* + \varepsilon$ and $\hat{\imath}_{t+1}^{\varepsilon} = i_{t+1}^* - (1-\delta)\,\varepsilon$ and $\hat{\imath}_s^{\varepsilon} = i_s^*$ for $s \neq t, t+1$. If we denote by $\hat{V}_0(\varepsilon)$ the value associated to the perturbed plan for each $\varepsilon \in O$, the optimality of the original plan implies $\hat{V}_0(\varepsilon) \leq V_0^*$ for all $\varepsilon \in O$, and $\hat{V}_0(0) = V_0^*$. Stated in other terms, $\varepsilon = 0$ is the optimal solution to

$$\max_{\varepsilon \in O} \hat{V}_0(\varepsilon).$$

The necessary first order condition of optimality is hence $\hat{V}_0'(0) = 0$. Since k_s^* are untouched, both for $s \leq t$ and $s \geq t+2$ the derivative with respect to ε of all terms are zero but period t and $t+1$ returns. We hence have:[5]

$$\begin{aligned}
&(1+r)^t \hat{V}_0'(\varepsilon) = \\
= \;& \frac{d}{d\varepsilon}\left[z_t f(k_t^*) - p_t(i_t^* + \varepsilon) + \left(\frac{1}{1+r}\right)\left(z_{t+1} f\left(k_{t+1}^* + \varepsilon\right) - p_{t+1}\left(i_{t+1}^* - (1-\delta)\,\varepsilon\right)\right)\right].
\end{aligned}$$

The FOC condition $\hat{V}_0'(0) = 0$ hence delivers the following Euler equation:

$$p_t = \frac{1}{1+r}\left[z_{t+1} f'\left(k_{t+1}^*\right) + p_{t+1}(1-\delta)\right]. \tag{8.2}$$

This condition determines the optimal level of next period capital (hence the optimal investment decision, given k_t). It states that the marginal cost of a unit of investment equals the marginal benefit. The marginal cost is the price of capital p_t. The marginal benefit accrues next period, so it's discounted by $(1+r)$. The next period marginal benefit is composed by two terms: *(i)* the increase in production associated to the higher stock of capital $z_{t+1} f'(k_{t+1})$ and *(ii)* the market value of one unit of capital after production. Equation (8.2) is sometimes called the Jorgenson's optimal investment condition, from the name of the Harvard's economist who advanced this theory.

If we assume that $p_t = 1$, i.e. that the price of capital is constant at one, the Euler equation (8.2) becomes

$$r + \delta = z_{t+1} f'\left(k_{t+1}^*\right),$$

[5] In fact, it is as if we solved the local problem:

$$\max_{\varepsilon} \; z_t f(k_t^*) - p_t(i_t^* + \varepsilon) + \left(\frac{1}{1+r}\right)\left(z_{t+1} f\left(k_{t+1}^* + \varepsilon\right) - p_{t+1}\left(i_{t+1}^* - (1-\delta)\,\varepsilon\right)\right)$$

which is the usual condition one gets in the static model of the firm. When the firm rents capital (instead of purchasing it), $r+\delta$ represents the *user cost of capital.* In words, the previous equality says that firms invest (purchase capital) up to the point where marginal product of capital (net of depreciation δ) equals the return on alternative assets, the real interest rate.[6]

Exercise 54 *Set $p_t = p$ and $z_t = z$ for all t, and state the problem in recursive form, carefully specifying what are the controls and the states of the dynamic problem, and what are the law of motion for the states. Now compute the FOC with respect to k_{t+1}, and the envelope condition. By rearranging terms, you should get the Euler equation for this problem. Perform the same exercise assuming that productivity $z \in \{z_h, z_l\}$ follows a Markov chain with transition matrix $\begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$, while the price of capital is fixed at p.*

When f is concave $f'(k_{t+1})$ is a decreasing function, hence invertible. If we denote by h the inverse function of $f'(k_{t+1})$, condition (8.2) can be written as

$$k_{t+1}^{*} = h\left(\frac{p_t\,(1+r) - p_{t+1}\,(1-\delta)}{z_{t+1}}\right), \tag{8.3}$$

with $h(\cdot)$ a decreasing function. We hence have that the optimal level of next period capital (hence investment as k_t is given) is increasing in z_{t+1} and p_{t+1}, while it decreases in p_t, r, and δ.

Exercise 55 *Explain intuitively, in economic terms, why according to (8.3) investment is increasing in z_{t+1} and p_{t+1}, and decreasing in p_t, r, and δ.*

Exercise 56 *(i) Assume $p_t = \bar{p}$ and $z_t = \bar{z}$ for all t and derive the steady state level of capital and investment. (ii) Now state the transversality condition problem (8.1) and verify that the optimal path converging to the steady state satisfies the transversality condition.*

8.4 Convex Adjustment Costs: The q-Theory of Investment

The neoclassical model has a couple of drawbacks. First, consider the case where firms are heterogeneous, say they have different marginal product of capital. As long as all

[6] See also Abel and Blanchard (1983).

firms face the same interest rate and prices for investment goods, all investment in the economy will take place in the firm with the highest marginal product of capital. This is clearly a counterfactual implication of the model.[7]

Another potential source of unrealistic behavior is that current investment is independent of future marginal products of capital. Recall that the equalization of marginal product to interest rate yields the desired level of capital and that investment is then equal to the difference between the existing and the desired capital stocks. Hence, investment is a function of both the existing capital stock and the real interest rate, but is independent of *future* marginal products of capital. If firms know that the marginal product will increase at some point T in the future, their best strategy is not to do anything until that moment arrives at which point they will discretely increase the amount of capital to the new desired level. In other words, because firms can discretely get the desired capital level at every moment in time, it does not pay them to plan for the future since future changes in business conditions will be absorbed by future discrete changes in capital stocks. Economists tend to think that future changes in business conditions have effects on today investment decisions. To get rid of this result we need a theory that makes firms willing to smooth investment over time. One way of introducing such a willingness to smooth investment is to make it costly to invest or disinvest large amounts of capital at once. This is the idea behind the concept of *adjustment costs*.

We will now imagine that firms behave exactly as just described, except that there are some installation or adjustment costs. By that we mean that, like in the neoclassical model, p units output can be transformed into one unit of capital. This capital (which we will call "uninstalled capital") is not useful until it is installed. Unlike the neoclassical model, firms have to pay some installation or adjustment costs in order to install or uninstall capital. These adjustment costs are foregone resources within the firm: for example computers can be purchased at price p but they cannot be used until they have been properly installed. The installation process requires that some of the workers stop working in the production line for some of the time. Hence, by installing the new computer the firm foregoes some resources, which we call *internal adjustment costs.*

The (cash-flow or profit) function Π_t will be modified as follows:

[7] A similar type of situation arises when we consider the world economy where all countries face the same "world real interest rate" but different countries have different levels of capital (so the poorest country has the highest marginal product of capital). If capital is free to move across borders, the neoclassical model of investment predicts that *all* the investment in the world will take place in the poorest country.

$$\Pi\left(k_t, k_{t+1}, i_t; p_t, z_t\right) = z_t f\left(k_t\right) - p_t\left(i_t + \phi\left(i_t, k_t\right)\right).$$

The only difference with respect to the Neoclassical model is hence the introduction of adjustment costs via the function $\phi\left(i_t, k_t\right)$. Since ϕ is multiplied by p_t, it is defined in physical units, just like its arguments i and k. We will assume that for all k, $\phi_1'\left(\cdot, k\right), \phi_{11}''\left(\cdot, k\right) > 0$, with $\phi\left(0, k\right) = \phi_1\left(0, k\right) = 0$. Intuitively, ϕ should decrease with k as congestion costs tend to be more proportional to the ratio i/k rather than the absolute value of i.

The problem of the firm hence specializes to

$$\begin{aligned}
V_0^* &= \max_{\{i_t, k_{t+1}\}} \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\left[z_t f\left(k_t\right) - p_t\left(i_t + \phi\left(i_t, k_t\right)\right)\right] \\
& \quad s.t. \\
k_{t+1} &= \left(1-\delta\right) k_t + i_t; \qquad \left(\frac{\lambda_t}{\left(1+r\right)^t}\right) \\
k_{t+1} &\geq 0, \text{ for all } t; \; k_0 \text{ given.}
\end{aligned}$$

We now compute the optimal program using (somehow heuristically) the standard Kuhn-Tucker theory. The first order conditions are

$$i_t \; : \; p_t\left(1 + \phi_1'\left(i_t^*, k_t^*\right)\right) = \lambda_t^* \tag{8.4}$$

$$k_{t+1} \; : \; \frac{1}{1+r}\left[z_{t+1} f'\left(k_{t+1}^*\right) - p_{t+1}\phi_2'\left(i_{t+1}^*, k_{t+1}^*\right) + \left(1-\delta\right)\lambda_{t+1}^*\right] = \lambda_t^*. \tag{8.5}$$

The (costate) variable λ_t^* represents the *present (i.e., at period t) value* of the marginal contribution of capital to profits (the period t shadow price). Condition (8.4) hence just equates costs (to the left hand side) to returns (to the left hand side) of a marginal unit of investment. Now define $q_t = \frac{\lambda_t^*}{p_t}$ the same marginal value normalized by the market price of capital. From (8.4) we obtain

$$1 + \phi_1'\left(i_t^*, k_t^*\right) \equiv g\left(i_t^*, k_t^*\right) = q_t.$$

Since $\phi_{11}'' > 0$, given k_t^* both ϕ_1' and g are increasing functions in of i_t^*. Denoting by h the inverse function of g conditional on k, we obtain

$$i_t^* = h\left(q_t, k_t^*\right),$$

with $h\left(1, k\right) = 0$ (since $\phi_1\left(0, k\right) = 0$). This is a very important relationship. First, since k_t^* is given, it means that the *only* thing that firms need to observe in order to make investment decisions in period t is q_t, the shadow price of investment. In other words, q_t is a "sufficient statistic" for fixed investment.[8] Second, the firm will make positive investment if and only if $q_t > 1$. The intuition is simple: When $q > 1$ (hence $\lambda^* > p$) capital is worth more inside the firm than in the economy at large; it is hence a good idea to increase the capital stock installed in the firm. Symmetrically, when $q < 1$ it is a good idea to reduce capital. Third, how much investment changes with q depends on the slope of h, hence on the slope of g. Since $g' = \phi''$ such slope is determined by the *convexity of the adjustment cost function* ϕ.

We now analyze the analogy of (8.5) to the Jorgenson's optimal investment condition (8.2). If we assume again $p_t = 1$, we obtain

$$\frac{1}{1+r}\left[z_{t+1}f'\left(k_{t+1}^*\right) - \phi_2'\left(i_{t+1}^*, k_{t+1}^*\right) + (1-\delta)\, q_{t+1}\right] = q_t. \tag{8.6}$$

The analogy with the standard condition is quite transparent. The introduction of adjustment costs into the neoclassical model creates a discrepancy between the market cost of investment p and the internal value λ of *installed* capital. The *shadow* (as opposed to market) marginal cost of capital is hence q_t (as opposed to one). The (discounted and deflated by $p_{t+1} = 1$) marginal benefit is again composed by two terms, where the first term now includes the additional component $-\phi_2'\left(i_{t+1}^*, k_{t+1}^*\right)$ (≥ 0) since capital also reduces adjustment costs. The internal value of one unit of capital in the next period, after production, is $(1-\delta)\, q_{t+1}$.[9]

Let's fix again p_t at one and z_t at a constant level z. The steady state level of investment will obviously be $i^{ss} = \delta k^{ss} > 0$, which implies that $q^{ss} = \lambda^{ss} > 1$. Since $\lambda^{ss} = q^{ss}$ is uniquely defined by (8.4), according to

$$\lambda^{ss} = g\left(\delta k^{ss}, k^{ss}\right),$$

[8] If h is linearly homogeneous in k, we have

$$\frac{i}{k} = \hat{h}(q) = h(q, 1),$$

that is, q is a sufficient statistic for the investment rate.

[9] When we allow time variations in prices, this condition becomes:

$$\frac{p_{t+1}}{p_t}\frac{1}{1+r}\left[\frac{z_{t+1}f'\left(k_{t+1}^*\right)}{p_{t+1}} - \phi_2'\left(i_{t+1}^*, k_{t+1}^*\right) + (1-\delta)\, q_{t+1}\right] = q_t,$$

which has the same interpretation as the above condition, with the additional deflationary term $\frac{p_{t+1}}{p_t}$ that keeps the benchmark value for q at one.

we only need to compute the steady state level of capital. From (8.5) (or (8.6)), we get

$$\frac{1}{1+r}\left[zf'\left(k^{ss}\right)-\phi_2'\left(\delta k^{ss},k^{ss}\right)\right]=\frac{r+\delta}{1+r}g\left(\delta k^{ss},k^{ss}\right).$$

Whenever the left hand side decreases with k^{ss} while the right hand side increases with k^{ss} (with at least one of the two conditions holds strictly) and some limiting conditions for $k=0$ and $k\to\infty$ are satisfied, there exists *one and only one* solution to this equation.

8.4.1 Marginal versus Average Tobin's q

Hayashi (1982) showed that under four key conditions the shadow price q_t (the *marginal* q) corresponds to the ratio between the value of the firm V_t^* divided by the replacement cost of capital $p_t k_t$. The latter ratio is often called Tobin's *average* q. Such conditions are: *(i)* the production function and the adjustment cost function are homogeneous of degree one, i.e. they display constant returns to scale; *(ii)* the capital goods are all homogeneous and identical; and *(iii)* the stock market is efficient, i.e. the stock market price of the firm equals the discounted present value of all future dividends; *(iv)* and the firms operates in a competitive environment, i.e. it takes as given prices and wages;

The intuition for such conditions is as follows. The first condition is a necessary condition as otherwise we obviously have a discrepancy between the returns of capita ad different firm's dimensions. The homogeneity of capital goods is also required since the marginal q refers to the last, newly installed (or about to be installed), capital, while the average also considers the value of all previously installed capital. If there is a large discrepancy between the two, because of the price of old equipment (say computers for example) decrease sharply, the average q tends to be well below the marginal q. Finally, the inefficiency of the stock market is clearly important. Recall that the marginal q considers the marginal value of future profits. The average q does not considers the average value of profits directly, it computes the ratio $\frac{V_t^*}{p_t k_t}$. Consider now phenomena that bring the value of the firm away from the fundamentals (such as some type of bubbles for example), then the two values (average and marginal q) can match only by chance. Finally, the competitive assumption is obviously important to maintain the linearities induces by the homogeneity of degree one. If a larger firm could grasp more profits by a stronger market power, this should be included while computing the marginal return to new installed capital.

To see it more formally, assume that $zf(k)=zk$ and that $\phi_2'(i,k)=0$ then from (8.6) we obtain:

$$\frac{1}{1+r}\left[z_{t+1}+(1-\delta)\,q_{t+1}\right]=q_t, \tag{8.7}$$

which implies

$$q_t = \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \frac{z_{t+s}}{1+r}. \tag{8.8}$$

That is, as long as productivity and adjustment costs do not depend on k, the shadow value of the the marginal unit of installed capital does not depend on the size of the firm k, i.e., the size of the firm is irrelevant at the margin.

Hayashi shows that the same idea holds true more generally, whenever the size of the firm is irrelevant at the margin.

Exercise 57 *Let $f(k) = F(k,1)$, assume inelastic labor supply normalized to one. Assume that both $F(k,n)$ and $\phi(i,k)$ are linearly homogeneous in their arguments, and amend the profit function to $\Pi = zf(k_t) - w_t - p_t(i_t + \phi(i_t,k_t))$. Show that under the stated assumptions the average and marginal q are equivalent.[Hint: Notice that $w_t = zF_n(k_t,1)$. Moreover, since F is homogeneous of degree one, we have $zf'(k)k + w = zF_k(k,1) + zF_n(k_t,1) = zf(k)$.]*

8.5 Linear Adjustment Costs and Employment Dynamics under Uncertainty

Following Bagliano and Bertola (2004), we now specify our model to address the issue of employment dynamics. The state variable will now be the stock of workers in a firm n_t, an we will completely abstract from capital. The evolution of the employment in a firm can be stated as follows

$$n_{t+1} = (1-\delta) n_t + h_t,$$

where δ indicates an exogenous separation rate, say due to worker quitting the firm for better jobs. The variable h_t indicates the *gross* employment variation in period t.

The cash flow function Π_t will be

$$\Pi(n_t, n_{t+1}, h_t; w_t, z_t) = z_t f(n_t) - w_t n_t - \phi(h_t),$$

where

$$\phi(h) = \begin{cases} hH & \text{if } h > 0 \\ 0 & \text{if } h = 0 \\ -hF & \text{if } h < 0. \end{cases}$$

The function $\phi(\cdot)$ represents the cost of hiring and firing, or *turnover,* which depends on gross employment variation in period t, but not on voluntary quits. In a stochastic environment the problem of the firm hence becomes:

$$\begin{aligned} W_0^* &= \max_{\{n_{t+1},h_t\}} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t [z_t f(n_t) - w_t n_t - \phi(h_t)] \right] \\ & \quad s.t. \\ n_{t+1} &= (1-\delta) n_t + h_t; \\ n_{t+1} &\geq 0, \text{ for all } t; \; n_0 \text{ given.} \end{aligned}$$

The analogy with the theory of investment is transparent. Notice however that w_t is not the 'price' of labor, it is a flow payment, rather than a stock payment such as p_t in the previous model. In fact, since it multiplies the stock of labor n_t, the wage is analogous to the *user or rental cost* of capital $(r+\delta)$ in the previous model.

If we denote by λ_t^* the shadow value of labor, defined as the marginal increase in discounted cash flow of the firm if it hires an additional unit of labor. When a firm increases the employment level by hiring an infinitesimal amount of labor while keeping the hiring and firing decisions unchanged, from the envelope conditions we have (have a look at how we derived (8.7) from (8.8)):

$$\lambda_t^* = \mathbf{E}_t \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \frac{[z_{t+s} f'(n_{t+s}) - w_{t+s}]}{1+r},$$

which can be written similar to our Euler equation as

$$\lambda_t^* = \frac{z_t f'(n_t) - w_t}{1+r} + \frac{1-\delta}{1+r} \mathbf{E}_t \left[\lambda_{t+1}^* \right]. \tag{8.9}$$

Give the structure of turnover costs the optimality condition for h gives $\lambda_t^* \in \partial \phi(h_t^*)$,[10] which implies

$$-F \leq \lambda_t^* \leq H$$

with $\lambda_t^* = H$ if $h_t > 0$ and $\lambda_t^* = -F$ if $h_t < 0$.[11] The idea is simple: the firms is actively changing the employment stock (on top of the exogenous separation rate δ) only when the marginal return compensates the cost, and when it his doing it, h will change so that to exactly equate turnover costs to returns.

[10] Recall from that the symbol '$\partial \phi(h)$' represents the subgradient of the function ϕ at point h.

[11] Note that $h = 0$ is the only point of non-differentiability, and $\partial \phi(0) = [-F, H]$.

Now we assume that $w_t = \bar{w}$ and that z_t follows a two states Markov chain with transition matrix Π. Denote by z_h, and z_l respectively the state with high and low productivity respectively. We are looking for a steady state distribution such that $\lambda_h^* = H$ while $\lambda_l^* = -F$. If for $i \in \{h, l\}$ we denote by $\mathbf{E}[\lambda'; i] = \pi_{ih}\lambda_h^* + \pi_{il}\lambda_l^*$ From (8.9) we get the stationary levels of n from

$$\lambda_h^* = H = \frac{z_h f'(n_h) - \bar{w}}{1+r} + \frac{1-\delta}{1+r}\mathbf{E}[\lambda'; h] \tag{8.10}$$

$$\lambda_l^* = -F = \frac{z_l f'(n_l) - \bar{w}}{1+r} + \frac{1-\delta}{1+r}\mathbf{E}[\lambda'; l]. \tag{8.11}$$

and the hiring/firing decision h can take four values, which solve

$$n_j = (1-\delta)\, n_i + h_{ij} \qquad \text{for } i, j \in \{h, l\}.$$

Exercise 58 *Consider the case with $\delta > 0$. What is the value for h_{ij} when $i = j$? Find conditions on the transition matrix Π so that a two-states steady state distribution exists for the model we just presented.*

Exercise 59 *Assume that the parameters of the model are such that a stationary steady state exists, set $\delta = 0$, and derive the values for $f'(n_h)$ and $f'(n_l)$ as functions of the parameters of the model: $H, F, \bar{w}, z, r$ and the entries of the matrix Π, which is assumed to take the form: $\begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$. Now perform the same calculations assuming that $F = H = 0$. Comment in economic terms your results in the two cases.*

8.5.1 The Option Value Effect

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