EFFICIENT TRADE MECHANISM WITH DISCRETE VALUES

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PRELIMINARY

ABSTRACT. We provide a treatment of bilateral trade with discrete values. We start with a one seller one buyer setup and characterize the necessary and sufficient condition on the distributions of valuations for the existence of a budget balanced efficient mechanism. The set of distributions satisfying the necessary and sufficient conditions is nonempty. Furthermore, we introduce a VCG' mechanism, which allocates the object efficiently and makes the buyer pay the smallest valuation he could report and induce the trade and gives the seller the payment equivalent to the highest value he could report and induce the trade, given fixed reports of the two agents. Efficient, budget balanced, incentive compatible and individually rational trade is possible if and only if the broker running the VCG' mechanism makes a non negative expected profit.

Our characterization extends to the setup with multiple buyers. For a class of distributions we show the following property. There exists a threshold such that if the number of buyers is above the threshold budget balanced efficient mechanism exists. On the other hand, we provide an example in which there exists a budget balanced efficient mechanism when there is one buyer, but not when there are two identical buyers. Therefore the existence of a budget balanced efficient mechanism is, so to say, not monotonic in the number of buyers.

Finally, we provide a treatment of ex ante maximization of gains from trade under budget balance. Among other results we fully characterize optimal mechanisms for the two-type case.

1. Introduction

Bilateral trade has attracted a lot of attention at least since the seminal work of Myerson and Satterthwaite (1983). Among several results they obtain impossibility of ex post efficient budget balanced bilateral trade left the deepest imprint in the literature, and beyond. This was a sharp contrast to the famous Coase' theorem, Coase (1960), claiming that even if the initial allocation is inefficient the market will take care of efficiency by themselves. Many economic policies had to be rethought, new possibilities considered.

Myerson and Satterthwaite (1983) pointed out themselves that the assumption of valuations being distributed over a compact interval with everywhere positive density is crucial for the nonexistence result. In particular, they provided an example with two types for the buyer and two types for the seller for which an ex post efficient budget balanced mechanism exists. A decade later Makowski and Mezzetti (1993) showed that efficient bilateral trade can obtain as soon as there is more than one potential buyer. Several other circumstances, too numerous for us to present here, have been explored in which the efficient trade does obtain.

Somewhat surprisingly not much, beyond the example of Myerson and Satterthwaite (1983), has been explored in the direction of bilateral trade with discrete values. To the best of our knowledge the only other characterization was provided by Matsuo (1989), who characterized the conditions under which the efficiency obtains for distributions with two values. In our opinion exploration of bilateral trade under the discrete types is crucial since results obtained under the hypothesis of a continuum of valuations may not hold here. Furthermore, for much of numerical computation in empirical work discrete valuation spaces are the natural environment to operate in.

We provide a general treatment of bilateral trade with discrete values. We provide the necessary and sufficient condition on distributions for which the efficient bilateral trade obtains; in the case of discrete distributions the set of distributions satisfying the condition is nonempty. The condition has several interpretations. The most intuitive one is that there exists an ex post efficient budget balanced mechanisms if and only if the VCG' mechanism runs a budget surplus. Two notions need special attention here. We say that a mechanism runs a budget surplus when the ex ante expected payment to the seller is no larger than the ex ante expected payment from the buyer. More important still, is the notion of the VCG' mechanism. VCG' mechanism is an ex post efficient mechanism that requires from the buyer to pay the smallest value he could have reported and still induced the trade, given the seller's report, while it pays to the seller the highest value he could have reported and induced the trade, given the buyer's report. VCG' mechanism extracts the most money from the buyer and gives the least to the seller among all efficient mechanisms. Furthermore, the VCG' as defined above can be embedded into a framework with any, suitably measurable, valuation spaces in $\mathbb{R}$, not necessarily discrete. In the case of interval valuations VCG' boils down to an expected externality mechanism, as for example Krishna and Perry (1998) define the VCG mechanism. In the case of discrete valuation spaces, however, VCG' mechanism can run a strictly higher budget surplus, and never lower. Therefore when considering general environments one ought to think of the appropriate efficient mechanism as the one in which the buyer pays the lowest valuation he could have and induce the trade, whereas the seller obtains the highest valuation he could have and induce the trade; and not as an expected externality mechanism.

Next we show how the analysis of efficiency extends to the case of more than one buyer. With discrete types it is easy to see that there exists, given the distributions, a number of buyer I^* such that if the number of buyer I is at least as large as I^* the budget balanced efficient mechanism exists; under some mild assumptions on the distributions. This result is a discrete version of the result obtained for interval valuation spaces by Makowski and Mezzetti (1993). Discrete setup, though, allows for additional insights. We provide an example in which with one buyer there exists a budget balanced efficient mechanism, while with two identical buyers it does not, rendering the existence of a budget balanced efficient mechanism non-monotonic in the number of buyers.

Finally, we provide a treatment of ex ante maximization of gains from trade under ex post budget balance. We show that the problem of ex ante maximization of gains from trade over ex post budget balanced mechanisms can be replaced with the problem of maximization of gains from trade over mechanisms that run a budget surplus. Even more, only tight mechanisms need to be considered, where by tight we mean the mechanism in which all the buyer's downward and seller's upward adjacent IC constraints, as well as buyer's lowest type's and the seller's highest type's IR constraints are binding. The alternative problem is easier to solve, which we demonstrate by fully characterizing the solution for the two-type case. Furthermore, we provide the necessary and sufficient condition for the payoff equivalence. More precisely, let x^* be some allocation rule appearing in an interim incentive compatible, individually rational and gains from trade maximizing mechanism. We provide a condition under which all the incentive compatible, individually rational gains from trade maximizing mechanisms with the allocation rule x^* are payoff equivalent.

Our contribution is two-fold. First, we provide several missing links in the theory of bilateral trade, like the necessary and sufficient condition for a budget balanced efficient mechanism to exist in the discrete values environment. Second, we develop methodological tools to deal with bilateral trade when values are not distributed over an interval, and therefore, incentive compatible mechanisms are not payoff equivalent.

1.1. **Related Literature.** The literature on bilateral trade is too vast for us to cover. We briefly outline the papers closely related to ours. The anchor to our paper is Myerson and Satterthwaite (1983). In spirit closest to ours are the papers by Krishna and Perry (1998) and Williams (1999). They show that a particular VCG mechanism extracts the largest budget surplus among all ex post efficient implementable mechanisms when the valuation spaces are connected. While their analysis thrives on payoff equivalence of incentive compatible mechanisms, our has to work around it.

The result that budget balanced efficient mechanisms exist, under the assumption of connected of valuation spaces, as soon as the number of buyers is large enough is due to Makowski and Mezzetti (1993).

The only paper, to the best of our knowledge, that deals with discrete valuation spaces in bilateral trade is Matsuo (1989). The paper characterizes necessary and sufficient conditions for budget balanced efficient trade in the 2 type case.

1.2. **Organization of the Paper.** In Section 2 we describe the framework. In Section 3 we introduce the VCG' mechanism and show it runs the highest budget surplus among all ex post efficient implementable mechanisms. Section 4 contains the characterization of necessary and sufficient conditions for existence of budget balanced efficient mechanisms. In Section 5 we extend the analysis to the environment in which a single seller is bargaining with multiple buyers. Section 6 provides the treatment of ex ante maximization of gains from trade. Section 7 concludes.

2. Framework

Consider a *bilateral trade* problem in which agent b wants to buy a *good* that agent s owns. The possible *values* of the buyer and seller for the good are $B_1 < B_2 < \ldots < B_m$, and $S_1 < S_2 < \ldots < S_n$, respectively ($m, n \geq 2$); let V_b and V_s denote the two sets of values. Suppose that $V_b \cap V_s = \emptyset$. [1] We assume the standard independent private value information structure. The valuations of the two agents are independently distributed random variables, with *probability mass functions* $p_s : V_s \to (0, 1]$ and $p_b : V_b \to (0, 1]$. Note that we require the support of p_b (p_s) be all of V_b (V_s).[2] At the time of the bargaining, the buyer (seller) knows his (her) own value, and believes that the seller's (buyer's) value is a random variable distributed according to the mass function p_s (p_b). Let $F_b : \mathbb{R} \to [0, 1]$ be the *cumulative distribution* corresponding to p_b, $F_b(v) = \sum_{B_i \leq v} p_b(B_i)$; F_s is defined analogously.

We assume that the agents are risk neutral, and have utility functions separable for money and the good. The mechanism designer needs to specify a game in order to determine the probability of trade and the payment for each of the two agents. In a *direct bargaining mechanism* the agents simultaneously report their values, and the outcome is determined by the triple of functions (x, t_b, t_s),

$$x : V_b \times V_s \to [0, 1],$$
$$t_b, t_s : V_b \times V_s \to \mathbb{R}.$$

When the reported valuations are (B_i, S_j), $x(B_i, S_j)$ is the *probability of trade* between the seller and the buyer, and $t_b(B_i, S_j)$ and $t_s(B_i, S_j)$ are the *monetary payments from* the buyer and respectively *to* the seller. Probability of trade functions are also called *allocations*.

For a mechanism (x, t_b, t_s), let

$$\overline{x}_b(B_i) = \sum_{j=1}^{n} p_s(S_j) x(B_i, S_j)$$

[1]Later in the paper we comment on how to deal with an environment in which this assumption is violated.

[2]The full support assumption is without loss of generality for our results.

be the *expected probability of trade* for a type B_i buyer and

$$\bar{t}_b(B_i) = \sum_{j=1}^{n} p_s(S_j) t_b(B_i, S_j)$$

be the *expected monetary transfer from* him; analogously, let $\overline{x}_s(S_j)$ be the expected probability of trade for a type j seller and $\bar{t}_s(S_j)$ be the monetary transfer *to* her.[3] The *expected gains from trade* for each agent type, given truthful reporting of the other agent, are given by

$$U_b(B_i) = \overline{x}_b(B_i) B_i - \bar{t}_b(B_i)$$

and

$$U_s(S_j) = \bar{t}_s(S_j) - \overline{x}_s(S_j) S_j.$$

A direct mechanism is (interim) *individually rational* if each agent receives non-negative expected gains from participating in the mechanism, conditional on each realization of his (her) value. A direct mechanism is (Bayesian) *incentive compatible* if honest reporting of the values forms a Bayesian Nash equilibrium of the induced bargaining game. By the *revelation principle*, the Bayesian Nash equilibrium of any bargaining game is outcome equivalent to the outcome of the truthful equilibrium in an incentive compatible direct mechanism, so the mechanism designer can restrict attention to incentive compatible direct mechanisms (Myerson (1979)). An allocation rule x is (interim) *implementable* if there exist t_b, t_s such that (x, t_b, b_s) is incentive compatible.

The *expected budget surplus* of (x, t_b, t_s) is

$$\Pi(x, t_b, t_s) \equiv \sum_{i=1}^{m} p_b(B_i) \bar{t}_b(B_i) - \sum_{j=1}^{n} p_s(S_j) \bar{t}_s(S_j).$$

A mechanism (x, t_b, t_s) is (ex post) *budget balanced* if $t_b(B_i, S_j) = t_s(B_i, S_j), \forall i \in \overline{1, m}, \forall j \in \overline{1, n}$. We use the shorthand (x, t) for the budget balanced mechanism (x, t, t).

Finally, the *ex post efficient allocation* x^* is defined by $x^*(B_i, S_j) = 1$ if $B_i > S_j$ and $x^*(B_i, S_j) = 0$ if $B_i < S_j$. (Recall that $V_b \cap V_s = \emptyset$.) A mechanism (x, t_b, t_s) is *ex post efficient* if $x = x^*$.

[3]For a budget balanced mechanism (x, t), $\bar{t}_b, \bar{t}_s$ are defined as the corresponding expected transfers for the mechanism (x, t, t).

3. The modified Vickrey-Clarke-Groves mechanism

One mechanism available to a designer interested in implementing the ex post efficient allocation is the *standard Vickrey-Clarke-Groves (VCG)* mechanism. The VCG mechanism is a direct revelation mechanism defined as follows.

Definition 1 (VCG mechanism $(x^{VCG}, t_b^{VCG}, t_s^{VCG})$)**.**

$$\begin{aligned} x^{VCG} &= x^* \\ t_b^{VCG}(B_i, S_j) &= \begin{cases} S_j & \text{if } B_i > S_j \\ 0 & \text{otherwise} \end{cases} \\ t_s^{VCG}(B_i, S_j) &= \begin{cases} B_i & \text{if } B_i > S_j \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Truthful behavior is a weakly dominant strategy in the game induced by the VCG mechanism. However, VCG is costly for the mechanism designer. For a pair of value reports (B_i, S_j), if $B_i > S_j$ the mechanism designer incurs an ex post deficit of $B_i - S_j$; otherwise he breaks even. Thus the VCG mechanism results in an expected deficit for any pair of buyer-seller value distributions.

The above described characteristics of the VCG mechanism were pointed out by Krishna and Perry (1998) and Williams (1999) for the case of interval type spaces. Furthermore, they showed that the VCG mechanism achieves the highest budget surplus among incentive compatible and individually rational mechanisms. In settings with discrete values, however, the mechanism designer can decrease the expected deficit (increase expected surplus) of VCG without renouncing weak dominant strategy incentive compatibility. Consider the following *modification of the Vickrey-Clarke-Groves mechanism (VCG')*. Fix a pair of value reports B_i, S_j. If $B_i > S_j$ trade occurs with probability 1, the buyer pays to the mechanism designer the smallest value in V_b that is larger than S_j, and the seller receives from the mechanism designer the highest value in V_s that is smaller than B_i. Otherwise, trade occurs with probability 0, and transfers are 0 for both agents.

Definition 2 (VCG' mechanism $(x^{VCG'}, t_b^{VCG'}, t_s^{VCG'})$).

$$\begin{aligned} x^{VCG'} &= x^* \\ t_b^{VCG'}(B_i, S_j) &= \begin{cases} min\{B_k : B_k > S_j\} & \text{if } B_i > S_j \\ 0 & \text{otherwise} \end{cases} \\ t_s^{VCG'}(B_i, S_j) &= \begin{cases} max\{S_k : B_i > S_k\} & \text{if } B_i > S_j \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

The VCG' mechanism is ex post efficient. The buyer receives the object when his reported value is larger than the seller's reported value, and in such instances he pays the lowest value he may have reported that would win him the object given the seller's report. The seller trades the object when her reported value is smaller than the buyer's reported value, and in such instances she is paid the highest value she may have reported that would lead to trade given the buyer's report. It follows that truthful behavior is a weakly dominant strategy in the game induced by VCG'. It should be noted that the transfers in a particular VCG' mechanism do depend on the distributions of buyer's and seller's valuations. Or to be more precise, on the support of the distribution. To choose the right VCG' the mechanism designer does not need to know the distribution of the valuations but he does have to know their support.

The VCG' mechanism is less costly (more profitable) than the VCG mechanism. For every pair of value reports, VCG' extracts at least as high payments from the buyer and provides at most as high payments to the seller as VCG. Unlike VCG, VCG' may result in an ex post surplus. Fix a pair of value reports $B_i > S_j$. Let $B_l = \min(V_b \cap [S_j, B_i])$, and $S_h = \max(V_s \cap [S_j, B_i])$. If $B_l > S_h$ (in particular, the inequality holds if $(V_b \cup V_s) \cap (S_j, B_i) = \emptyset$, in which case $B_l = B_i, S_h = S_j$) the mechanism designer realizes an ex post surplus of $B_l - S_h$.

Our first goal is to show that VCG' achieves the largest expected budget surplus among ex post efficient, incentive compatible and individually rational mechanisms. Then we proceed to characterize the pairs of value distributions for which the maximum expected budget surplus of VCG' is positive.

In VCG', buyer B_i receives the object from all sellers with values smaller than B_i, and pays B_k to all seller types in the interval (B_{k-1}, B_k) for $k \leq i$; under the disjoint values assumption, the probability that the seller's value is in (B_{k-1}, B_k) is $F_s(B_k) - F_s(B_{k-1})$. Hence, the expected payment from buyer B_i is[4] $\sum_{k=1}^{i}(F_s(B_k) - F_s(B_{k-1}))B_k$, and the expected payment from all buyer types is $\sum_{i=1}^{m} p_b(B_i)\sum_{k=1}^{i}(F_s(B_k)-F_s(B_{k-1}))B_k$. A similar expression obtains for the expected payment to all seller types. Therefore, the expected budget surplus in the VCG' mechanism is $\sum_{i=1}^{m} p_b(B_i)\sum_{k=1}^{i}(F_s(B_k) - F_s(B_{k-1}))B_k - \sum_{j=1}^{n} p_s(S_j)\sum_{k=j}^{n}(F_b(S_{k+1}) - F_b(S_k))S_k$ which can be rewritten as

$$\Pi^{VCG'} \equiv \sum_{i=1}^{m}(1 - F_b(B_{i-1}))(F_s(B_i) - F_s(B_{i-1}))B_i - \sum_{j=1}^{n} F_s(S_j)(F_b(S_{j+1}) - F_b(S_j))S_j.$$

Proposition 1. *The VCG' mechanism achieves the maximum expected budget surplus among ex post efficient, incentive compatible and individually rational mechanisms.*

Proof. Proofs are in the Appendix. □

Since in our environment the revenue equivalence principle does not apply expected budget surplus of a mechanism is not pinned down by incentive compatibility constraints and binding IR constraints. That is, generally there will exist efficient incentive compatible mechanisms which leave the lowest type seller and the highest type buyer with the expected utility zero but will achieve strictly lower budget surplus than VCG' does. An alternative interpretation of VCG' is the following. Suppose the mechanism designer delegates the implementation of the efficient allocation rule to a profit maximizing broker. Such a broker will implement it via VCG' or a mechanism that is interim equivalent to VCG'.

The main technical difference, and complication, between obtaining our result and the one in Krishna and Perry (1998) and Williams (1999) is that we can not resort to the revenue equivalence principle. Incentive constraints under a discrete state space namely leave plenty of room to specify the transfers. While incentive compatible transfers are not pinned down in general, the incentive compatible transfers of concern to us are. More precisely, for any implementable allocation rule, and transfers that implement it, there exist transfers which leave buyer's lowest type and seller's highest type expected utility

[4]The following formal definition is used routinely to simplify notation. Let B_0 and S_{n+1} be two reals satisfying $B_0 < \min(B_1, S_1)$ and $S_{n+1} > \max(B_n, S_n)$. For example, the term corresponding to the index $k = 1$ in the expression $\sum_{k=1}^{i}(F_s(B_k) - F_s(B_{k-1}))B_k$ equals $F_s(B_1)B_1$, and the term corresponding to the index $k = n$ in the expression $\sum_{k=j}^{n}(F_b(S_{k+1}) - F_b(S_k))S_k$ equals $(1 - F_b(S_n))S_n$.

unaltered and make all the buyer's downward adjacent and all the seller's upward IC constraints binding. Moreover, these transfers achieve the highest expected budget surplus given the expected utilities of the lowest buyer's and the highest seller's type. Therefore one can restrict attention to the mechanisms in which the relevant IC constraints are binding. The highest budget surplus is then achieved by extracting all the surplus from the lowest buyer's and the highest seller's type. It is easy to verify that VCG' does precisely that for the ex post efficient allocation rule.

4. Ex post efficient budget balanced mechanisms

The sign of $\Pi^{VCG'}$ provides a necessary and sufficient condition for an optimal budget balanced implementable mechanism to be ex post efficient. For the case $m = n = 2$ the corresponding condition was provided by Matsuo (1989).

Theorem 1. *There exists an ex post efficient budget balanced implementable mechanism if and only if* $\Pi^{VCG'} \geq 0$.

The following result is a consequence of Proposition 1 and Theorem 1.

Corollary 1. *There exists an ex post efficient budget balanced implementable mechanism if and only if the VCG' mechanism does not run an expected deficit.*

By perturbing types of the buyer above the highest seller's type, or adding a buyer's type above the highest seller's type, one can obtain a pair of distributions for which ex post efficient, budget balanced, incentive compatible and individually rational trade is possible. Similarly, one can perturb the seller't lowest types. Next Remark shows such a perturbation when one starts from a pair of distributions for which the seller's highest type is above the buyer's highest type.

Remark 1. Fix a pair of value distributions (p_b, p_s) with $B_m < S_n$ and $\varepsilon \in (0,1)$. Let $V_b' = V_b \cup \{B_{m+1}\}, p_b'(B_i) = (1-\varepsilon)p_b(B_i), \forall i = \overline{1,m}, p_b'(B_{m+1}) = \varepsilon$. Then the expected surplus of the VCG' mechanism when the values are distributed according to (p_b', p_s) is $\Pi' = \varepsilon(1 - F_s(B_m))B_{m+1} + \kappa$, where κ depends on (p_b, p_s) and ε, but not on B_{m+1}. For sufficiently large B_{m+1}, $\Pi' > 0$, so for the corresponding pair of value distributions (p_b', p_s)

there exists an ex post efficient implementable mechanism (VCG') that does not run an expected deficit.

The above result extends the result of Makowski and Mezzetti (1994), showing that ex post budget balance can be substituted by ex ante budget balance, to an environment that does not require convexity of the valuation space. Furthermore, we identify the mechanism, VCG', that delivers the highest ex ante budget surplus, among ex post efficient incentive compatible and individually rational mechanisms. Thus being what Krishna and Perry (1998) call the salient mechanism.

5. Multiple Buyers

The above results extend to the setting with multiple buyers. Suppose there are I buyers. Buyer i's valuation is denoted by a superscript i, and the profile of valuations by $(B,S) \equiv \left(B^1, B^2, ..., B^I, S\right)$. The maximum of all the buyers' valuations but buyer i's is denoted by Y^i. Furthermore, let $\overline{Y^i} \equiv \min\{B \in V_b : B > Y^i\}$.

For the ease of exposition we assume buyers have the same distribution over valuations. It is easy to show, following the methodology of the single buyer case, that ex post efficiency and ex post budget balance can be achieved in an incentive compatible and individually rational mechanism if and only if a VCG' mechanism adapted to the multiple buyer environment does not run an ex ante budget deficit. Due to buyers having the same distribution there is a large set of ex post efficient allocation rules depending on how the ties between them are broken. Let x^* be some such allocation rule. The corresponding buyer i's transfer in a VCG' mechanism with the allocation rule x^* is[5]:

$$t_i^{VCG'}(B,S;x^*) = \begin{cases} 0 & \text{if } \max\{Y^i, S\} > B^i \\ B^i x_i^*(B,S) & \text{if } B^i = Y^i > S \\ \overline{Y}^i \left(1 - x_i^*(Y^i, B^{-i}, S)\right) + Y^i x_i^*(Y^i, B^{-i}, S) & \text{if } B^i > Y^i > S \\ \min\{B_h : B_h > S\} & \text{if } B^i > S > Y^i \end{cases}.$$

The first case is familiar, when the buyer does not trade his transfer is zero. The second corresponds to the case when the buyer is tied for the highest value in which case he pays

[5]We preserve the assumption that the set of buyers' and seller's valuations has an empty intersection.

his value multiplied by the probability that he wins in the tie break. The third is the case in which buyer i's value is the highest and there is at least one more buyer with a value above the seller's. This one is somewhat trickier since one has to take into account that the buyer could under-report, tie, and still win with a positive probability. The last case corresponds to buyer i's value being the only larger than the seller's value.

The seller's transfer is

$$t_s^{VCG'}(B,S;x^*) = \begin{cases} 0 & \text{if } \max\left\{B^1,...,B^I\right\} < S \\ \max\left\{S_h : S_h < max\left\{B^1,...,B^I\right\}\right\} & \text{if } \max\left\{B^1,...,B^I\right\} > S \end{cases}.$$

While the transfers clearly depend on the tie-breaking rule it is easy to verify that the ex ante budget surplus of a VCG' mechanism does not. More precisely, the sum of expected transfers over all the buyers is independent of the tie-breaking rule, as is the seller's transfer.

Let $\Pi^{VCG'}$ denote the ex ante budget surplus of a VCG' mechanism. An ex post budget balanced, incentive compatible and individually rational mechanism can be constructed when $\Pi^{VCG'} \geq 0$ the following way. Buyer i pays the transfer

$$t_i(B,S) = \bar{t}_i^{VCG'}(B^i) + \frac{1}{I}\left[\bar{t}_s^{VCG'}(S) - \bar{\bar{t}}_s^{VCG'}\right] - \alpha_i \Pi^{VCG'},$$

where $\alpha_i \in [0,1]$ for all $i \in \{1,...,I\}$ and $\sum_i \alpha_i \leq 1$. The seller, on the other hand, receives

$$t_s(B,S) = \bar{t}_s^{VCG'}(S) - \bar{\bar{t}}_s^{VCG'} + \sum_i \bar{t}_i^{VCG'}(B^i) - \left(\sum_i \alpha_i\right)\Pi^{VCG'}.$$

The expected transfers in the above budget balanced mechanism are equal to expected transfers in the VCG' minus a positive constant for each buyer and plus a positive constant for the seller. Therefore the incentive compatibility and individual rationality are preserved.

Next we consider two clarifying examples.

Example 1. Let $B_H > S_H > B_L > S_L$ and let buyers' valuations be distributed with a probability mass function p_b and seller's with p_s. When there is only one buyer ex ante budget surplus, $\Pi^{VCG'}$, is given by the analysis in the previous section; here we denote it by $\Pi_1^{VCG'}$. Now consider there are two ex ante identical buyers. We use the following tie breaking rule: when both buyers and the seller are of low type the object is allocated to buyer 2 with probability one: $x_{b2}(B_L, B_L, S_L) = 1$; whereas, when both buyers are of the high type the object is allocated to buyer 1: $x_{b1}(B_H, B_H, \cdot) = 1$. [6] Buyers' transfers in the VCG', given the allocation rule, are:

$$\begin{aligned} t_{b1}(B_H, B_H, \cdot) &= B_H, \\ t_{b1}(B_H, B_L, \cdot) &= B_H, \\ t_{b2}(B_L, B_H, S_H) &= B_H, \\ t_{b2}(B_L, B_H, S_L) &= t_{b2}(B_L, B_L, S_L) = B_L; \end{aligned}$$

for all the other profiles of valuations buyers' transfers are zero. Notice that when the profile of valuations is (B_L, B_H, S_L) buyer 2, who obtains the good, pays B_L, because he could report B_L and still clinch the good. The seller receives the highest value he could have reported and induced trade.

The ex ante expected budget surplus in the mechanism is now easily computed:

$$\Pi_2^{VCG'} = p_b(B_L)\,\Pi_1^{VCG'} + p_b(B_H)\,[B_H - S_H].$$

The reasoning behind the above equation is the foolowing. When buyer 1 is the low type, which happens with probability $p(B_L)$, he never obtains the good, therefore the situation for buyer 2 is just as if buyer 1 did not exist, hence the expected budget surplus Π_1. When buyer 1 is the high type he either wins or buyer 2 is also the high type and buyer 2 wins. In any case one of the buyers pays B_H whereas the seller gets S_H.

One can show more generally:

$$\Pi_I^{VCG'} = p_b(B_L)\,\Pi_{I-1}^{VCG'} + p_b(B_H)\,[B_H - S_H],$$

for $I > 1$. It is easy to see that $\Pi_{I-1}^{VCG'} \geq 0$ implies $\Pi_I^{VCG'} > 0$, and furthermore, that $\Pi_I^{VCG'}$ converges to $B_H - S_H$ as I goes to infinity. Therefore, no matter with how low $\Pi_1^{VCG'}$ one starts, if the number of buyers increases sufficiently one will be able to obtain ex post efficient and ex post budget balanced mechanism. The intuition is clear, as the number of buyers increases the probability that at least two buyers have a high valuation goes to 1 which means that the expected budget surplus converges to $B_H - S_H$.

[6] We choose a particular tie breaking rule here because it simplifies the computation of $\Pi_2^{VCG'}$. $\Pi_2^{VCG'}$, however, does not depend on the tie breaking rule of an efficient mechanism.

In the two type case adding an additional buyer will not destroy the possibility of efficient, budget balanced, incentive compatible and individually rational trade. With more than two types, however, this is not the case. It can happen that with one buyer one can achieve ex post efficiency and budget balance, while with two buyers one can not.

Example 2. Let $I = 2$ and $B_H > S_H > B_M > S_M > B_L > S_L$. Since the tie-breaking rule does not influence the ex ante budget surplus we assume, for easier calculations, that all ties are broken in favor of buyer 2. Then

$$\begin{aligned}\Pi_2^{VCG'} &= p_b(B_L)\,\Pi_1^{VCG'} + p_b(B_H)\left[p_b(B_H) + p_b(B_M) + p_s(S_H)\left(p_b(B_M) + p_b(B_L)\right)\right](B_H - S_H) \\ &\quad + p_b(B_M)\left(p_b(B_M) + p_b(B_L)\right)\left(p_s(S_M) + p_s(S_L)\right)[B_M - S_M] \\ &\quad + p_b(B_H)\left(p_b(B_M) + p_b(B_L)\right)\left(p_s(S_M) + p_s(S_L)\right)[B_M - S_H],\end{aligned}$$

where $\Pi_1^{VCG'}$ is the expected budget balance of the mechanism with a single buyer. The expression for $\Pi_2^{VCG'}$ is computed by summing the budget surplus over buyer 1's valuations and collecting the terms. I.e., when buyer 1's valuation is B_L the conditional budget surplus is $\Pi_1^{VCG'}$ (as if he was not there). When buyer 1's valuation is B_M, there are nine cases to consider, depending on buyer 2's and the seller's valuation, out of which there is no trade in two of them, etc.

Now, let $p_b(B_i) = p_s(S_i) = \frac{1}{3}$, for all $i \in \{L, M, H\}$, $S_L = 3k, S_M = 8k, S_H = 9k$, and $B_L = 7k, B_M = 8k + 1, B_H = 9k + 1$ where $k > 1$ is a parameter of the mechanism. Irrespective of k $\Pi_1^{VCG'} = \frac{1}{3}$, implying possibility of ex post efficient, budget balanced, incentive compatible and individually rational trade. On the other hand

$$\Pi_2^{VCG'}(k) = \frac{1}{3}\Pi_1^{VCG'} + \frac{8}{27} + \frac{4}{27} + \frac{4}{27}(1 - k).$$

Therefore by setting k high enough one can make $\Pi_2^{VCG'}$ arbitrarily negative. Ex post efficiency can be restored, however, by increasing the number of buyers: as one increases I, the probability of at least two buyers having valuation B_H goes to 1, therefore the expected budget surplus converges to $B_H - S_H$. Hence there exists an I^* such that for all $I \geq I^*$ one can achieve ex post efficiency and ex post budget balance.

The last observation holds for general finite distributions. Let $B_m > S_1$ and $V_b \cap V_s = \emptyset$. Furthermore, let $\Delta = min\{B_m - S_j : B_m > S_j\}$. As the number of buyers goes to infinity, the probability that at least two of them are of the highest type, i.e. B_m, goes to one, implying that the expected ex ante budget surplus of a VCG' mechanism converges to Δ. Therefore for large enough I ex post efficient, BB, IC and IR trade is possible. Notice that this is in sharp contrast with results obtained under connected valuation space, where one needs the highest buyer's valuation to be above the highest seller's; see Williams (1999).

6. Gains from Trade

6.1. **Maximizing Gains from Trade.** Thus far we were concerned with characterizing conditions under which ex post efficiency and budget balance can be satisfied simultaneously; in an incentive compatible and individually rational mechanism. The above analysis, however, tells us nothing about what can be achieved when full efficiency does not obtain. Broader understanding of the model is to be gained by considering the problem of ex ante maximization of total gains from trade under the standard conditions of budget balance, incentive compatibility and individual rationality. The expected total gains from trade in a mechanism (x, t_b, t_s) are

$$\sum_{i,j} p_b(B_i)p_s(S_j)x(B_i, S_j)[B_i - S_j].$$

The approach to solving maximization of gains from trade in an environment with connected valuation spaces is to apply the revenue equivalence and reduce the problem by the dimension of transfers. The discrete nature of our setup, however, renders the revenue equivalence principle non-applicable. We navigate around the revenue equivalence by showing that maximization of gains from trade only needs to be considered over a narrow set of mechanisms with revenue equivalence-like properties. We start by characterizing the set of allocation rules x which can be implemented in a budget balanced, incentive compatible and individually rational mechanism.

Let x be an implementable allocation rule. We call a mechanism (x, t_b, t_s) *tight* if the buyer's downward adjacent IC constraints, the seller's upward adjacent IC constraints as well as the buyer's lowest type's IR and the seller's highest type's IR constraints are binding; we do not impose budget balance at this point. Notice that a tight mechanism is uniquely pinned down, given the implementable allocation x, at the interim stage:

$$\begin{aligned} \bar{t}_b(x)(B_k) &= \sum_{j=1}^{k} \left[\overline{x}_b(B_j) - \overline{x}_b(B_{j-1})\right] B_j, \\ \bar{t}_s(x)(S_l) &= \sum_{j=l}^{n} [\overline{x}_s(S_l) - \overline{x}_s(S_{l+1})] S_l. \end{aligned}$$

A mechanism yielding the above expected transfers can be specified as follows:

$$\begin{aligned} t_b^T(x)(B_k, S_l) &\equiv \sum_{j=1}^{k} [x(B_j, S_l) - x(B_{j-1}, S_l)] B_j, \\ t_s^T(x)(B_k, S_l) &\equiv \sum_{j=l}^{n} [x(B_k, S_j) - x(B_k, S_{j+1})] S_j. \end{aligned}$$

Notice that the above mechanism evaluated at the efficient allocation rule x^* yields the VCG' mechanism. In the reminder of the section we use $\Pi(x)$ as the shorthand for the ex ante budget surplus of a tight mechanism with the implementable allocation rule x.

Lemma 1.

(6.1)

$$\Pi(x) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\left[\frac{(1-F(B_{i-1}))B_i - (1-F(B_i))B_{i+1}}{p_B(B_i)}\right] - \left[\frac{S_j F_s(S_j) - S_{j-1}F_S(S_{j-1})}{p_S(S_j)}\right]\right] x(B_i, S_j) p_B(B_i) p_S($$

The first term is the buyer's virtual valuation, namely:

$$\frac{(1-F(B_{i-1}))B_i - (1-F(B_i))B_{i+1}}{p_B(B_i)} = B_i - (B_{i+1} - B_i)\frac{1-F_B(B_i)}{p_B(B_i)};$$

and the second the seller's virtual valuation:

$$\frac{S_j F_s(S_j) - S_{j-1}F_S(S_{j-1})}{p_S(S_j)} = S_j + (S_j - S_{j-1})\frac{F_S(S_{j-1})}{p_S(S_j)}.$$

Notice, however, that the left hand-sides of the two equalities allow for a more intuitive economic interpretation. Suppose that after (B_i, S_j) is reported trade occurs with probability 1. Accounting for individual rationality constraints at most B_i can be charged to the buyer and at least S_j given to the seller. But then all the types of the buyer above B_i could also report B_i which corresponds to the expected revenue of $(1 - F(B_{i-1}))B_i$. On the other hand, requiring that the buyer reports at least B_{i+1} for the trade to occur would account for the expected revenue of $(1-F(B_i))B_{i+1}$. The marginal revenue of facilitating trade at B_i thus being

$$\frac{(1-F(B_{i-1}))B_i-(1-F(B_i))B_{i+1}}{p_B(B_i)}.$$

Likewise

$$\frac{S_jF_s(S_j)-S_{j-1}F_S(S_{j-1})}{p_S(S_j)}$$

can be interpreted as the marginal cost of allowing trade when the seller reports S_j rather than S_{j-1}. This interpretation is closely related to the one of Bulow and Roberts (1989). While Bulow and Roberts (1989) redefine some objects as quantities and prices, the interpretation here is immediate due to the nature of the discrete valuation space.

The following theorem shows how the tight mechanisms can be used to characterize the set of allocation rules that can be implemented in a budget balanced, incentive compatible and individually rational mechanism.

Theorem 2. *Let x be an implementable allocation rule. Then there exists a budget balanced transfer t such that (x,t) is budget balanced, incentive compatible and individually rational if and only if* $\Pi(x)\geq 0$.

An alternative interpretation of Theorem 2 arises if one introduces an additional agent into the economy. In particular, suppose there is profit maximizing broker who specializes in designing mechanisms. $\Pi(x)$ can now be viewed as the largest ex ante profit that such a broker can make, given the IC and IR constraints of the buyer and the seller, if implementation of the allocation rule x is delegated to him. Theorem 2 can now be reinterpreted to read that an allocation rule x can be implemented in an budget balanced, IC and IR mechanism if and only if a profit maximizing broker can make a profit when the implementation of x is delegated to him.

Theorem 1 states that full efficiency is attainable in a budget balanced, IC and IR mechanism whenever $\Pi(x^*)\geq 0$. Thus the only interesting case concerning maximization of gains from trade is the one with $\Pi(x^*)<0$.

Theorem 3. *Let (F_B,F_S) be such that $\Pi(x^*)<0$. Then there exists a budget balanced transfer rule t such that the mechanism (x,t) maximizes gains from trade among budget* *balanced, incentive compatible and individually rational mechanisms if and only if the allocation rule x is a solution to the problem*

$$\begin{aligned} \max_x \quad & \sum_{i,j} p_b(B_i)p_s(S_j)x(B_i,S_j)[B_i - S_j] \\ s.t. \quad & x \textit{ is implementable}, \\ & \Pi(x) = 0. \end{aligned}$$

Proof. Theorem 2 establishes the Theorem 3 when the equality $\Pi(x) = 0$ is replaced by an inequality $\Pi(x) \geq 0$. Therefore we only need to argue that at the optimum the broker makes zero expected profit from implementing x. Suppose there exists a mechanism (x, t) that maximizes gains from trade subject to the usual conditions and leaves the broker with the positive profit; i.e. $\Pi(x) > 0$. By assumption $\Pi(x^*) < 0$. The set of allocation rules is convex. So is the set of implementable allocation rules. Indeed, suppose the allocation rules x and x' are implementable. Let $\alpha \in [0, 1]$ and $x'' = \alpha x + (1 - \alpha)x'$. To verify that x'' is implementable it is enough to show that

$$\begin{aligned} \overline{x''}_B(B_{i+1}) &\geq \overline{x}_B(B_i'') \\ \overline{x''}_S(S_j) &\geq \overline{x}_S(S_{j+1''}), \end{aligned}$$

for all $i \in 1, ..., m$ and $j \in 1, ..., n$. But this follows immediately from the fact that $\overline{x''}_B = \alpha\overline{x}_B + (1 - \alpha)\overline{x'}_B$, the equivalent equation for the seller's possibility of trade and the fact that x and x' satisfy the above inequalities.

Since gains from trade and $\Pi(x)$ are linear in x it follows that there exists an $\alpha \in [0, 1]$ such that $x' = \alpha x^* + (1 - \alpha)x$ achieves higher gains from trade than x and satisfies $\Pi(x') > 0$. By the above observation x' is also implementable, which brings us to a contradiction with the optimality of (x, t). □

More can be said about the solution to the problem. Whenever $B_i < S_j$ the coefficient on $x(B_i, S_j)$ in $\Pi(x)$ is negative. Since the coefficient in the expression for the expected gains of trade is also negative, $x(B_i, S_j)$ in such a case is optimally set to zero. That is, there is no trade in the optimal mechanism when the trade is inefficient.

In the next example we provide the full solution of maximization of gains from trade for the environment in which there are two types of the buyer and two types of the seller.

Example 3. The only interesting case is $B_H > S_H > B_L > S_L$. In all the other cases the ex post efficient allocation rule can be implemented by posting a fixed price. For completeness we provide the formula for the budget surplus of a tight mechanism with an allocation rule x:

$$\begin{aligned}\Pi(x) &= p_b(B_2)p_s(S_2)\left[B_2 - \frac{S_2 - p_s(S_1)S_1}{p_s(S_2)}\right]x(B_2, S_2)\\ &+ p_b(B_2)p_s(S_1)[B_2 - S_1]x(B_2, S_1)\\ &+ p_b(B_1)p_s(S_2)\left[\frac{B_1 - p_b(B_2)B_2}{p_b(B_1)} - \frac{S_2 - p_s(S_1)S_1)}{p_s(S_2)}\right]x(B_1, S_2)\\ &+ p_b(B_1)p_s(S_1)\left[\frac{B_1 - p_b(B_2)B_2}{p_b(B_1)} - S_1\right]x(B_1, S_1).\end{aligned}$$

If F_b and F_s are such that

$$\Pi(x^*) = p_b(B_2)p_s(S_2)B_2 + p_s(S_1)B_1 - p_b(B_2)S_2 - p_b(B_1)S_1 \geq 0,$$

the efficient allocation can be implemented and we are done.

From here on we consider the case of $\Pi(x^*) < 0$. We solve the problem by at first neglecting the monotonicity constraints. At the end we verify that they are satisfied at the optimum. Since the coefficients on $x(B_2, S_1)$ are positive both in the equation for gains from trade as well as in $\Pi(x)$, $x(B_2, S_1)$ is optimally set to 1. We already argued above that $x(B_1, S_2)$ is optimally set to 0. The remainder of the analysis is covered in the proof of the following proposition. Define

$$M \equiv [B_1 - S_1]\left[B_2 - \frac{S_2 - p_s(S_1)}{p_s(S_2)}\right] - [B_2 - S_2]\left[\frac{B_1 - p_b(B_2)B_2}{p_b(B_1)} - S_1\right].$$

Theorem 4. *Let $m = n = 2$, $B_2 > S_2 > B_1 > S_1$ and $\Pi(x^*) < 0$. Then the allocation rule in an optimal budget balanced implementable mechanism is given by*

$$\begin{aligned}
x(B_2, S_2) &= \begin{cases} 1 & \text{if } M < 0 \\ \frac{p_s(S_1)[B_1 - S_1]}{p_b(B_2)[S_2 - p_s(S_2)B_2 - p_s(S_1)S_1]} & \text{if } M > 0 \end{cases} \\
x(B_2, S_1) &= 1 \\
x(B_1, S_2) &= 0 \\
x(B_1, S_1) &= \begin{cases} \frac{p_b(B_2)[B_2 - S_2]}{p_s(S_1)[p_b(B_2)B_2 + p_b(B_1)S_1 - B_1]} & \text{if } M < 0 \\ 1 & \text{if } M > 0 \end{cases},
\end{aligned}$$

where

$$M \equiv [B_1 - S_1]\left[B_2 - \frac{S_2 - p_s(S_1)}{p_s(S_2)}\right] - [B_2 - S_2]\left[\frac{B_1 - p_b(B_2)B_2}{p_b(B_1)} - S_1\right].$$

When $M = 0$ there is a continuum of optimal pairs $x(B_2, S_2)$, $x(B_1, S_1)$.

7. Some Loose Ends

7.1. **Optimal Tie-breaking Rule.** When we allow for ties, i.e. when $V_b \cap V_s$ is nonempty, there exist several efficient allocation rules. The above results extend to this setting in the following sense. An efficient allocation rule x^* is implementable in an BB, IC and IR mechanism if and only if $\Pi(x^*) \geq 0$. Furthermore, when $B_i = S_j$, marginal revenue from the trade is strictly lower than the marginal cost (equivalently, virtual valuation of the buyer is lower than the virtual valuation of the seller). That, in turn, means that the ex ante budget balance is maximized by setting $x(B_i, S_j) = 0$ when $B_i = S_j$. Let x^{**} be the efficient allocation rule that leaves the object to the seller in the case of a tie. Then

$$\Pi(x^{**}) \geq \Pi(x^*),$$

for every efficient allocation rule x^*. Finally, efficient, BB, IC and IR trade is possible if and only if $\Pi(x^{**}) \geq 0$.

7.2. **Multiple Buyers and Multiple Sellers.** Our analysis extends to models with multiple buyers and multiple sellers. VCG' mechanism, for the case $V_b \cap V_s = \emptyset$, is defined by an efficient allocation rule and by the transfers requiring that the buyer, who wins the object, pays the smallest valuation he could have reported and there was still trade (some care is needed to be taken when ties among buyers occur) and the seller pays the highest valuation he could have reported and still traded (again some more care needs to be taken in the case of ties between sellers). As long as such a VCG' mechanism runs a budget surplus efficient, BB, IC and IR trade is possible.

Such a VCG' mechanism complements the asymptotically optimal mechanism, without a fixed fee, in Tatur (2005).

7.3. **General Distributions with Bounded Support.** Notice that our definition of the VCG' extends to the case of general distribution functions with bounded support when the set of valuations is taken to be the support of the distribution function. It is easy to verify that the VCG' mechanism, which in a case of a tie between a buyer and a seller leaves the object to the seller, creates the largest ex ante budget surplus among all efficient, IC and IR mechanisms.

8. Conclusion

We provide an extensive study of bilateral trade with discrete values. A point to note is that the mechanisms we derive through most of the paper are interim individually rational but not necessarily ex post. While seldomly required in the bilateral trade literature ex post individual rationality is a desirable feature of a mechanism. After all, a potential buyer could be deterred from participating in a mechanism that can leave him worse off then he was to start with. An interesting question for the further research is under what conditions does an ex post budget balanced, efficient, individually rational and interim incentive compatible mechanism exist. In the case of bilateral bargaining with two types both of the buyer and the seller ex post individual rationality can be obtained always when the budget balanced efficient (interim individually rational) mechanism exists; see the proofs in Matsuo (1989). On the other hand, environments with more than two types can be provided in which a budget balanced efficient mechanism that is interim individual rationality exists while such an ex post individual rational mechanism does not exist.

Appendix A. Proofs

The following result is used to prove Proposition 1

Lemma 2. *The expected transfers in any ex post efficient implementable mechanism* (x^*, t_b, t_s) *satisfy the following inequalities,*

$$\bar{t}_b(B_i) \leq \sum_{k=1}^{i}(F_s(B_k) - F_s(B_{k-1}))B_k, \forall i \in \overline{1,m} \tag{A.1}$$

$$\bar{t}_s(S_j) \geq \sum_{k=j}^{n}(F_b(S_{k+1}) - F_b(S_k))S_k, \forall j \in \overline{1,n}. \tag{A.2}$$

Proof. Note that under the disjoint values assumption for an ex post efficient implementable mechanism (x^*, t_b, t_s), $IC_{B_i \to B_{i-1}}$ can be rewritten

$$\bar{t}_b(B_i) - \bar{t}_b(B_{i-1}) \leq (\overline{x}_b^*(B_i) - \overline{x}_b^*(B_{i-1}))B_i = (F_s(B_i) - F_s(B_{i-1}))B_i.$$

Then $IR_{B_1}, IC_{B_2 \to B_1}, \ldots, IC_{B_n \to B_{n-1}}$ imply that

$$\begin{aligned}
\bar{t}_b(B_1) &\leq F_s(B_1)B_1 \\
\bar{t}_b(B_2) - \bar{t}_b(B_1) &\leq (F_s(B_2) - F_s(B_1))B_2 \\
&\cdots \\
\bar{t}_b(B_n) - \bar{t}_b(B_{n-1}) &\leq (F_s(B_n) - F_s(B_{n-1}))B_n.
\end{aligned}$$

For $i = \overline{1,m}$, adding up the top i inequalities above we obtain

$$\bar{t}_b(B_i) \leq \sum_{k=1}^{i}(F_s(B_k) - F_s(B_{k-1}))B_k.$$

The set of inequalities corresponding to the expected transfers for the seller follow analogously. □

Proof of Proposition 1. From Lemma 2, the expected surplus $\sum_{i=1}^{m} p_b(B_i)\bar{t}_b(B_i)-\sum_{j=1}^{n} p_s(S_j)\bar{t}_b(S_j)$ of an ex post efficient implementable mechanism (x^*, t_b, t_s) is at most

$$\sum_{i=1}^{m} p_b(B_i)\sum_{k=1}^{i}(F_s(B_k)-F_s(B_{k-1}))B_k-\sum_{j=1}^{n} p_s(S_j)\sum_{k=j}^{n}(F_b(S_{k+1})-F_b(S_k))S_k,$$

which is equal to Π by **??**-**??**. □

Proof of Theorem 1. **Only if part.** Suppose that (x,t) is an ex post efficient budget balanced implementable mechanism. Then (x,t,t) is an ex post efficient implementable mechanism with zero expected surplus. By Proposition 1, $\Pi \geq 0$.

If part. Assume that $\Pi \geq 0$ and define

$$t(B_i, S_j) = \bar{t}_b^{VCG'}(B_i) - \bar{\bar{t}}_b^{VCG'} + \bar{t}_s^{VCG'}(S_j) + \alpha\Pi,$$

where $\bar{\bar{t}}_b^{VCG'} = \sum_l p_b(B_l)\bar{t}_b^{VCG'}(B_l)$; any α in $[0,1]$ will do. Since the VCG' is implementable,

$$\bar{t}_b(B_i) = \bar{t}_b^{VCG'}(B_i) - (1-\alpha)\Pi(x, t_b, t_s)$$

and

$$\bar{t}_s(S_j) = \bar{t}_s^{VCG'}(S_j) + \alpha\Pi(x, t_b, t_s),$$

i.e. the interim transfers in the two mechanism differ by a constant which is negative in the case of the buyer and positive in the case of the seller, (x,t) is implementable too.

□

Proof of Theorem 2. **If part.** Suppose $\Pi(x, t_b^T(x), t_s^T(x)) \geq 0$. We can define a budget balanced mechanism by setting

$$t(B_i, S_j) = \bar{t}_b^T(x)(B_i) - \sum_k p_b(B_k)\bar{t}_b^T(x)(B_k) + \bar{t}_s^T(x)(S_j) + \alpha\Pi(x, t_b, t_s),$$

with $\alpha \in [0,1]$. Clearly (x,t) is implementable. Indeed $(x, t_b^T(x), t_s^T(x))$ is implementable,

$$\bar{t}_b(B_i) = \bar{t}_b^T(x)(B_i) - (1-\alpha)\Pi(x, t_b, t_s)$$

and

$$\bar{t}_s(S_j) = \bar{t}_s^T(x)(S_j) + \alpha\Pi(x, t_b, t_s);$$

i.e. the interim transfers in the two mechanism differ by a constant which is negative in the case of the buyer and positive in the case of the seller.

Only if part. Suppose (x, t) is an optimal budget balanced implementable mechanism, and suppose that there exists some implementable tight mechanism $(x', t_b^T(x), t_s^T(x))$ that runs a budget surplus and achieves strictly higher gains from trade. Then by an argument used in the (If part) we could construct a budget balanced implementable mechanism (x', t') achieving higher gains from trade than (x, t), contradicting the optimality of the latter. □

Lemma 3. *$B_2 > S_2 > B_1 > S_1$ and $\Pi(x^*) < 0$ imply*

$$\begin{aligned} p_s(S_2)B_2 + p_s(S_1)S_1 &< S_2, \\ p_s(S_2)(B_1 - p_b(B_2)B_2) &< p_b(B_1)(S_2 - p_s(S_1)S_1), \\ B_1 &< p_b(B_2)B_2 + p_b(B_1)S_1. \end{aligned}$$

Proof. Assume $B_2 > S_2 > B_1 > S_1$ and $\Pi(x^*) < 0$. The first inequality follows by

$$\begin{aligned} 0 &> p_b(B_2)p_s(S_2)B_2 + p_s(S_1)B_1 - p_b(B_2)S_2 - p_b(B_1)p_s(S_1)S_1 \\ &> p_b(B_2)p_s(S_2)B_2 + p_s(S_1)S_1 - p_b(B_2)S_2 - p_b(B_1)p_s(S_1)S_1 \\ &= p_b(B_2)[p_s(S_2)B_2 + p_s(S_1)S_1 - S_2]. \end{aligned}$$

For the second notice

$$\begin{aligned} p_b(B_1)[S_2 - p_s(S_1)S_1] &> p_b(B_1)p_s(S_2)B_2, \\ &= p_s(S_2)[B_2 - p_b(B_2)B_2], \\ &> p_s(S_2)[B_1 - p_b(B_2)B_2, \end{aligned}$$

where the first inequality, in the chain, follows from the first result of this lemma. Finally, the third inequality of the lemma is implied by

$$
\begin{aligned}
0 &> p_b(B_2)p_s(S_2)B_2 + p_s(S_1)B_1 - p_b(B_2)S_2 - p_b(B_1)p_s(S_1)S_1 \\
&> p_b(B_2)p_s(S_2)S_2 + p_s(S_1)B_1 - p_b(B_2)S_2 - p_b(B_1)p_s(S_1)S_1 \\
&= p_s(S_1)B_1 - p_b(B_2)p_s(S_1)S_2 - p_b(B_1)p_s(S_1)S_1.
\end{aligned}
$$

□

Proof of Theorem 4. At first we characterize optimal budget balanced implementable allocation rules neglecting implementability constraints (monotonicity of interim allocation rules). These are verified in the end.

We showed, Lemma 3, that $\Pi(x)$ is increasing in $x(B_2, S_1)$, but so is welfare. $x(B_2, S_1)$ is therefore optimally set to 1. By similar reasoning, both $\Pi(x)$ and the welfare are decreasing in $x(B_1, S_2)$, $x(B_1, S_2)$ is optimally set to 0. From now on we fix those two values. Now the problem reduces to maximizing

$$p_b(B_2)p_s(S_2)[B_2 - S_2]x(B_2, S_2) + p_b(B_1)p_s(S_1)[B_1 - S_1]x(B_1, S_1)$$

subject to

$$
\begin{aligned}
p_b(B_2)[p_s(S_2)B_2 + p_s(S_1)S_1 - S_2]x(B_2, S_2) + p_b(B_2)p_s(S_1)[B_2 - S_1] \\
+p_s(S_1)[B_1 - p_b(B_1)S_1 - p_b(B_2)B_2]x(B_1, S_1) = 0.
\end{aligned}
$$

One should keep increasing $x(B_2, S_2)$, at the cost of decreasing $x(B_1, S_1)$, when

$$p_s(S_2)[B_2 - S_2][p_b(B_2)B_2 + p_b(B_1)S_1 - B_1] > p_b(B_1)[B_1 - S_1][S_2 - p_s(S_2)B_2 - p_s(S_1)S_1]$$

and vice versa when the opposite strict inequality holds. Suppose that the above inequality holds. Notice that $\Pi(x) > 0$ when $x(B_2, S_2) = 1$ and $x(B_1, S_1) = 0$, therefore $x(B_2, S_2)$ is optimally set to 1. The binding budget surplus constraint now yields

$$x(B_1, S_1) = \frac{p_s(S_1)[B_1 - S_1]}{p_b(B_2)[S_2 - p_s(S_2)B_2 - p_s(S_1)S_1]}.$$

The other two possibilities are obtained similarly.

We are left to show that the optimal allocation rules are indeed implementable. Monotonicity requirements are

$$\begin{aligned} p_s(S_2)x(B_2,S_2)+p_s(S_1)x(B_2,S_1) &\geq p_s(S_2)x(B_1,S_2)+p_s(S_1)x(B_1,S_1) \\ p_b(B_2)x(B_2,S_1)+p_b(B_1)x(B_1,S_1) &\geq p_b(B_2)x(B_2,S_2)+p_b(B_1)x(B_1,S_2). \end{aligned}$$

The two monotonicity conditions are readily verified after plugging in $x(B_2,S_1)=1$ and $x(B_1,S_2)=0$. □

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