

Notes for Exercise Session 3

1 No-Arbitrage Restrictions on Bond Pricing Parameters

1.1 State variable dynamics.

1. Transition equation for X_t follows VAR(1):

$$X_t = \mu + \Phi X_{t-1} + v_t, \quad v_t \text{ is i.i.d. } N(0, \Omega).$$

1.2 Short rate equation.

$$r_t = \delta_0 + \delta_1' X_t$$

δ_0 : a scalar.

δ_1 : $K \times 1$ vector.

1.3 Time-varying prices of risk

(associated with the sources of uncertainty v_t).

$$\Lambda_t = \lambda_0 + \lambda_1' X_t$$

Λ_t : $K \times 1$ vector.

λ_0 : $K \times 1$ vector.

λ_1 : $K \times K$ matrix.

If investors are risk-neutral, $\lambda_0 = 0$ and $\lambda_1 = 0$, hence $\Lambda_t = 0$, no risk adjustment. If $\lambda_0 \neq 0$ and $\lambda_1 = 0$, then price of risk is constant.

1.4 Pricing kernel.

1. No arbitrage opportunity between bonds with different maturities implies that there is a discount factor m linking the price of yield of maturity n this month with the yield of maturity $n - 1$ next month.

$$P_t^{(n)} = E_t \left[m_{t+1} P_{t+1}^{(n-1)} \right]$$

The stochastic discount factor is related to the short rate and risk perceived by the market,

$$m_{t+1} = \exp \left(-r_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t - \Lambda'_t v_{t+1} \right)$$

No-arbitrage recursive relation can be derived from the above equations as:

$$\begin{aligned} P_t^{(n)} &= E_t \left[m_{t+1} P_{t+1}^{(n-1)} \right] = E_t \left[m_{t+1} m_{t+2} P_{t+2}^{(n-2)} \right] \\ &= E_t \left[m_{t+1} m_{t+2} \dots m_{t+n} P_{t+n}^{(0)} \right] = E_t \left[m_{t+1} m_{t+2} \dots m_{t+n} \cdot 1 \right] \\ &= E_t \left[\exp \left(- \sum_{i=0}^{n-1} \left(r_{t+i} + \frac{1}{2} \Lambda'_{t+i} \Omega \Lambda_{t+i} + \Lambda'_{t+i} v_{t+1+i} \right) \right) \right] \\ &= E_t \left[\exp (A_n + B'_n X_t) \right] = E_t \left[\exp (-n y_{t,n}) \right] \\ &= E_t^Q \left[\exp \left(- \sum_{i=0}^{n-1} r_{t+i} \right) \right] \end{aligned}$$

E_t^Q denotes the expectation under the risk-neutral probability measure, under which the dynamics of the state vector X_t are characterised by the risk-neutral vector of constants and autoregressive matrix:

$$\begin{aligned} \mu^Q &= \mu - \Omega \lambda_0 \\ \Phi^Q &= \Phi - \Omega \lambda_1 \end{aligned}$$

Affine functions of the state variables for yields are:

$$\begin{aligned} p_{t,t+n} &\equiv \ln P_t^{(n)} = A_n + B'_n X_t \\ y_{t,t+n} &= a_n + b'_n X_t = \frac{-1}{n} (A_n + B'_n X_t) \end{aligned}$$

where the coefficients and follow the difference equations:

$$\begin{aligned} A_{n+1} &= A_n + B'_n (\mu - \Omega \lambda_0) + \frac{1}{2} B'_n \Omega B_n + A_1 \\ B'_{n+1} &= B'_n (\Phi - \Omega \lambda_1) + B'_1 \end{aligned}$$

with $a_1 = \delta_0 = -A_1$ and $b_1 = \delta_1 = -B_1$.

These can be derived from the pricing kernel equation.

$$\begin{aligned} P_t^{(n+1)} &= E_t \left[m_{t+1} P_{t+1}^{(n)} \right] \\ &= E_t \left[\exp \left\{ -r_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t - \Lambda'_t v_{t+1} \right\} \exp \{ A_n + B'_n X_{t+1} \} \right] \end{aligned}$$

$$\begin{aligned}
&= \exp \left\{ -r_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t + A_n \right\} E_t \left[\exp \left\{ -\Lambda'_t v_{t+1} + B'_n X_{t+1} \right\} \right] \\
&= \exp \left\{ -\delta_0 - \delta'_1 X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t + A_n \right\} \\
&\quad \cdot E_t \left[\exp \left\{ -\Lambda'_t v_{t+1} + B'_n (\mu + \Phi X_t + v_{t+1}) \right\} \right] \\
&= \exp \left\{ -\delta_0 - \delta'_1 X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t + A_n + B'_n (\mu + \Phi X_t) \right\} \\
&\quad \cdot E_t \left[\exp \left\{ -\Lambda'_t v_{t+1} + B'_n v_{t+1} \right\} \right] \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t \right\} \\
&\quad \cdot E_t \left[\exp \left\{ (-\Lambda'_t + B'_n) v_{t+1} \right\} \right] \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t \right\} \\
&\quad \cdot \exp \left\{ E_t \left[(-\Lambda'_t + B'_n) v_{t+1} \right] + \frac{1}{2} \text{var} \left[(-\Lambda'_t + B'_n) v_{t+1} \right] \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t \right\} \\
&\quad \exp \left\{ \frac{1}{2} \text{var} \left[(-\Lambda'_t + B'_n) v_{t+1} \right] \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t \right\} \\
&\quad \cdot \exp \left\{ \frac{1}{2} E_t \left[(-\Lambda'_t + B'_n) v_{t+1} v'_{t+1} (-\Lambda_t + B_n) \right] \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \Lambda'_t \Omega \Lambda_t \right\} \\
&\quad \cdot \exp \left\{ \frac{1}{2} \left[\Lambda'_t \Omega \Lambda_t - 2 B'_n \Omega \Lambda_t + B'_n \Omega B_n \right] \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - B'_n \Omega \Lambda_t + \frac{1}{2} B'_n \Omega B_n \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - B'_n \Omega \Lambda_t + \frac{1}{2} B'_n \Omega B_n \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n \mu + (B'_n \Phi - \delta'_1) X_t - B'_n \Omega (\lambda_0 + \lambda_1 X_t) + \frac{1}{2} B'_n \Omega B_n \right\} \\
&= \exp \left\{ -\delta_0 + A_n + B'_n (\mu - \Omega \lambda_0) + \frac{1}{2} B'_n \Omega B_n + (B'_n \Phi - B'_n \Omega \lambda_1 - \delta'_1) X_t \right\} \\
&= \exp \left\{ [A_1 + A_n + B'_n (\mu - \Omega \lambda_0) + \frac{1}{2} B'_n \Omega B_n] + [B'_n \Phi - B'_n \Omega \lambda_1 + B'_1] X_t \right\}
\end{aligned}$$

5. An alternative presentation for the no-arbitrage coefficients.

In order to understand intuitively how these restrictions are imposed directly on the coefficients a_n , b_n in the yield equation, we can write them in the following affined form.

Given that

$$\begin{aligned}
p_{t,t+n} &= A_n + B'_n X_t \\
y_{t,t+n} &= a_n + b'_n X_t = \frac{-1}{n} (A_n + B'_n X_t)
\end{aligned}$$

we can derive

$$\begin{aligned}
b_{n+1} &= \frac{1}{(n+1)} \left[\sum_{i=0}^n (\Phi' - \lambda'_1 \Omega)^i \right] b_1 \\
a_{n+1} &= a_1 - \frac{1}{(n+1)} \sum_{i=1}^n B^{(i)}
\end{aligned}$$

where $B^{(i)} = B'_i (\mu - \Omega \lambda_0) + \frac{1}{2} B'_i \Omega B_i$.

2 The likelihood function with Chen-Scott (1993) method

(The likelihood function representation follows closely to Ang, Piazzesi (2003).)

In order to be able to extract factors under no-arbitrage restrictions, we employ the method by Chen and Scott (1993). Assume that there are K factors in the state equation and that among them, K_2 factors are unobserved. When the number of yields N exceeds number of unobserved factors, K_2 , following Chen and Scott (1993), we assume that K_2 yields, y_t^{NE} , are observed without measurement errors, and that $N - K_2$ yields, y_t^E , are measured with error u_t^m . The state vector contains both observed variables X_t^o and latent factors X_t^u , thus $X_t = [X_t^o; X_t^u]$

The measurement equation can be written as following:

$$y_t = a + b^o X_t^o + b^u X_t^u + b^m u_t^m$$

$$\text{where } y_t = \begin{bmatrix} y_t^{NE} \\ y_t^E \end{bmatrix}, a = \begin{bmatrix} a^{NE} \\ a^E \end{bmatrix}, b^o = \begin{bmatrix} b^{NE,o} \\ b^{E,o} \end{bmatrix}, b^u = \begin{bmatrix} b^{NE,u} \\ b^{E,u} \end{bmatrix}, \text{ and } b^m = \begin{bmatrix} \mathbf{0}_{(K_2 \times (N-K_2))} \\ b^{E,m} \end{bmatrix}.$$

For a given parameter vector $\theta = (\mu, \Phi, \Omega, \delta_0, \delta_1, \lambda_0, \lambda_1)$, the unobserved factors X_t^u will be solved from the yields and the observed variables X_t^o as: $X_t^u = (b^{NE,u})^{-1} [Y_t^{NE} - a^{NE} - b^{NE,o} X_t^o]$.

Denoting the normal density functions of the state variables X_t^u and the error u_t^m as f_X and f_{u^m} respectively, the joint likelihood $\mathcal{L}(\theta)$ of the observed data on zero coupon yields Y_t and the observable factors X_t^o is given by:

$$\begin{aligned} \mathcal{L}(\theta) &= \prod_{t=2}^T f(y_t, X_t^o | y_{t-1}, X_{t-1}^o) \\ \log(\mathcal{L}(\theta)) &= \sum_{t=2}^T \log |\det(J)| + \log f_X(X_t^o, X_t^u | X_{t-1}^o, X_{t-1}^u) + \log f_{u^m}(u_t^m) \\ &= -(T-1) \log |\det(J)| - \frac{(T-1)}{2} \log(\det(\Omega)) \\ &\quad - \frac{1}{2} \sum_{t=2}^T (X_t - \mu - \Phi X_{t-1})' \Omega^{-1} (X_t - \mu - \Phi X_{t-1}) \\ &\quad - \frac{(T-1)}{2} \log \sum_{i=1}^{N-K_2} \sigma_i^2 - \frac{1}{2} \sum_{t=2}^T \sum_{i=1}^{N-K_2} \frac{(u_{t,i}^m)^2}{\sigma_i^2} \end{aligned}$$

(The constant terms like $\frac{(T-1)}{2} \log(2\pi)$ are ignored.)

The Jacobian term is:

$$J = \begin{pmatrix} I_{K-K_2} & 0_{(K-K_2) \times K_2} & 0_{(K-K_2) \times (N-K_2)} \\ B^o & B^u & B^m \end{pmatrix}$$

3 Kalman Filter

State space model:

$$\begin{aligned} y_t &= A + BX_t + e_t, & e_t & i.i.d. N(0, \sigma^2 I) \\ X_t &= \mu + \Phi X_{t-1} + D\varepsilon_t, & \varepsilon_t & i.i.d. N(0, \Omega) \end{aligned}$$

3.1 Initial values (at steady state) of $X_{0|0}$ and $\Sigma_{0|0}$

1) $X_{0|0}$: Set to unconditional mean.

2) $\Sigma_{0|0}$: the unconditional covariance matrix of stationary X_t , described as:

$$\begin{aligned} Cov(X_t) &= \Phi \cdot Cov(X_t) \cdot \Phi' + D \cdot Cov(\varepsilon_t) \cdot D' \\ \Sigma_{0|0} &= \Phi \cdot \Sigma_{0|0} \cdot \Phi' + D \cdot \Omega \cdot D' \\ vec(\Sigma_{0|0}) &= vec(\Phi \cdot \Sigma_{0|0} \cdot \Phi') + vec(D \cdot \Omega \cdot D') \\ vec(\Sigma_{0|0}) &= (\Phi \otimes \Phi') vec(\Sigma_{0|0}) + vec(D \cdot \Omega \cdot D') \\ vec(\Sigma_{0|0}) &= (I - (\Phi \otimes \Phi')^{-1}) vec(D \cdot \Omega \cdot D') \end{aligned}$$

3.2 Prediction.

For $t = 1, \dots, T$:

$$\begin{aligned} X_{t|t-1} &= \mu + \Phi X_{t-1|t-1} \\ \Sigma_{t|t-1} &= \Phi \cdot \Sigma_{t-1|t-1} \cdot \Phi' + D \cdot \Omega \cdot D' \\ \eta_{t|t-1} &= y_t - y_{t|t-1} = y_t - A - BX_{t|t-1} \\ f_{t|t-1} &= B \cdot \Sigma_{t|t-1} B' + \sigma^2 I \end{aligned}$$

Likelihood function:

$$\begin{aligned} l(\Theta) &= l(\Theta) - \frac{1}{2} \ln \left((2\pi)^n |f_{t|t-1}| \right) - \frac{1}{2} \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1} \\ &= l(\Theta) - \frac{n}{2} \ln(2\pi) - \frac{1}{2} |f_{t|t-1}| - \frac{1}{2} \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1} \end{aligned}$$

3.3 Updating.

Kalman gains: $K_t = \Sigma_{t|t-1} B f_{t|t-1}^{-1}$

$$X_{t|t} = X_{t|t-1} + K_t \eta_{t|t-1}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - K_t B \Sigma_{t|t-1}$$

3.4 Likelihood function.

By repeating step 2-3, the likelihood function for the regression is obtained as:

$$\begin{aligned} l(\Theta) &= -\frac{1}{2} \sum \ln \left((2\pi)^n |f_{t|t-1}| \right) - \frac{1}{2} \sum \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1} \\ &= -\frac{nT}{2} \ln(2\pi) - \frac{1}{2} \sum |f_{t|t-1}| - \frac{1}{2} \sum \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1} \end{aligned}$$