

Solution Report – Exercises 3

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Essentially Affine Term Structure Model.

The model consists of the following building blocks:

$$\begin{aligned} 1. X_t &= \mu + \Phi X_{t-1} + \nu_t \quad \nu_t \sim i.i.d.N(0, \Omega) \\ 2. r_t &= \delta_0 + \delta_1 X_t \\ 3. \Lambda_t &= \lambda_0 + \lambda_1 X_t \\ 4. m_{t+1} &= \exp(-r_t - \frac{1}{2} \Lambda_t' \Omega \Lambda_t - \Lambda_t' \nu_{t+1}) \end{aligned}$$

These setting leads to the econometric representation in the state-space general framework:

$$\begin{aligned} y_{t,t+n} &= -\frac{1}{n} (A_n + B_n' X_t) + \varepsilon_{t,t+n} \quad \varepsilon_{t,t+n} \sim N(0, \sigma_n^2) \\ X_t &= \mu + \Phi X_{t-1} + \nu_t \quad \nu_t \sim i.i.d.N(0, \Omega) \end{aligned}$$

The no-arbitrage assumption imposes the following structure on the coefficients of the measurement equation (for $n \geq 1$):

$$\begin{aligned} A_{n+1} &= A_n + B_n' (\mu - \Omega \lambda_0) + \frac{1}{2} B_n' \Omega B_n + A_1 \\ B_{n+1}' &= B_n' (\Phi - \Omega \lambda_1) + B_1' \\ A_1 &= -\delta_0, B_1 = -\delta_1 \\ a_n &= -\frac{1}{n} A_n, b_n = -\frac{1}{n} B_n \end{aligned}$$

Clearly, the recursive coefficients in the measurement equation are highly correlated with the specification of unknown parameters.

In order to implement MLE, we need to:

1. Specify parameter space: $\{\mu, \Phi, \Omega, \delta_0, \delta_1, \lambda_0, \lambda_1\}$.
2. Given the coefficients available for both state equation and measurement equation, calculate likelihood function, either by Chen-Scott method, or Kalman Filter, then implement MLE.

1 Question 1-4 One Factor Model

1.1 Q1. Pricing yields for representative agents with different characteristics

Assume that the underlying factor is the short rate

$$r_t = x_t \implies \delta_0 = 0, \delta_1 = 1$$

There are 3 cases for state factors:

$$\begin{aligned}
\text{Random walk} &: x_t = x_{t-1} + v_t \nu_t \sim i.i.d.N(0, \sigma_v^2) \\
\mu &= 0, \phi = 1, \sigma_v = 0.0006 \\
\text{AR}(1) &: x_t = \mu + \phi x_{t-1} + v_t \nu_t \sim i.i.d.N(0, \sigma_v^2) \\
\mu &= 0.0001, \phi = 0.98, \sigma_v = 0.0006 \\
i.i.d.\text{with mean } \bar{x}, x_t &= \bar{x} + v_t \nu_t \sim i.i.d.N(0, \sigma_v^2) \\
\mu &= \bar{x} = 0.001, \phi = 0, \sigma_v = 0.003
\end{aligned}$$

Meanwhile, for different risk aversion behaviors of agents, we can get based on different specification of risk premium.

Agent one : risk neutral. ($\lambda_0 = \lambda_1 = 0$)

Agent two : risk aversion with constant risk price ($\lambda_0 = -800, \lambda_1 = 0$)

Agent three : risk aversion with time varying risk price ($\lambda_0 = 500, \lambda_1 = -10^5$)

For maturity $n = 2 : 120$, we plot $[A_n, B_n]$ and $[a_n, b_n]$. Plot them for the three different agents, we can find the specification of unknown parameters are crucial for the results of the loadings (see other plots after you run the code). Figure 1 plot the yield curve by using 0.009 as the initial value in different cases.

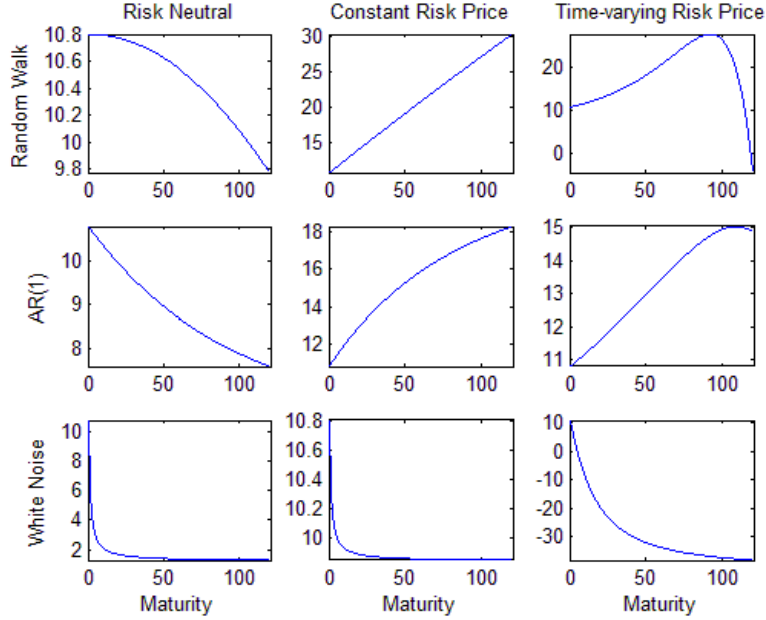


Figure 1: Yield curve, $x_0 = 0.009$

The more interesting story will happen if we use the long-run mean (here we can adopt sample mean) as our initial value, we would get flat term structure curve.

1.2 Q2. One factor (short rate) model estimation in two steps

From question 2 to 4, we will use different approaches to estimate one factor model in which short interest rate is the only factor. The approaches are: 2 step estimation; Chen-Scott method and Kalman Filter (Ex 2). Given the data set of yields from 1975:02-2001:12.

Assume short rate 1-month yield follows an AR(1) process. Given the data set of yields from 1975:02-2001:12, estimate the state space model in two steps:

1. Using OLS estimation to find out the coefficient $\{\mu, \Phi, \sigma_v\}$
2. Given $\{\hat{\mu}, \hat{\Phi}, \hat{\sigma}_v\}$, using MLE to estimate the risk price $\{\lambda_0, \lambda_1\}$. We can simply write down the code for likelihood function and use maximization command in MATLAB to get the optimal solution.

1.3 Q3. One factor model estimation in one step with Chen-Scott Method

1.3.1 Chen-Scott Method

Chen-Scott (1993) provide approach to do maximum likelihood estimation for a multifactor equilibrium model. By using Chen-Scott method, we are able to extract factors under no-arbitrage restrictions. Assume there are K factors in the state equation and that among them, K_2 factors are unobserved. When the number of yields N exceeds number of unobserved factors K_2 , we assume that K_2 yields, y^{NE} , are observed without measurement errors, and that $N - K_2$ yields, y^{ER} , are measured with error u^m . Thus, $X_t = [X_t^o; X_t^u]$

The measurement equation can be written as:

$$y_t = a + b^o X_t^o + b^u X_t^u + b^m u_t^m$$

where $y_t = \begin{bmatrix} y_t^{NE} \\ y_t^{ER} \end{bmatrix}$, $a = \begin{bmatrix} a^{NE} \\ a^{ER} \end{bmatrix}$, $b^o = \begin{bmatrix} b_t^{NE,o} \\ b_t^{ER,o} \end{bmatrix}$, $b^u = \begin{bmatrix} b_t^{NE,u} \\ b_t^{ER,u} \end{bmatrix}$ and

$$b^m = \begin{bmatrix} 0 \\ b_{ER,m} \end{bmatrix}$$

$$X_t^u = (b^u)^{-1} \left(y_t^{NE} - a^{NE} - b_t^{NE,o} X_t^o \right)$$

$$\mathcal{L}(\theta) = \prod_{t=2}^T f(y_t, X_t^o | y_{t-1}, X_{t-1}^o)$$

$$\begin{aligned}
\log(\mathcal{L}(\theta)) &= -(T-1) \log |\det(J)| - \frac{T-1}{2} \log(\det(\Omega)) \\
&\quad - \frac{1}{2} \sum_{t=2}^T (X_t - \mu - \Phi X_{t-1})' \Omega^{-1} X_t - \mu - \Phi X_{t-1} \\
&\quad - \frac{1}{2} \sum_{i=1}^{N-K_2} \log(\sigma_i^2) - \frac{1}{2} \sum_{t=2}^T \sum_{i=1}^{N-K_2} \frac{(u_{t,i}^m)^2}{\sigma_i^2}
\end{aligned}$$

The Jacobian term is: $J = \begin{pmatrix} I_{K-K_2} & 0_{(K-K_2) \times K_2} & 0_{(K-K_2) \times (N-K_2)} \\ b^o & b^u & b^m \end{pmatrix}$

Assume short rate 1-month yield follows an AR(1) process, $\mu = \bar{r}(1 - \phi)$. Given the data set of 11 yields from 1975:02-2001:12.

1.4 Q4. One factor model estimation in one step with Kalman Filter

1.4.1 Kalman Filter

State space model:

$$\begin{aligned}
y_t &= A + BX_t + e_t & \varepsilon_{t,t+n} &\sim N(0, \sigma^2 I) \\
X_t &= \mu + \Phi X_{t-1} + D\varepsilon_t & \varepsilon_t &\sim i.i.d. N(0, \Omega)
\end{aligned}$$

Initial values $X_{0|0}$ and $\Sigma_{0|0}$. Set $X_{0|0}$: unconditional mean and $\Sigma_{0|0}$: the unconditional covariance matrix of stationary X_t , described as:

$$\begin{aligned}
cov(X_t) &= \Phi cov(X_t) \Phi' + D \cdot cov(\varepsilon_t) \cdot D \\
\Sigma_{0|0} &= \Phi \Sigma_{0|0} \Phi' + D \cdot \Omega \cdot D
\end{aligned}$$

Prediction: for $t=1, \dots, T$

$$\begin{aligned}
X_{t|t-1} &= \mu + \Phi X_{t-1|t-1} \\
\Sigma_{t|t-1} &= \Phi \Sigma_{t-1|t-1} \Phi' + D \cdot \Omega \cdot D \\
\eta_{t|t-1} &= y_t - y_{t|t-1} \\
f_{t|t-1} &= B \cdot \Sigma_{t|t-1} \cdot B' + \sigma^2 I
\end{aligned}$$

Updating:

$$\begin{aligned}
K_t &= \Sigma_{t|t-1} B f_{t|t-1}^{-1} \\
X_{t|t} &= X_{t|t-1} + K_t \eta_{t|t-1}, \eta_{t|t-1} = y_t - y_{t|t-1} \\
\Sigma_{t|t} &= \Sigma_{t|t-1} - K_t B \Sigma_{t|t-1}
\end{aligned}$$

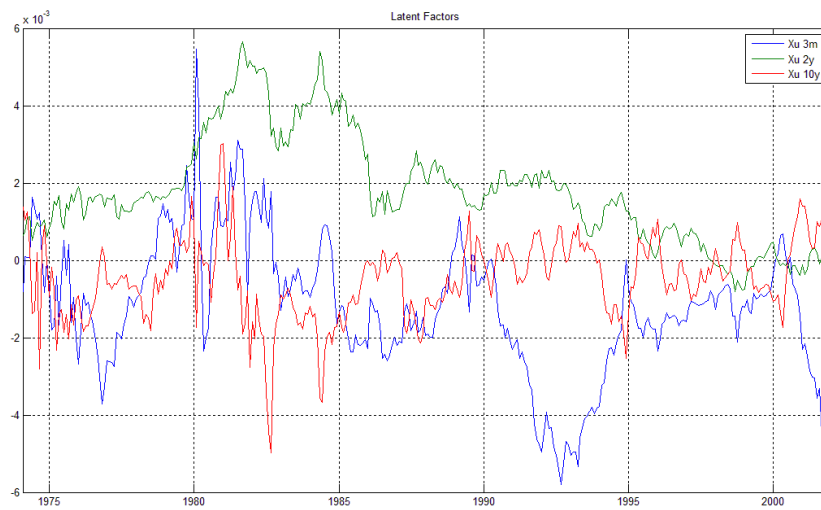
Likelihood function

$$\begin{aligned}
l(\Theta) &= -\frac{1}{2} \sum \ln((2\pi)^n |f_{t|t-1}|) - \frac{1}{2} \sum \eta_{t|t-1}' f_{t|t-1}^{-1} \eta_{t|t-1} \\
&= -\frac{nT}{2} \sum \ln(2\pi) - \frac{1}{2} \sum |f_{t|t-1}| - \frac{1}{2} \sum \eta_{t|t-1}' f_{t|t-1}^{-1} \eta_{t|t-1}
\end{aligned}$$

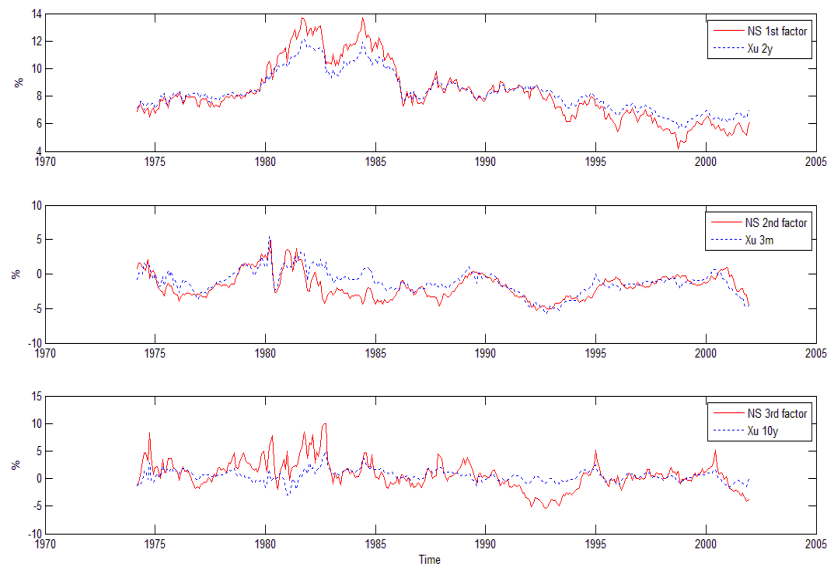
2 Question 5-6 Estimation and forecast for three factor model

2.1 Q5. Three factor model estimation with Chen-Scott method

Generalise the above model with three independent latent factors (Φ, Ω, λ_1 diagonal). Plot the three factors.

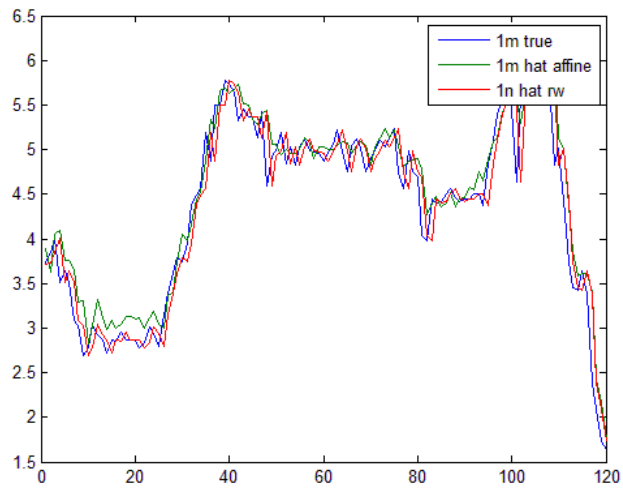


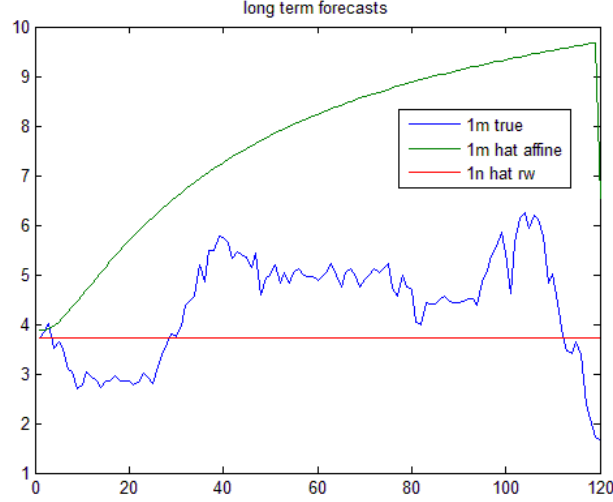
And compare the three factors with the Nelson-Siegel factor.



2.2 Q6. Forecast

Use the model specification in question 5), we make real time forecast for the last ten year period 1992:01-2001:12





If we consider h-step ahead forecasts, particularly for large h, the results of affine model and random-walk are quite different. However both of them are not good forecasting model. For instance, we use the value at 1991:12 as initial value and use the first window estimators to do the long term forecasts. Then we can get the above figure.

3 Question 7 Mixed factors

Assume that the short rate is affine in output gap, inflation and monetary policy shocks. We use unemployment rate, CPI inflation to proxy the first two concepts. $X_t = [u_t, \pi_t, f_t^u]'$. Meanwhile, X_t follows VAR(1) process.

3.1 Additional materials: Quasi Maximum Likelihood Estimates Standard Errors

If the sample size T is sufficiently large, it often turns out that the distribution of the maximum likelihood estimate $\hat{\theta}$ can be well approximated by the following distribution:

$$\hat{\theta} \approx N(\theta_0, T^{-1}\mathcal{I}^{-1})$$

Where θ_0 is the true value of parameter vector and \mathcal{I} is the information matrix.

The second-derivative estimate of the information matrix is

$$\hat{\mathcal{I}}_{2D} = -T^{-1} \frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta \partial \theta'} \Big|_{\theta=\hat{\theta}}$$

A second estimate of the information matrix is called the outer-product estimate

$$\begin{aligned}\hat{\mathcal{I}}_{OP} &= -T^{-1} \sum_{t=1}^T \left[h(\hat{\theta}, y_t) \right] \left[h(\hat{\theta}, y_t) \right]' \\ h(\hat{\theta}, y_t) &= \left. \frac{\partial \log f(y_t | y_{t-1}, y_{t-2}, \dots, \theta)}{\partial \theta} \right|_{\theta=\hat{\theta}}\end{aligned}$$

Although maximum likelihood may be yielding a reasonable estimate of θ , when the innovations are not *i.i.d.*. Gaussian, the standard errors proposed above may no longer be valid. An approximate variance-covariance matrix for $\hat{\theta}$ that is sometimes valid even if the probability density is misspecified is given by

$$E(\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)' \cong T^{-1} \left\{ \hat{\mathcal{I}}_{2D} \hat{\mathcal{I}}_{OP}^{-1} \hat{\mathcal{I}}_{2D} \right\}^{-1}$$

This variance-covariance matrix was proposed by White (1982), who described this approach as *quasi-maximum likelihood estimation*.

As long as we know the information of estimated standard error for each parameter, we can easily to calculate the relevant statistics.