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SUPPLEMENT: ARCH IN OPTION PRICING

OVERVIEW

1) Quick Review of Black-Scholes-Merton Results for IID Normal Returns
2) Smiles, smirks, and term structures of implied volatility
3) Option pricing under skewed and fat-tailed returns
4) Temporal aggregation of ARCH volatility forecasts
5) GARCH option pricing
6) Modelling and predicting the implied volatility surface (IVS)

GENERALITIES

- In this lecture we shall price plain vanilla European call and put options under a variety of Normal, non-Normal, and conditionally heteroskedastic processes
  - A European call option gives the owner the right but not the obligation to BUY a unit of the underlying asset at time t+T (or, T periods from the current period, when the underlying will have price S_{t+T}) at a (strike or exercise) price X
  - A European put option gives the owner the right but not the obligation to SELL a unit of the underlying asset at time t+T (or, T periods from the current period, when the underlying will have price S_{t+T}) at a (strike or exercise) price X
  - We call $c_t(X,T)$ the price of a European call, and $p_t(X,T)$ the price of a put
    - American calls and puts are similar, but they can be exercised at any time before the expiration date T

Generalities: Black & Scholes Formula

- We shall only deal with European options
  - At maturity, the payoff of a call option is $c_t(X,0) = \max[S_{t+T} - X, 0]$ while the payoff of a put option is $p_t(X,0) = \max[X - S_{t+T}, 0]$
- As you know very well, when (daily, continuously compounded) returns on the underlying asset are Gaussian IID $N(\mu,\sigma^2)$, we have that:

$$r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) \sim N(\mu,\sigma^2)$$
$$r_{t+1:t+T} \equiv \ln(S_{t+T}) - \ln(S_t) \sim N(T\mu,T\sigma^2)$$

  - We are either ignoring all dividends and cash distributions or assuming the underlying price has been adjusted by deducting the present expected value of such cash flows
  - Also assume the riskless interest rate (r) is constant
- Under IID normal returns, applying the risk neutral evaluation principle, we obtain that:

$$c^{BS}_t(X,T) = S_t\Phi(d) - \exp(-rT)X\,\Phi(d - \sigma T^{1/2})$$

Generalities: Black & Scholes Formula

- Here d is a parameter defined as

Expected rate of cash flow per day

$$d \equiv [\sigma T^{1/2}]^{-1}[\ln(S_t/X)+T(r - q + 0.5\,\sigma)]$$

and $\Phi(d)$ is the CDF of a standard normal N(0,1) evaluated at d

- Notice that σ is expressed in daily terms

- BS formula can be interpreted as:

  $c^{BS}_t(X,T)$ = {Risk neutral expected value of the underlying asset conditioning on exercise} - {Risk neutral expected cost of exercise} = $S_t\Phi(d) - \exp(-rT)X\,\Phi(d - \sigma T^{1/2})$

  - $\Phi(d - \sigma T^{1/2})$ is the risk-neutral probability of exercise at maturity
  - $\Phi(d)$ is delta, the sensitivity of the option price to changes in S_t

- By the no arbitrage condition called put-call parity,

$$S_t + p^{BS}_t(X,T) = c^{BS}_t(X,T) + \exp(-rT)$$

one shows that $p^{BS}_t(X,T) = \exp(-rT)X\Phi(\sigma T^{1/2} - d) - S_t\Phi(-d)$

  - Please review the proof of the put-call parity at p. 125 of Christoffersen

Generalities: Black & Scholes Formula

- What is the financial econometrics of BS formula?
- Notice that the price of a call option may be written as $c^{BS}_t(X,T,S_t,r,q,\sigma)$, where X and T are stated in the contract
  - S_t and r are easy to retrieve from commonly available data sources, where r usually being a riskless rate that (either exactly or by interpolation) matches the maturity T
    - For instance, LIBOR USD rates; appropriate notation is then r(t,T)
  - q is a bit more problematic and sometimes is derived from historical data, other times it is estimated
- The only estimable parameter is indeed the (daily) volatility σ: how do you that?
  - For instance given a cross-section at time t on prices on options along different strikes (X_1, X_2, …, X_J) and maturities (T_1, T_2, …, T_N), one will estimate σ by solving the Nonlinear Least Squares (NLS) problem:

The Econometrics of Black & Scholes

$$\hat{\sigma}_t^{NLS} \equiv \arg\min_{\sigma} \left\{ \frac{1}{JN} \sum_{j=1}^{J} \sum_{n=1}^{N} \left[c^{BS}(X_j, T_n, S_t, r(t,T_n), q(t,T_n), \sigma) - c^{mkt}(X_j, T_n) \right]^2 \right\}$$

- Basically one minimizes the MSE of the pricing errors at time t
- However, nothing compels you to compute such a NLS estimate day by day. E.g., given a long sample of data,

$$\hat{\sigma}_t^{NLS} \equiv \arg\min_{\sigma} \left\{ \frac{1}{\Upsilon} \sum_{t=1}^{N} \sum_{j=1}^{J} \sum_{n=1}^{N} \left[c^{BS}(X_{tj}, T_{tn}, S_t, r(t,T_n), q(t,T_{tn}), \sigma) - c^{mkt}(X_{tj}, T_{tn}) \right]^2 \right\}$$

$$\Upsilon \equiv \sum_{t=1}^{N} \sum_{j=1}^{J} \sum_{n=1}^{N} 1$$

- This estimation problem is very different from computing BS implied volatility
  - Implied volatility corresponds to the solution of the NLS problem contract by contract, one option at the time
  - But this one (nonlinear) equation in one unknown, $\sigma^{IV}_t(X,T,S_t,r(t,T),q(t,T), c^{mkt}_t)$ which is simply the volatility that

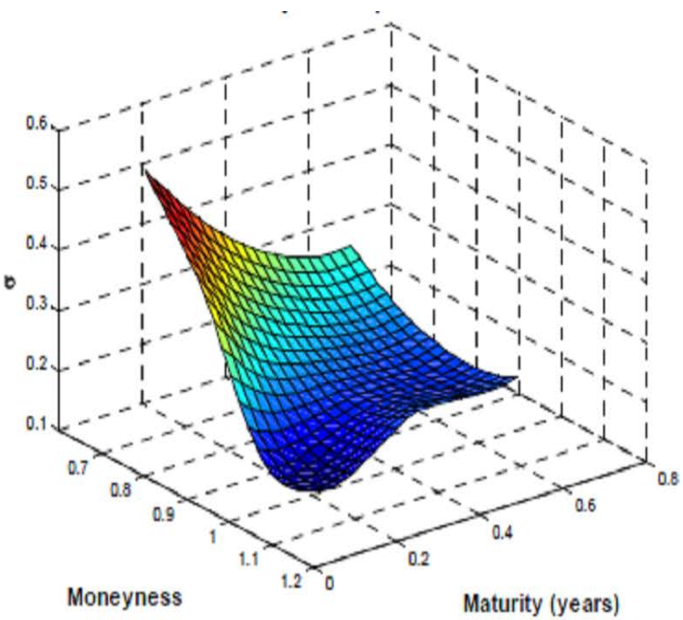
Implied Volatilities in Black & Scholes Formula

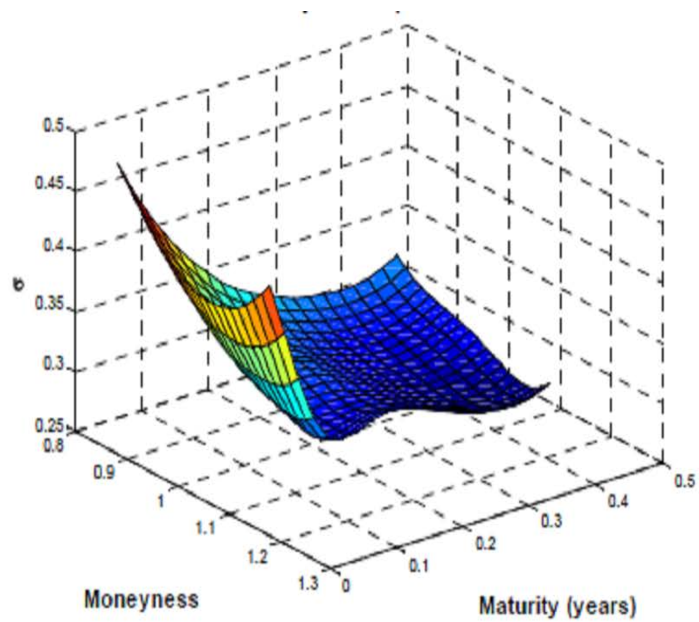
solves the equation:

$$c^{BS}_t(X, T, S_t, r, q, \sigma^{IV}_t) = c^{mkt}_t(X, T)$$

- Does the solution exist? Yes, because $c^{BS}_t(X, T, S_t, r, q, \sigma)$ is monotone increasing in σ (please make sure you prove that)
- Basically, σ^{IV}_t is obtained by "inverting" BS formula
  - This is done using a numerical equation solver

- Under the normality assumption of BS, σ^{IV}_t should be constant as postulated in BS model
- In particular, it should not depend on either strike or maturity





SMIRKS AND SMILES IN IMPLIED VOLATILITIES

- The data reveal enormous deviations of IVs in the options data from what BS implies
  - Equity options typically contain "smirks" (also called skews)
  - Forex options contain "smiles"
  - There are also term structure of IVs
  - We sometimes speak of an implied volatility surface (IVS)
- This is evidence of misspecifications in BS: this is where our story begins
- We will explore two modifications of BS that capture/fit the existence of an IVS:
  - Pricing options under skewed and thick-tail process for the returns on the underlying asset
  - Pricing options under ARCH
- These two avenues match our discussion on how it would be possible to capture non-normality

Option Pricing with Skewness & Excess Kurtosis

- Definitions of skewness and excess kurtosis:

$$SK_1 \equiv \frac{E[(r_{t+1}-\mu)^3]}{\sigma^3} \qquad xK_1 \equiv \frac{E[(r_{t+1}-\mu)^4]}{\sigma^4} - 3$$

  - These are one-period ahead definitions
  - Skewness is an indicator of asymmetry of the (unconditional) distribution of returns (it is 0 under a normal)
  - Excess kurtosis measures the differential thickness of the tails of the (unconditional) distribution of returns vs. a Gaussian
    - It can be both positive or negative but -3 is the lower bound
- Assume that returns on the underlying asset remain IID, but they are no longer normal, so that SK $\neq$ 0 and *x*K $\neq$ 0
- From IID-ness, we have that at horizon T:

$$SK_T = \frac{SK_1}{\sqrt{T}} \qquad xK_T = \frac{xK_1}{T}$$

  - This means that both SK and *x*K $\rightarrow$ 0 as T $\rightarrow \infty$

Option Pricing with Skewness & Excess Kurtosis

- At this point consider the (unconditional) density of the standardized T-horizon returns:

$$\epsilon_{t+1:t+T} \equiv [\ln(S_{t+T}) - \ln(S_t) - T\mu]/[T^{1/2}\sigma]$$

  - Please check that by definition, SK(r) = SK(ϵ) and xK(r) = xK(ϵ)
- Under mild assumptions, take a fourth-order Gram-Charlier expansion around the normal density function $\phi(\epsilon_{t,T})$, where $\epsilon_{t,T} \equiv \epsilon_{t+1:t+T}$:

Gaussian N(0,1) density

$$f(\epsilon_{t,T}) = \phi(\epsilon_{t,T}) - (1/6)SK_T D^3\phi(\epsilon_{t,T}) + (1/24)xK_T D^4\phi(\epsilon_{t,T})$$

  - The terms $D^3\phi(\epsilon_{t,T})$ and $D^4\phi(\epsilon_{t,T})$ equal:

$$D^3\phi(z) \equiv (3z - z^3)\phi(z) \qquad D^4\phi(z) \equiv (z^4 - 6z^2 + 3)\phi(z)$$

  - These are modified Gaussian density functions obtained by taking the jth differential of a Gaussian density: e.g., $D^3\phi(z) \equiv d^3\phi(z)/dz^3 = (2\pi\sigma^2)^{-1/2}(3z - z^3)\exp(-z^2/\sigma^2)$
  - Same logic as a Taylor series expansion, but for densities

Option Pricing with Skewness & Excess Kurtosis

- A Gram-Charlier (CG) expansion may approximate arbitrarily well a rather wide class of densities
- Importantly, it nests a standard normal density of asset returns, in the sense that when $SK_T = xK_T = 0$, $f(\epsilon_{t,T}) = \phi(\epsilon_{t,T})$
- In general, finding the price of a call under a CG expansion is a headache (but possible)
- However, if ready to assume that $\sigma^3 \cong \sigma^4 = 0$ (i.e. volatility is small enough that raising it to high powers it vanishes), then the following simple approximation holds:

$$c^{GC}_t(X,T) = c^{BS}_t(X,T) + (1/6)S_t\phi(d)\sigma SK_1(2T^{1/2}\sigma - d) + \\ - (1/24)S_t\phi(d)\sigma(xK_1/T^{1/2})[1 - d^2 + 3dT^{1/2}\sigma - 3T\sigma^2]$$

where $d \equiv [\sigma T^{1/2}]^{-1}[\ln(S_t/X)+T(r - q + 0.5\,\sigma)]$

  – Clearly, $SK_1 = xK_1 = 0$, $c^{GC}_t(X,T) = c^{BS}_t(X,T)$
  – The CG formula depends on SK_1 and xK_1 because $SK_T = SK_1/T^{1/2}$

Option Pricing with Skewness & Excess Kurtosis

$xK_T = xK_1/T$ have been plugged in already

- By put-call parity it is then easy to compute $p^{CG}_t(X,T) = c^{CG}_t(X,T) + \exp(-rT) - S_t$

- How do you implement this model empirically?
- The logical approach is identical to what we have reviewed already
  - Given a cross-section at time t on prices on options along different strikes ($X_1, X_2, \ldots, X_J$) and maturities ($T_1, T_2, \ldots, T_N$), one will estimate σ, SK_1 and xK_1 by NLS:

$$\hat{\sigma}^{NLS}, \widehat{SK_1^{NLS}}, \widehat{xK_1^{NLS}} \equiv \underset{\sigma, SK_1, xK_1}{\arg\min} \left\{ \frac{1}{JN} \sum_{j=1}^{J} \sum_{n=1}^{N} [c^{CG}(X_j, T_n, S_t, r(t,T_n), q(t,T_n), \sigma, SK_1, xK_1) + -c^{mkt}(X_j, T_n)]^2 \right\}$$

  - What about implied volatility? Technically, you understand that the concept stops making sense because one should speak of the triplet IV, implied skewness, and implied kurtosis

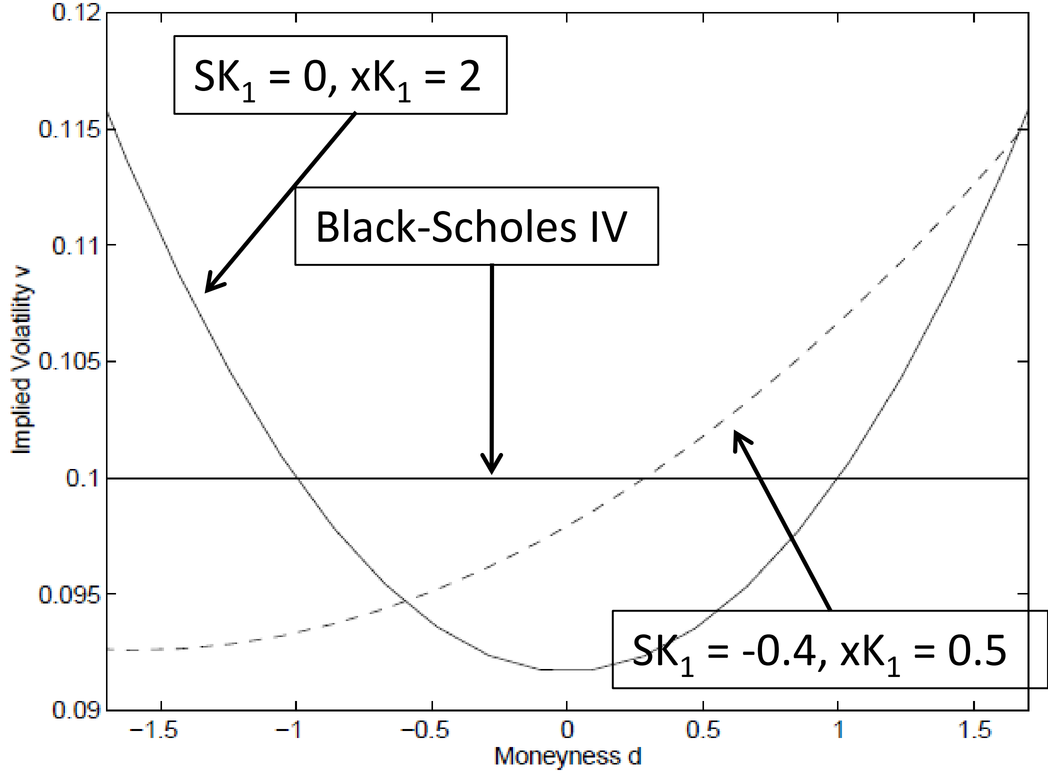
Option Pricing with Skewness & Excess Kurtosis

- However, the concept of IV is so "ingrained" in the mentality of option traders, that another approximation is often used:

$$\sigma_{GC}^{IV} \simeq \sigma^{IV}\left[1-\frac{SK_1}{6\sqrt{T}}d-\frac{xK_1}{24T}(1-d^2)\right]$$

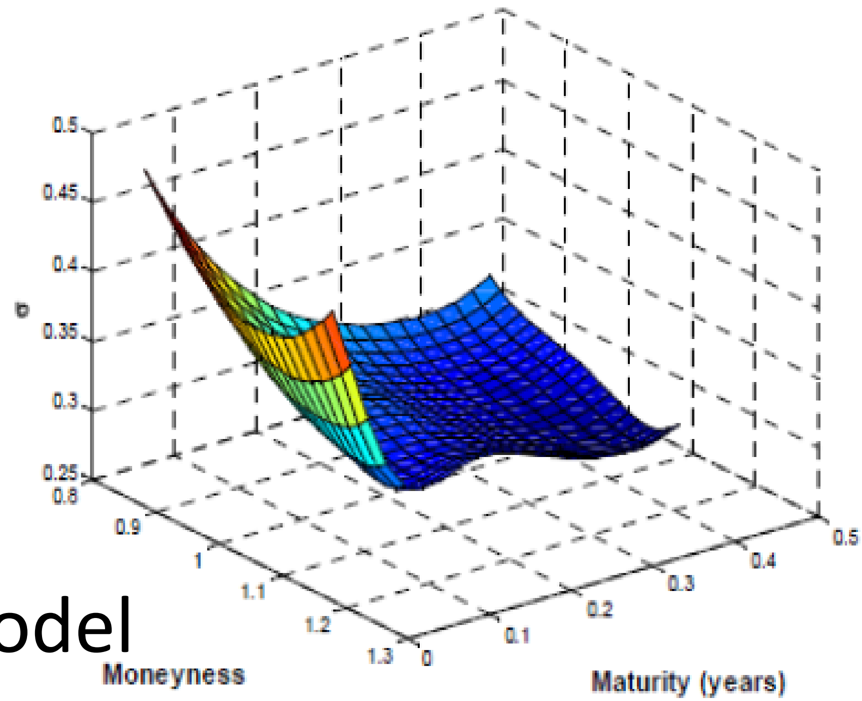
  where σ^{IV} is the standard BS IV, and SK and *x*K are values obtained from the historical distribution of asset returns

- The GC model may go a long way to explain the existence of a IVS
- The plot shows that smiles originate from the case of zero skewness but positive excess kurtosis
- Smirks from the presence of negative skewness (here T=1 month, σ = 10% p.a.)



ARCH Option Pricing: Introduction

- The GC model may do a great job at a given time t because it is very flexible
- However, we know that not only return densities are non-normal, but that such non-normalities are predi-ctable over time (e.g. variance)
  - Such predictability is probably responsible for the existence of a term structure of IV
- Solution: try and use our GARCH models for predictable variance in underlying returns to also generate:
  - Multi-period volatility forecasts
  - Option prices
- Two cases: (i) Ad-hoc, extended BS-style pricing; (ii) Structural model



Temporal Aggregation in GARCH Models

- To deal with ad-hoc case, we have to first review a few issues of time aggregation of GARCH variance forecasts
- Scaling is often done by taking estimated variance (st. deviation) and multiplying it by the horizon h ($h^{1/2}$)
  - Formally, this means $\sigma^2_{(h)t} = h\sigma^2_{(1)t}$ and $\sigma_{(h)t} = [h\sigma_{(1)t}]^{1/2}$
- But this simple way of scaling works in IID environments but fails otherwise
  - In fact, comparing forecasts of long-run variance obtained from these simple rules with actual long-run variances is often used as a test for whether returns are IID or not
  - These are the so-called variance ratio tests: under IID returns, $\sigma^2_{(h)t}/h\sigma^2_{(1)t} = 1$ for all *ts*
- We know that the presence of ARCH implies that returns are no longer IID: therefore the simple scaling must fail
- Yes, but how should we proceed instead?

TEMPORAL AGGREGATION IN GARCH MODELS

– Write a standard GARCH(1,1) model as:

$$\sigma^2_{(1)t} = \alpha_0 + \alpha_1\epsilon^2_{t-1} + \beta_1\sigma^2_{(1)t-1}$$

- Drost and Nijman (1993, ECMA) study the temporal aggregation of GARCH(1,1) processes and obtain a conversion formula of 1-day variance to *h*-day volatility completely different from the IID scaling formula:

$$\sigma^2_{(h)t} = \alpha_0(h) + \alpha_1(h)\, e^2_{(h)t-1} + \beta_1(h)\,\sigma^2_{(h)t-1}$$

$$\alpha_0(h) = h\alpha_0 \frac{1-(\beta_1+\alpha_1)^h}{1-(\beta_1+\alpha_1)}$$

$$\alpha_1(h) = (\beta_1+\alpha_1)^h - \beta_1(h)$$

and $|\beta_1(h)| < 1$ solves the quadratic equation

$$\frac{\beta_1(h)}{1+\beta_1^2(h)} = \frac{a(\beta_1+\alpha_1)^h - b}{a\left(1+(\beta_1+\alpha_1)^{2h}\right) - 2b}$$

Temporal Aggregation in GARCH Models

$$a = h(1-\beta_1)^2 + 2h(h-1)\frac{(1-\beta_1-\alpha_1)^2(1-\beta_1^2-2\beta_1\alpha_1)}{(k-1)\left(1-(\beta_1+\alpha_1)^2\right)}$$
$$+4\frac{\left(h-1-h(\beta_1+\alpha_1)+(\beta_1+\alpha_1)^h\right)(\alpha_1-\beta_1\alpha_1(\beta_1+\alpha_1))}{1-(\beta_1+\alpha_1)^2}$$
$$b = (\alpha_1-\beta_1\alpha_1(\beta_1+\alpha_1))\frac{1-(\beta_1+\alpha_1)^{2h}}{1-(\beta_1+\alpha_1)},$$

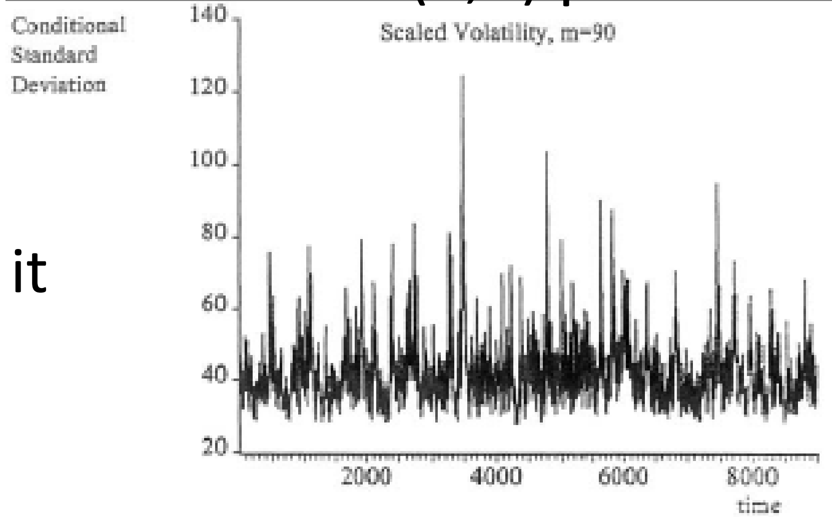
where *k* is the one period kurtosis of returns

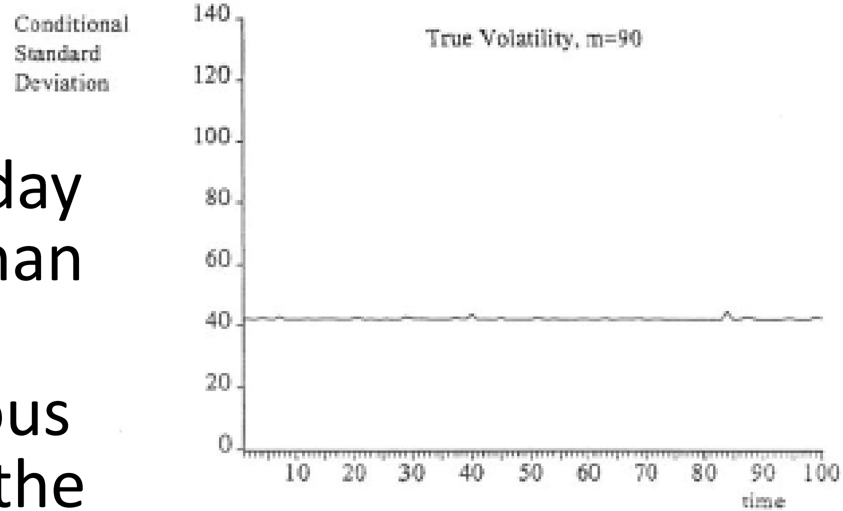
- This is the correct conversion formula of 1-day volatility to h-day variance: a bit more complicated than $\sigma^2_{(h)t} = h\sigma^2_t$ or
$$\sigma^2_{(h)t} = h\alpha_0 + h\alpha_1\epsilon^2_{t-1} + h\beta_1\sigma^2_{(1)t-1}$$
- As $h \rightarrow \infty$, these formulas reveal that $\alpha_1(h)$ and $\beta_1(h)$ go to 0: temporal aggregation produces gradual disappearance of volatility fluctuations, $\sigma^2_{(\infty)t} = h\alpha_0$
- The standard $\sigma^2_{(h)t} = h\sigma^2_{(1)t}$ scaling, in contrast, magnifies

TEMPORAL AGGREGATION IN GARCH MODELS

volatility fluctuations

– Consider one example: we parameterize a GARCH(1,1) process to be realistic for daily returns by setting $\alpha_1 = 0.1$ and $\beta_1 = 0.85$, which are typical point estimates
  - The choice of α_0 is arbitrary, we set it equal to 1
– The GARCH(1,1) process $\sigma^2_{(1)t} = 1 + 0.1\epsilon^2_{t-1} + 0.85\sigma^2_{(1)t-1}$ governs 1-day volatility
  - This is shown in the plot on top
– We obtain the second (correct) 90-day volatility by applying the Drost-Nijman formula
– Scaling in general produces erroneous conclusions of large fluctuations in the





TEMPORAL AGGREGATION IN GARCH MODELS

conditional variance of long-horizon returns, when in fact the opposite is true
  - Moreover the scaled volatility estimates are sometimes too high and sometimes too low
  - If estimate a short-horizon GARCH, compute long-horizon volatilities by the D&N formula
- Unfortunately, D&N's formula only applies to the plain vanilla GARCH(1,1) case
  - If the estimated model is not in the class considered by D&N, estimate a h-day model, in the sense that returns should be long-horizon h-day returns
  - Notice that unfortunately, if the short-horizon model is not correctly specified, the conversion formula can give very different values from what one gets by applying the D&N formula

AD-HOC OPTION PRICING

- A GARCH model is used to forecast the path of conditional variance over the life of the option, (t,T)
- Such a sum of variances over time is then plugged into BS formula to obtain the option price
- Why is such an approach "ad hoc" (which is a derogatory remark to make)?
- Simple: under a GARCH model, we know that the underlying asset returns will not be normal, and such in general BS does not obtain
  - The expression for the (total, sum) variance forecast k-periods ahead can be used in order to price options, representing the expected variance during the life of the option:

$$(1/T) \sum_{k=0}^{T-1} \mathbb{E}(\sigma^2_{t+k} \,|\, \mathcal{F}_t)$$

  - This quantity is also called integrated variance

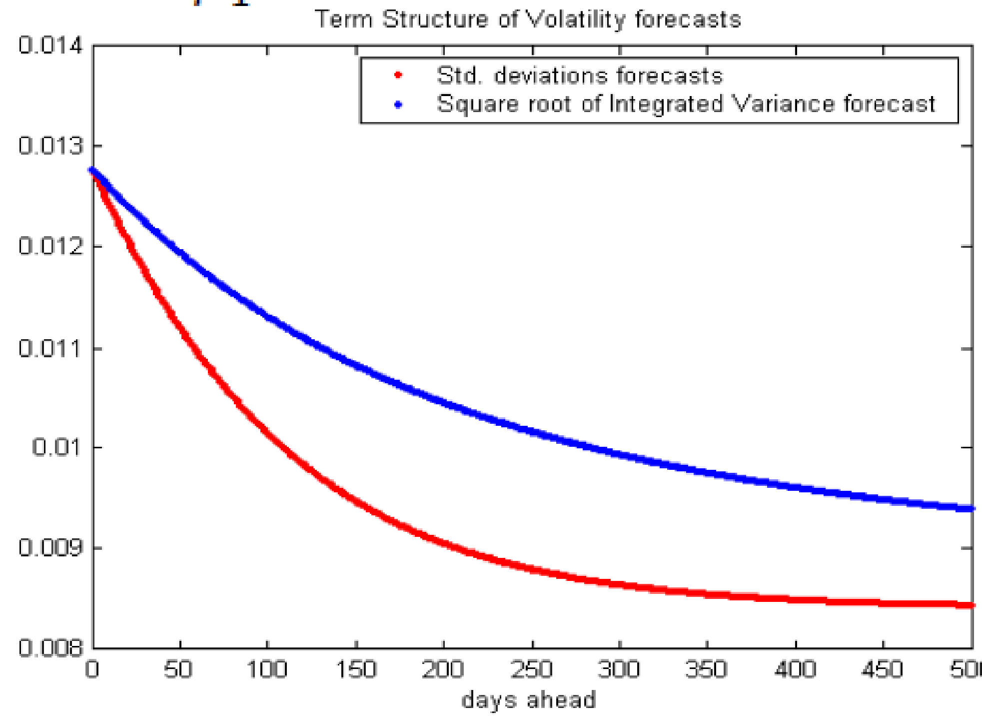
Ad-Hoc Option Pricing

– This expression can be easily computed in a GARCH(1,1) model
– We just replace the expected values previously computed and we obtain:

$$\sigma^2_{AV} = \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}(\sigma^2_{t+k}|\mathcal{F}_t) = \bar{\sigma}^2 + (\sigma^2_{t+1} - \bar{\sigma}^2)\frac{1-(\alpha_1+\beta_1)^{T-1}}{1-(\alpha_1+\beta_1)}$$

$$\bar{\sigma}^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1}$$

– For instance, even if the point forecasts of volatility decline quickly, their cumulant has a flatter slope
– At this point:
$c^{BS}_t(X,T) = S_t\Phi(d) - \exp(-rT)X\,\Phi(d - \sigma_{AV}T^{1/2})$
where $d \equiv [\sigma_{AV}T^{1/2}]^{-1}[\ln(S_t/X)+ +T(r - q + 0.5\sigma_{AV})]$



NGARCH(1,1) Structural Option Pricing

- The model is structural in the sense that the option price is derived within a consistent risk-neutral pricing framework, introduced by Heston and Nandi (2000, RFS)
- Suppose the process for the returns on the underlying is:

$$r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r + \lambda\sigma_{t+1} - 0.5\sigma^2_{t+1} + \sigma_{t+1}v_{t+1}$$

$$\text{with } v_{t+1} \sim N(0,1),\ \sigma^2_{t+1} = \alpha_0 + \alpha_1\sigma_t(v_t - \theta)^2 + \beta_1\sigma^2_t$$

  - You will recognize a special NGARCH(1,1) model
  - λ is the unit risk premium on volatility, an obvious ARCH-M term, and $-0.5\sigma^2_{t+1}$ is a Jensen's inequality correction terms
    - Please check: while it is obvious that $E_t[r_{t+1}] = r + \lambda\sigma_{t+1} - 0.5\sigma^2_{t+1}$, this term make it such that $E_t[S_{t+1}/S_t] = \exp(r + \lambda\sigma^2_{t+1})$
- Under the risk neutral measure (*), we know that the process of r_{t+1} must be such that:

$$E^*_t[S_{t+1}/S_t] = \exp(r) \qquad Var^*_t[r_{t+1}] = \sigma^2_{t+1}$$

- Consider the following variation of the original process:

NGARCH(1,1) STRUCTURAL OPTION PRICING

(*) $r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r - 0.5\sigma^2_{t+1} + \sigma_{t+1}v^*_{t+1}$
with $v^*_{t+1} \sim N(0,1)$, $\sigma^{*2}_{t+1} = \alpha_0 + \alpha_1\sigma^2_t (v^*_t - \lambda - \theta)^2 + \beta_1\sigma^2_t$

– In essence, the conditional mean function has changed and v_{t+1} has been replaced by v^*_{t+1}
– Let's check now that such a new process is the risk-neutralized version of the original NGARCH(1,1):

$$E^*_t[S_{t+1}/S_t] = E^*_t[\exp(r_{t+1})] = E^*_t[\exp(r - 0.5\sigma^2_{t+1} + \sigma_{t+1}v^*_{t+1})]$$
$$= \exp(r - 0.5\sigma^2_{t+1})E^*_t[\exp(\sigma_{t+1}v^*_{t+1})]$$
$$= \exp(r - 0.5\sigma^2_{t+1})\exp(0.5\sigma_{t+1}) = \exp(r)$$

$$Var^*_t[r_{t+1}] = E^*_t[\sigma^{*2}_{t+1}] = E^*_t[\alpha_0 + \alpha_1\sigma^2_t (v^*_t - \lambda - \theta)^2 + \beta_1\sigma^2_t]$$
$$= E^*_t[\alpha_0 + \alpha_1(r_t - r - 0.5\sigma^2_t - \lambda\sigma_t - \theta\sigma_t)^2 + \beta_1\sigma^2_t]$$

$= \sigma_t v_t$

$$= E^*_t[\alpha_0 + \alpha_1\sigma_t (v_t - \theta)^2 + \beta_1\sigma^2_t] = \sigma^2_{t+1}$$

– Here we have exploited the fact that if $v^*_{t+1} \sim N(0,1)$, then $E^*_t[\exp(\sigma_{t+1}v^*_{t+1})] = \exp(0.5\sigma_{t+1})$ because of log-normality

NGARCH(1,1) Structural Option Pricing

  - But this establishes that the process (*) has been risk-neutralized, as desired
- So what? On the one hand, this is good news because the process (*),

$$r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r - 0.5\sigma^2_{t+1} + \sigma_{t+1}v^*_{t+1}$$

$$\text{with } v^*_{t+1} \sim N(0,1),\ \sigma^{*2}_{t+1} = \alpha_0 + \alpha_1\sigma^2_t(v^*_t - \lambda - \theta)^2 + \beta_1\sigma^2_t$$

is now risk neutral and can be used to compute the value of a call option simply as:

$$c^{HN}_t(X,T) = \exp(-rT)E^*_t[\max(S_{t+T} - X, 0)]$$

- On the other hand, there is no closed-form solution and one computes prices by simulation (Duan, 1995, MF)
  - Please see in Christoffersen (p. 136) what is meant for that
  - Given S_t and σ^2_t, one needs to simulate M paths of risk-neutralized returns $r^m_{t+j} = r - 0.5\sigma^2_{m,t+j} + \sigma_{m,t+j}v^*_{m,t+j}$, m=1, 2, ..., M j=1, 2, ..., T and $S^m_{t+j} = S^m_{t+j-1}(1+r^m_{t+j})$, where $v^*_{m,t+j} \sim N(0,1)$

NGARCH(1,1) Structural Option Pricing

- There are two problems with such an approach: (i) it is slow because of the need to simulate M paths over T periods: (ii) unless $M \to \infty$, simulations introduce further errors that do not derive only from misspecification or measurement problems

- However, HN show that if slightly changes the conditional model to become

$$r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r + \lambda\sigma^2_{t+1} + \sigma_{t+1}v_{t+1}$$

$$\text{with } v_{t+1} \sim N(0,1),\ \sigma^2_{t+1} = \alpha_0 + \alpha_1(\sigma_t v_t - \theta)^2 + \beta_1\sigma^2_t$$

then BS-style closed form solutions are available

  - In this case, the risk-neutralized process is

$$r_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r - 0.5\sigma^2_{t+1} - \lambda\sigma^2_{t+1} + \sigma_{t+1}v^*_{t+1}$$

$$\text{with } v^*_{t+1} \sim N((\lambda - 0.5)\sigma^2_{t+1},1),\ \sigma^{*2}_{t+1} = \alpha_0 + \alpha_1(\sigma_t v_t - \theta - \lambda - 0.5)^2 + \beta_1\sigma^2_t$$

which clearly does not work

  - However, if $\lambda = -0.5$, then $r_{t+1} = r + \sigma_{t+1}v^*_{t+1}$

$$\text{with } v^*_{t+1} \sim N(0,1),\ \sigma^{*2}_{t+1} = \alpha_0 + \alpha_1(\sigma_t v_t - \theta)^2 + \beta_1\sigma^2_t$$

NGARCH(1,1) STRUCTURAL OPTION PRICING

then it is easy to check that $E^*_t[S_{t+1}/S_t] = \exp(r)$ and $Var^*_t[r_{t+1}] = \sigma^2_{t+1,}$ the process has been risk-neutralized

- In this case, HN's closed form solution is:

$$c^{HN}_t(X,T) = S_t P_1 - \exp(-rT) X P_2$$

where P_1 and P_2 can be spotted from the formula:

$$c^{HN}_t(X,T) = S_t \left(1/2 + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{X^{-i\varphi} f^*(i\varphi+1)}{i\varphi f^*(1)}\right] d\varphi\right) - Xe^{-rT}\left(1/2 + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{X^{i\varphi} f^*(i\varphi)}{i\varphi}\right] d\varphi\right)$$

  - Further details can be found in Christoffersen, but it is pointless unless you are about ready to code this formula up
    - Some technical slang: HN's paper is interesting because (under $\lambda = -0.5$) it is based on the conditional characteristic function and the link btw. CCF and the conditional density function (inverse Fourier transform) $$\int_A^\infty p(x)\,dx = 1/2 + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\varphi A) f(i\varphi)}{i\varphi}\right] d\varphi$$

NGARCH(1,1) Structural Option Pricing

- Last remark is: when T → 0, $c^{HN}_t(X,T) \to c^{BS}_t(X,T)$, i.e., the GARCH option pricing formula is LOCALLY Black-Scholes
- How do you deal with empirical implementation?
- In the usual way: in this case the unknown parameters are α_0, α_1, θ, β_1, and in some cases also λ (when it is not constrained to equal -0.5). For instance:

$$\hat{\alpha}_0^{NLS}, \hat{\alpha}_1^{NLS}, \hat{\beta}_1^{NLS}, \hat{\theta}^{NLS}, \hat{\lambda}^{NLS} \equiv \underset{\alpha_0,\alpha_1,\beta_1,\theta,\lambda}{\arg\min} \left\{ \frac{1}{JN} \sum_{j=1}^{J} \sum_{n=1}^{N} [c^{HN}(X_j, T_n, S_t, r, q, \alpha_0, \alpha_1, \beta_1, \theta, \lambda) + -c^{mkt}(X_j, T_n)]^2 \right\}$$

- Do markets price options this way? Yes and no…
  - Yes, there is a keen understanding that BS is a biased formula that systematically over-prices out of the money puts and under-prices out of the money calls
  - No, a number of trading desks refrain from using either computationally-intensive methods or from making tight parametric assumptions

IVS Empirical Models

- The idea of empirical IVS models is that most of the time the IVS is sufficiently stable that simple, static parametric models of its shape at time t may be useful to forecast its shape at time t+1 (see Dumas et al., 1998)
  - Also called "practitioners' Black-Scholes" (PBS)
  - How do you capture the IVS shape at time t? There are many alternative ways to proceed, for instance through a simple spline regression (smoother):

    $$\sigma^{IV}(X_{tj},T_{tn}) = a_0 + a_1(S_t/X_{tj}) + a_2(S_t/X_{tj})^2 + a_3(T_{tn}/365) + a_4(T_{tn}/365)^2 + \\ + a_5(S_t/X_{tj})(T_{tn}/365) + e_{t,jn} \qquad j=1,2,\ldots,J, \quad t=1,2,\ldots,T$$

    where the $\sigma^{IV}(X_{tj},T_{tn})$s represent BS IVs over a cross section of strikes and maturities
  - At this point, the fitted value from the regression are used to forecast option prices by plugging them into BS formula
  - It is also possible to estimate a_0, a_1, a_2, a_3, a_4, and a_5 by using a NLS approach similar to what we have used above:

IVS Empirical Models

$$\left\{\hat{a}_j^{NLS}\right\}_{j=0}^{5} \equiv \underset{\left\{a_j^{NLS}\right\}_{j=0}^{5}}{\arg\min} \left\{ \frac{1}{\Upsilon} \sum_{t=1}^{N} \sum_{j=1}^{J} \sum_{n=1}^{N} [c^{PBS}(X_{tj}, T_{tn}, S_t, r, q, \hat{\sigma}^{PBS}(X_{tj}, T_{tn})) - c^{mkt}(X_j, T_n)]^2 \right\}$$

$$\hat{\sigma}^{PBS}(X_{tj}, T_{tn}) = \hat{a}_0 + \hat{a}_1 \frac{S_t}{X_{tj}} + \hat{a}_2 \left(\frac{S_t}{X_{tj}}\right)^2 + \hat{a}_3 \left(\frac{T_{tn}}{365}\right) + \hat{a}_4 \left(\frac{T_{tn}}{365}\right)^2 + \hat{a}_5 \left(\frac{S_t}{X_{tj}}\right)\left(\frac{T_{tn}}{365}\right)$$

– Other researchers have experimented with a simpler version:

$$\left\{\hat{a}_j^{NLS}\right\}_{j=0}^{5} \equiv \underset{\left\{a_j^{NLS}\right\}_{j=0}^{5}}{\arg\min} \left\{ \frac{1}{\Upsilon} \sum_{t=1}^{N} \sum_{j=1}^{J} \sum_{n=1}^{N} [\hat{\sigma}^{PBS}(X_{tj}, T_{tn}) - \sigma^{IV}(c^{mkt}(X_j, T_n))]^2 \right\}$$

– The typical empirical finding is that PBS performs MUCH better than BS and considerably better than many other option pricing models. It may "tie" with ARCH option pricing, SVJM etc.

– Guidolin and Goncalves (2006, JBus) have shown that there is more: the very daily coefficients estimates of a_{0t}, a_{1t}, a_{2t}, a_{3t}, a_{4t}, and a_{5t} are strongly predictable over time, e.g. according to

$$a_{kt} = b_{k1} + b_{k2}a_{kt-1} + \xi_{kt} \qquad k=0,\ldots,5 \quad t=1,\ldots,N$$

and this may even improve the performance of a dynamic PBS

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