Just Enough or All: Selling a Firm

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April 9, 2015

Abstract

We consider the problem of selling a firm to a single buyer. The magnitude of the post-sale cash flow rights as well as the benefits of control are the buyer's private information. In contrast to research that assumes the private information of the buyer is one-dimensional, the optimal mechanism is a menu of tuples of cash-equity mixtures. We provide sufficient conditions for an optimal mechanism to attain one of the following forms: *i)* a take-it or leave-it offer for the smallest fraction of the company that facilitates the transfer of control, or *ii)* a take-it or leave-it offer for *all* the shares of the company. The first case prevails when the seller wants to screen more precisely with respect to the private benefits, the latter when the seller wants to screen finely with respect to cash flows.

JEL Code: D82, D86.
Keywords: Multidimensional mechanism design, negotiated block trades, private benefits, privatization, takeovers, bilateral trade, asymmetric information, cash-equity offers.

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[§]We would like to thank Josh Cherry, Wioletta Dziuda, Boyan Jovanovich, Samuel Lee, Qingmin Liu, Johannes Horner and Yeon-Koo Che for their useful comments. Nenad gratefully acknowledges the support of ERC through Advanced Grant 295835.

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1 Introduction and Related Literature

A fundamental question in economics is how the terms of trade are reached in exchange of assets. Although there has been a substantial amount of research that relates the form of negotiations to agreement outcomes for assets with consumption value, much less is known about the optimal mixture of equity and cash when selling corporate control. Consider a large company that wants to buy the control rights of a small start up firm, with the intention of restructuring the organization to enhance its value and profit from its future cash flows, and the externalities generated on its primary business. Typically the buyer's ability to increase the value of the target firm is highly uncertain from the perspective of the outsiders. Moreover, the benefits of the takeover to the buyer are partly determined by its financing and operational costs, which are also not known to the outsiders. How then should the owners of the target firm sell their firm? Should they claim rights to the post-takeover profits by retaining some shares of the company, or should they sell the whole firm? The same question arises, for instance, when a revenue-maximizing government must privatize a company, when a successful startup that has grown large sells itself, when a large blockholder sells his stake, or when a government sells mineral rights. In all these examples, it is the ability to retain a share of the profits that would be generated by ceding control to a buyer that distinguishes the sale of a firm from many other assets.

In our setting a single agent—the seller—who owns the firm can sell any fraction of his shares. The seller is faced with a single buyer (she).[1] The buyer adds value (v) to the target firm if she gains a controlling stake of the firm. In addition she enjoys a private benefit from controlling the firm (b). We allow the private benefits of control to be *negative*, which then represents the opportunity cost of controlling the company, or in the example of mineral rights sale, the private cost of drilling. Similarly, we allow v to be *negative*, which represents the possibility that the buyer destroys value. In the example of a large company buying a small company, this may represent a situation in which the large company simply uses the operations of the small company solely for the purposes of benefiting the operations of the large company. Both v and b are known only to her. Each share has one vote, and for convenience we assume that the controlling stake is 50% of the shares. As a result, the buyer controls the firm only when she obtains at least 50% of the firm.[2] We provide a characterization of the optimal

[1]The assumption of a single potential buyer in negotiations is realistic and commonly observed. For example, Boone and Mulherin (2007) found in a sample of 400 large merger transactions from the 1990s, involving a total of over one trillion dollars in transaction value, that half were executed through bilateral negotiations.

[2]The analysis remains the same except for some minor changes for any threshold different than 50%.

mechanism the seller should employ to sell the firm when he can offer the buyer a menu of tuples of fractions of shares and a price per share. We assume that the seller has all the bargaining power and unlimited commitment. This assumption, while restrictive, is for example plausible in privatization or government sale of mineral rights.

Assuming the value of the firm under new ownership is the buyer's private information, theory predicts that the sold share should be the minimal necessary to transfer control of the firm to the buyer (see, for example, Cremer (1987) and Samuelson (1987)). The intuition is that retaining a share in the firm allows the seller to capture the full value of the firm on the retained share and limit the information rents of the buyer to the share that was sold. This intuition breaks down when the buyer enjoys a private benefit of control or when the buyer has an opportunity cost of controlling the company.

To see this, suppose that there are two possible types A and B with arbitrary probabilities. Type A can generate a post-takeover value 1 and has no private benefits, i.e., $v_A = 1, b_A = 0$, while type B cannot improve managing of the company but does have private benefits of control. In particular, $v_B = 0$ and $b_B = 0.8$. The seller can extract the full surplus from the buyer by selling him fraction 0.8 of the company at the price 0.8. The offer clearly extracts the full surplus from type B. In addition, when type A buys 80% of the company the seller's payoff is the price 0.8 plus the worth 0.2 of the shares that he kept, thus 1 in total. It is easy to verify that full surplus cannot be extracted by an offer for half of the company. The above result generalizes to saying that the seller can extract the full surplus by selling the fraction b_B as long as $b_B \in [0.5, 1]$. That is, as long as the slope of the line between the two types in the coordinate system where v is on the horizontal axis and b on the vertical is between -1 and -0.5. The reason is that all the types on the line with the slope $-\gamma$ have the same value for fraction γ of the company. On the fraction the seller keeps the value automatically increases after the transfer of control. On the other hand, when $b_B > 1$ the optimal mechanism can always be implemented as a take-it-or-leave-it offer for the whole company. In some cases it will be the unique optimal mechanism. For example, when b_B is above but sufficiently close to 1 and the probabilities of the two types are equal, the seller does not want to exclude either type. Since type B values any controlling stake more than type A, the seller will make an offer that extracts the full surplus from type A and charges as much as possible for the control to type B. This is precisely the offer for the whole company at the price 1.

With more than two types the optimal mechanism need not be a take-it-or-leave-it offer for a fixed share of the company. Varying the share of the company, in addition to the price, becomes a powerful screening tool. To see this consider the following three type example. Let $(v_A, b_A) = (1, 0), (v_B, b_B) = (0.9, 0.1)$ and $(v_C, b_C) = (0, 0.55)$. The seller can extract all the surplus by offering a menu with two offers; but not with a take-it-or-leave-it offer. One for the whole company at price 1, and the other for the half of the company at price 0.55. Type A strictly prefers the first one, type C the second one, while type B is indifferent. Indeed the foundation of our analysis is a characterization of incentive compatible mechanisms through their exclusion boundary. The exclusion boundary is a curve in the space of post-takeover values and private benefits that separates the types who obtain a controlling stake in the given mechanism from those who do not. The share of the company the buyer obtains when she gains control defines the slope of the exclusion boundary.[3]

Our main results provide sufficient conditions on the joint distribution of v and b such that the optimal mechanism takes one of the following forms:

1. A take-it-or-leave-it offer for the smallest fraction of the company that facilitates the transfer of control (Proposition 3).

2. A take-it-or-leave-it offer for *all* the shares of the company (Proposition 4).

Roughly speaking, in the first case the seller wants to screen finely over private benefits. The finest screening over those that the buyer's incentive compatibility constraints permit is attained by an offer for the sale of half of the company. In the second case the seller wants to screen finely over the post-takeover cash flows, which is achieved by an offer for the whole company.

We also show that under some conditions the seller is able to extract the full value, v, per share (Proposition 5), whereas the buyer earns information rents only on the private benefits of control. This presents an interesting contrast to Zingales (1995) where it is argued that the two components of firm value (v and b) should be sold using two separate mechanisms.

An important feature of optimal mechanisms in our model is that the larger the v, the larger is the fraction of shares sold when the control is transfered. In other words, when the seller retains a smaller fraction of equity, then post-transfer share prices increase more. This is consistent with the empirical studies that document that in mergers and takeovers, if the cash transfer is higher in the cash-equity mixture, the

[3]Characterization of boundaries arising from and defining incentive-compatible mechanisms is strictly speaking not a full characterization of incentive compatibility, as opposed to the one in Rochet (1987), because there is not a unique incentive-compatible mechanism that induces a given exclusion boundary. Implementability conditions have also been provided in more specialized environments, for example, by McAfee and McMillan (1988). For a recent take on implementability in multi-dimensional environments, see Carbajal and Ely (2013), Kos and Messner (2013) and Rahman (2010).

target share price increases more (see Travlos (1987) and Franks et al. (1988)). In our model, this result is obtained due to the two-dimensional private information that the buyer holds. Other models explain this empirical finding either via competition effects on the buyers' side or by two-sided asymmetric information (Hansen (1987), Fishman (1989), Berkovitch and Narayanan (1990), Eckbo et al. (1990)).

1.1 Related Literature

Our paper contributes to the literature on contingent contracts. Hansen (1985) studies auctions with contingent payments, while Cremer (1987) and Samuelson (1987) show that by using auction forms which combine contingent payments and cash transfers, the seller can do almost as well as under full information. DeMarzo et al. (2005) and Gorbenko and Malenko (2010) analyze bidding with securities. In a recent paper Deb and Mishra (2014) provide a general characterization of implementability with contingent contracts. We study a more specific environment and as a consequence obtain a more detailed characterization of implementable contracts. Unlike in our model, the information in those models is single-dimensional.

The closest model to ours is the one analyzed in Cornelli and Li (1997) and Cornelli and Felli (2012). In their model the seller is selling a good to several potential buyers. Buyers' payoff is comprised of the post-takeover value and private benefits. Private benefits are linearly related to the post-takeover value, and the latter is buyer's private information. In their model, unlike in ours, knowing the buyer's post-takeover value implies knowing his private benefits, therefore, the private information is single-dimensional. Both papers, however, make a mistake in the analysis. We provide an analysis of this environment and a discussion of the two papers in Section 4.

Our analysis resembles the analysis of multi-dimensional mechanism design settings in Lewis and Sappington (1988) and Laffont et al. (1987) in the sense that we reduce a multi-dimensional mechanism design problem to a single-dimensional object, namely an exclusion boundary. However, incentive constraints put restrictions on the shape of the exclusion boundaries, which is particular to our environment. Strong results about the structure of optimal mechanisms have remained rather elusive in general multi-dimensional environments. Some notable exceptions are Laffont et al. (1987), Lewis and Sappington (1988), Rochet (2009) and the results in the literature on budget constraints as in Che and Gale (2000) and Pai and Vohra (2008). Deneckere and Severinov (2014) provide an analysis of an interesting class of multidimensional problems. Their analysis differs from ours in that it relies on differentiability of the utility function, which is not satisfied in our environment. For an detailed outline of the multi-dimensional mechanism design literature, see Rochet and Stole (2003) and the literature review in Manelli and Vincent (2007).

The key feature of our model is that both the post-takeover value of the firm and the private benefits of control are the buyer's private information. Beginning with Berle and Means (1932), there has been a substantial literature arguing that control of the firm allows the controller to enjoy benefits not shared with minority shareholders (see Jensen and Meckling (1976), Dodd and Warner (1983) and Johnson et al. (2000)). These benefits can be monetary, such as excess salary, or non-monetary, such as amenities like professional sports teams and newspapers.[4] For example, Dyck and Zingales (2004) estimate that on average the private benefit of control is worth 14% of the equity value of a firm. See Barak and Lauterbach (2011) for a brief summary of the empirical literature devoted to estimating the magnitude of the private benefit.[5]

Given the magnitude of the private benefits of control, there has been a great deal of interest in understanding its impact on the sale of corporate control (see, for example, Bebchuk (1994) and Zingales (1995)). We focus on firms whose shares are concentrated in the hands of a controlling shareholder. Many publicly traded small and mid-sized companies in the US and across the world have concentrated ownership (Betton et al. (2008), Holderness (2009)). The sale of firms in which no shareholder has a controlling interest is interesting and has been studied in the context of public tender offers.[6]

Our analysis of the optimal mechanism highlights a difficulty in one of the empirical approaches used to estimate private benefits of takeovers (see Barclay and Holderness (1989)). It estimates the private benefits of control via the difference between the share price of a publicly traded firm on the day of the transfer and the following day. Let P be the price paid for control, Q the number of shares transferred and w the value of the shares the following day. Then, the private benefit of control, b, is assumed to be $P - wQ$. This assumption is valid if we believe the seller has the ability to make a take-it-or-leave-it offer and there is no private information. Dyck and Zingales

[4]Private benefits can be negative because of personal monitoring costs or lawsuits brought by government officials. Another way to think of a negative private benefit is that it can be the outside option of the acquirer or foregone opportunities due to managing the target firm.

[5]There are plenty of other reasons why an acquiring company may have private benefits of control that are not appropriated by the target company's shareholders. For example, the target company may have good distribution capabilities in new areas, which the acquiring company can use for its own products as well. Alternatively, the target company may allow the acquiring company to enter a new market without having to take on the risk, time and expense of starting a new division. Finally, a takeover can facilitate the acquiring company to reduce its redundant functions.

[6]This raises free-riding issues that can emanate from the transfer of control. See, for example, Grossman and Hart (1980), Marquez and Yılmaz (2008), Bagnoli and Lipman (1988), Burkart and Lee (2010) and Marquez and Yılmaz (2012). Ekmekci and Kos (2015) and Ekmekci and Kos (2014) study takeovers when a minority stake of the firm is owned by large shareholders and a majority stake of the firm is widely dispersed.

(2004) suggest a modification assuming that bargaining power is shared between buyer and seller but continue to assume full information. Suppose, instead, we maintain the assumption that the seller has all the bargaining power but the buyer has private information about b and the increase in the share value, v. Then, the approach taken in Barclay and Holderness (1989) ignores the informational rent that goes to the buyer. If we account for the informational rent the buyer earns, our model predicts that if v and b are private information but independent of each other, the expected value of b conditional on control being acquired declines with v.

Below we introduce the notation we use as well as a description of the model. The remaining sections contain the analysis of the model. All proofs not contained in the main text are in the Appendix.

2 The Model

The value of the company under the current owner is commonly known and normalized to 0. Namely, his plans and management skills are well understood. If, however, the buyer obtains at least a 0.5 share in the company, she obtains control rights. We call this event a successful takeover. The value of the company after a successful takeover (the post-takeover value) is the buyer's private information and resides in an interval $V = [\underline{v}, \overline{v}]$. We allow for $\underline{v} < 0$ in which case the acquisition is value destroying.

In addition to the post-takeover value of the company, conditional on a takeover, the buyer obtains a private benefit $b \in [\underline{b}, \overline{b}] = B$, which is also the buyer's private information. We allow for $\underline{b} < 0$. Negative b can be interpreted to mean that control is costly or that the buyer has an unknown outside option that pays off $-b$.

The values of v and b are drawn from some commonly known distribution on $[\underline{v}, \overline{v}] \times [\underline{b}, \overline{b}]$. It will be convenient to think of nature as first choosing the post-takeover value v from a distribution F. We assume that F has density f positive everywhere on $[\underline{v}, \bar{v}]$. Conditional on v, nature chooses b according to distribution $G(b; v)$. The latter has a density $g(b; v)$, which is strictly positive on all of $[\underline{b}, \overline{b}]$ for all v.

If the buyer obtains a share $x \geq 0.5$, she controls the company and the value of her share is vx. In addition she enjoys a private benefit b, independently of the shares owned. If she obtains a share $x < 0.5$, that share is worth 0, which is the value of the company under the current owner; in that case she does not have a private benefit. Formally, if the current owner sells to the buyer a share x at a price t, the buyer's utility is

$$u(x,v,b)-t=\begin{cases} vx+b-t & \text{if} \quad x\geq 0.5 \\ -t & \text{if} \quad x<0.5 \end{cases}.$$

If the owner sells less than 50% of the firm, he receives the transfer t and retains control of the company, the value of which remains 0. If, however, he sells at least a share $x \geq 0.5$, the value of the remaining share is $(1-x)v$. Formally, the seller's payoff is

$$w(x,v)+t=\begin{cases} (1-x)v+t & \text{if} \quad x\geq 0.5 \\ t & \text{if} \quad x<0.5 \end{cases}.$$

Prior work has examined three special case of the setting just described.

1. Post-takeover cash flows v are common knowledge and private benefits b are private information.

   If v is common knowledge, the seller can set a price of v per share and capture the entire post-takeover value of the company. That is, he can sell a fraction $x \geq 0.5$ for xv and in addition enjoy $(1-x)v$ on the share he keeps. Thus, the seller is indifferent about what fraction of the firm beyond 50% is to be given up. The only question is how much of the private benefit of control can be captured. The answer follows from Mussa and Rosen (1978) or Myerson (1981). The optimal mechanism can be seen as a price of v per share and a take-it or leave-it offer for control of the firm. Notice that the model is silent about the fraction of shares beyond 50% that would change hands.

2. Private benefits b are common knowledge and v is private information.

   In this case the seller will charge b for control and must decide what fraction of shares above 50% to part with and the price to charge for them. An answer was provided by Cremer (1987) and Samuelson (1987) in an environment introduced by Hansen (1985) and generalized in Cornelli and Li (1997). The seller should part with the smallest fraction of shares necessary to yield control for a posted price. This is akin to the observation that it is better to run an auction with contingent payments, i.e., shares. See DeMarzo et al. (2005), Che and Kim

(2010) and Skrzypacz (2012) for a more extensive discussion. Limiting the fraction of shares that goes to the buyer limits the informational rents she earns.

3. Values v and b are private information but linearly related.

This case is discussed in Section 4.

3 Analysis

We examine the question of how the seller should design an optimal mechanism to sell his firm. Invoking the revelation principle, we can restrict attention to direct mechanisms (Q, T), where

$$Q : V \times B \to [0, 1],$$

and

$$T : V \times B \to \mathbb{R}.$$

Q is an allocation rule mapping each announced type (v, b) into a fraction of shares of the firm, $Q(v, b)$, the buyer receives. T are the transfers to the seller. Transfers can be assumed deterministic without loss of generality. The assumption that the allocation rule is deterministic is standard in multi-dimensional mechanism design literature, but possibly not without loss.[7]

Let

$$P(v, b) = 1_{[Q(v,b)\geq 0.5]}$$

for all (v, b), where 1 is the indicator function. $P(v, b) = 1$ if the buyer upon reporting (v, b) acquires a controlling share of the company and 0 otherwise.

Let (Q, T) be a mechanism. Type (v, b)'s payoff when she reports type (v', b') is

$$P(v', b') \left[vQ(v', b') + b\right] - T(v', b').$$

[7] In our model, neither private benefits b nor cash flows v are explicitly contractible. The tools that the seller has to his disposal are the fraction of shares he transfers and a transfer price, both of which depend solely on the buyer's report and not on the actual value of v or b. A similar assumption restricting contractability of cash flows is implicitly made throughout the related literature as for example in Zingales (1995), Cornelli and Li (1997), DeMarzo et al. (2005) and Deb and Mishra (2014). Unlike in DeMarzo et al. (2005), securities such as debt contracts are superfluous in our model, because there is no moral hazard and neither of the participants is liquidity or budget constrained.

Incentive compatibility can then be expressed as

$$P(v,b)\left[vQ(v,b)+b\right]-T(v,b)\geq P(v',b')\left[vQ(v',b')+b\right]-T(v',b') \tag{1}$$

for all (v,b) and (v',b'), and individual rationality as

$$P(v,b)\left[vQ(v,b)+b\right]-T(v,b)\geq 0 \tag{2}$$

for all (v,b). Let $U(v,b)\equiv P(v,b)\left[vQ(v,b)+b\right]-T(v,b)$. The seller's problem is to maximize

$$E[T(v,b)+vP(v,b)(1-Q(v,b))],$$

where the expectation is with respect to v and b, over all the mechanisms (Q,T) satisfying (1) and (2).

3.1 The Seller's Profit from Incentive Compatible Mechanisms

We adopt a somewhat unconventional approach in that we first characterize the seller's payoff from incentive compatible mechanisms, only then do we proceed to characterize more detailed properties of such mechanisms. Benefits of such sequencing will become clear later. First we characterize the buyer's payoff when the seller screens her only along the dimension of private benefits.

Lemma 1. *Let (Q,T) be an incentive-compatible mechanism. Then,*

$$U(v,b)=U(v,\underline{b})+\int_{\underline{b}}^{b}P(v,x)dx.$$

Proof. After observing that

$$U(v,b)\geq U(v,b')+[b-b']P(v,b'),$$

the proof follows the approach of Myerson (1981). □

Lemma 1 shows that along the dimension of private benefits, the indicator function P plays the same role as the probability of obtaining the object in a standard auction setting. Moreover, one should notice that this relation is independent of the fine details of the allocation Q, beyond the P.

The next result gives an expression for the seller's profit arising from an incentive-compatible mechanism (Q,T), solely in the terms of the allocation rule Q and the payoff to the lowest type, thereby establishing a revenue-equivalence result.[8]

Proposition 1. *Let (Q, T) be an IC mechanism. Then the seller's payoff from mechanism (Q, T) can be written as*

$$\pi = -U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)} \right\} f(v) dv. \tag{3}$$

Proof. Proof of the above and subsequent results can be found in the Appendix. □

The expression for the seller's profit can be decomposed into two parts. The term $v + b - \frac{1-G(b;v)}{g(b;v)}$ has the standard interpretation of a conditional virtual value. The seller is unable to capture all of the buyer's private benefits b as he needs to incentivize the buyer to truthfully reveal the component b by leaving her some information rent. The seller's revenue is further diminished by the informational advantage the buyer has about the post-takeover value of the firm v. The additional information rent he has to pay for the revelation of v is reflected by the term $-P(v, \underline{b})Q(v, \underline{b})\frac{1-F(v)}{f(v)}$.

The seller's payoff is, at least directly, dependent on Q only along the ray $b = \underline{b}$. However, we show below that the controlling stake in the firm that the buyer receives, Q, does play a significant role in the characterization of incentive-compatible mechanisms. Namely, the shape of Q puts restrictions, through incentive compatibility, on the shape of P, which enters the payoff directly.

3.2 Characterization of Incentive Compatible Mechanisms

In a single-dimensional mechanism design problem, under standard single-crossing conditions, an allocation rule Q is implementable if and only if it is non-decreasing. In our setting with two-dimensional types, incentive compatibility implies that Q is non-decreasing in each component (holding the other fixed). In addition, one can show that the fraction of the company the buyer obtains in nondecreasing in v, even when b is not held fix, provided the buyer obtains the control. These weak forms of monotonicity are insufficient to characterize all incentive-compatible allocation rules. The do, however, allow us to establish the existence of a boundary that separates the set of types who obtain control from the set of the types who do not.

[8]The analysis follows standard arguments. See, for example, the derivation in Armstrong (1996).

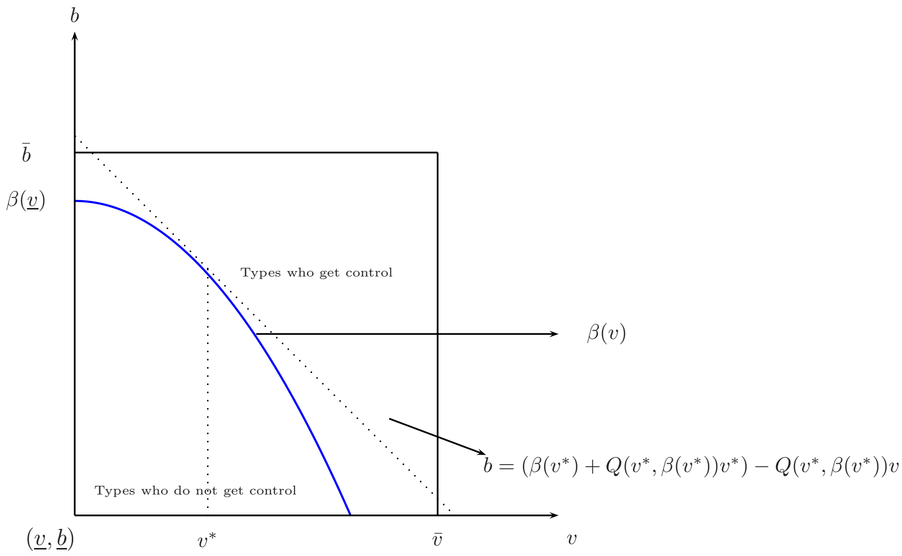

Figure 1: The curve $\beta(v)$ is the exclusion boundary that separates the types who get control of the firm and the types who get excluded from taking control of the firm. The slope of $\beta(v)$ at $v = v^*$ is equal to $-Q(v^*, \beta(v^*))$.

Lemma 2. *Let (Q, T) be an incentive-compatible mechanism. Define*

$$\beta(v) = \inf\{b : Q(v, b) \geq 0.5\}$$

for all v. Then $P(v, b) = 1$ if $b > \beta(v)$, and $P(v, b) = 0$ if $b < \beta(v)$, for all (v, b).[9] *Moreover, β is nonincreasing.*

For each incentive-compatible mechanism (Q, T), we call the associated function $\beta(\cdot)$ the *exclusion boundary* of the mechanism. The types above the exclusion boundary of an incentive compatible mechanism obtain the controlling stake, while the types below do not. Proposition 1 implies that the seller's profit from an IC mechanism depends only on P, the allocation along the ray $\underline{b}$, $Q(\cdot, \underline{b})$, and the payoff $U(\underline{v}, \underline{b})$. Since P is fully determined by the exclusion boundary β, the seller's payoff can be determined by the knowledge of β, $Q(\cdot, \underline{b})$, and $U(\underline{v}, \underline{b})$ alone. The other details of the mechanism are irrelevant.

In what follows we take a closer look at the properties of exclusion boundaries arising from incentive compatible mechanisms. We say that an exclusion boundary β is *implementable* if there is an incentive-compatible mechanism (Q, T) whose exclusion boundary is β.

Definition 1. *An exclusion boundary β is regular if there are numbers $a, c \in [\underline{v}, \bar{v}]$ such that $\beta(v) = \bar{b}$ for $v < a$; $\beta(v) = \underline{b}$ for $v > c$; β is continuous everywhere and is concave*

[9] We adopt convention $\beta(v) = \bar{b}$ if $Q(v, b) < 0.5$ for all b.

in $[\underline{v}, c]$; *and, at every differentiability point* $v \in [a, c]$,

$$\beta'(v) \in [-1, -0.5].$$

The following proposition gives a characterization of all incentive-compatible mechanisms through their exclusion boundaries.

Proposition 2. *An exclusion boundary* β *is implementable if and only if it is regular. If* (Q, T) *is incentive compatible with an exclusion boundary* β*, and if* β *is differentiable at some* $v \in [a, c]$ *for which* $Q(v, \beta(v)) \geq 0.5$*, then*

$$\beta'(v) = -Q(v, \beta(v)).^{10}$$

While the types on the exclusion boundary need not obtain a controlling stake, we will assume they do to simplify the presentation. The assumption is without loss of generality as far as the optimal mechanisms for the seller are concerned, since the exclusion boundary is of measure zero.

Proposition 2 states that if a mechanism (Q, T) is incentive compatible, its exclusion boundary is regular. Conversely, any regular exclusion boundary is implementable. We show the necessity part of the proposition by first establishing an intermediate result (Lemma 3 in the Appendix). This intermediate result shows that if (Q, T) is an incentive-compatible mechanism and (v, b) is a type such that $Q(v, b) \geq 0.5$, then all the types (v', b') who do not receive a controlling stake must lie below the line with a slope $-Q(v, b)$ that passes through the point (v, b). This is the line that represents the set of types whose payoffs would be identical to the payoff of type (v, b) if they mimicked (v, b). Therefore, any type above this line who does not get a controlling stake in the company would prefer to represent herself as the type (v, b). This reasoning, applied to the exclusion boundary β, yields the second claim and the necessity result stated in Proposition 2 (see Figure 1 for a depiction).

Sufficiency is obtained by constructing a mechanism that allocates the fraction $-\beta'(v^-)$ to types (v, b) such that $v \leq c$ and $b \geq \beta(v)$,[11] fraction $-\beta'(c^-)$ when $v > c$, and fraction 0 otherwise. The transfers are, $T(v, b) = 0$ for $v < a$, or $v \in [a, c]$ and $b < \beta(v)$; $T(v, b) = vQ(v, \beta(v)) + \beta(v)$ for $v \in [a, c]$ and $b \geq \beta(v)$; and $T(v, b) = cQ(c, \underline{b}) + \underline{b}$ for $v > c$.

The above result relates the shape of the exclusion boundary β arising from an

[10] Strictly speaking, if v is a differentiability point of β and if $v \in [a, c]$, then $\beta'(v) = -\tilde{Q}$ for any $\tilde{Q}$ that is a limit point of $Q(v, b')$ as $b' \downarrow b$. We keep the formulation with $Q(v, \beta(v)) \geq 0.5$ for expositional simplicity.

[11] $\beta'(v^-)$ denotes the left-hand derivative of β at v.

incentive-compatible mechanism and the allocation rule. In particular, an exclusion boundary β arising from an incentive-compatible mechanism allows us to partition $[\underline{v}, \bar{v}]$ into three intervals. If $a > \underline{v}$, then $\beta(v) = \bar{b}$ when $v \leq a$. On the second interval, which we call the interior segment, $\beta(v)$ is strictly decreasing with a derivative equal to $-Q(v, \beta(v))$ almost everywhere on this interval. If $c < \bar{v}$, then $\beta(v) = \underline{b}$ for all $v \in [c, \bar{v}]$. Given an incentive-compatible mechanism, one or more of the segments may be empty. Moreover, standard monotonicity considerations imply that $Q(v, \beta(v))$ is nondecreasing in v. Therefore, β is concave on the interior segment.

3.3 Structure of Optimal Mechanisms

Let the conditional virtual valuation of type (v, b) be

$$\phi(v, b) \equiv v + b - \frac{1 - G(b; v)}{g(b; v)}.$$

As mentioned earlier, $\phi(v, b)$ is the virtual valuation of type (v, b) if the seller knew the common value component, v. We make the following single-crossing assumption: $\phi(v, \cdot)$ crosses 0 at most once, and if it crosses it, it does so from below, for every $v \in V$.

Let

$$\alpha(v) \equiv \inf\{b : \phi(v, b) \geq 0\}$$

for all $v \in V$. The term $\alpha(v)$ is the smallest b, given a fixed v, for which the virtual valuation $\phi(v, b)$ is nonnegative. With other words, $\alpha(\cdot)$ is the boundary separating types with nonegative virtual valuations from the types whose virtual valuation is negative; see Figure 2 for a depiction. If, for a given v, $\phi(v, b) < 0$, for all b we set $\alpha(v) = \underline{b}$.

Assumption 1. α is absolutely continuous and piecewise differentiable, and $\alpha'(v) \geq -0.5$ at each point of differentiability v.

Assumption 1 is a restriction on the distribution of types, $G(b; v)$. Below we give examples of environments in which Assumption 1 is satisfied. Our first main result provides a characterization of optimal mechanisms under Assumption 1.

Proposition 3. *If Assumption 1 is satisfied, a take-it-or-leave-it offer for half of the firm is an optimal mechanism for the seller.*

When Assumption 1 is satisfied the seller would like to screen more finely with respect to b than with respect to v. The intuition is most clear in the extreme case

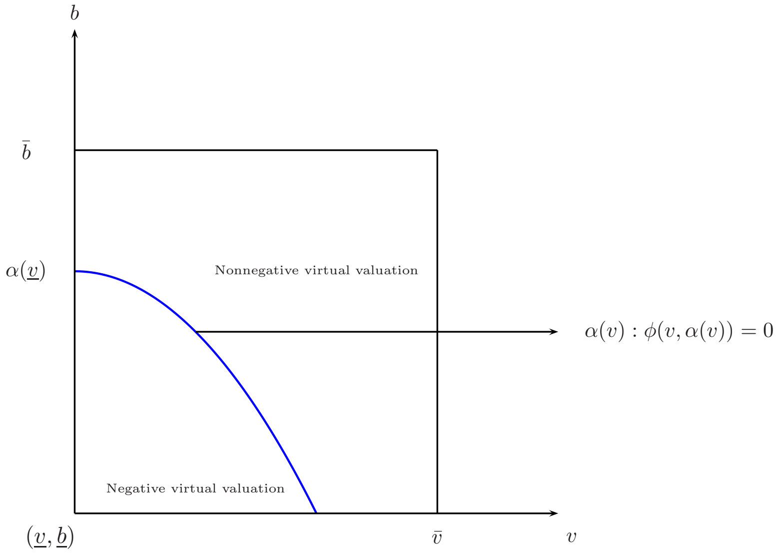

Figure 2: The curve $\alpha(v)$ separates the types whose conditional virtual valuations are nonnegative from those with virtual valuations that are negative.

where α is flat ($\alpha' = 0$). That is, when all the types with negative virtual valuations are below some flat boundary. In such a case the seller would like to screen solely on the basis of b. Namely, he would like to sell only to a buyer with a positive virtual valuation, which here translates into selling only to a buyer whose b exceeds some threshold. The seller can, however, not distinguish between a type that has a high v and low b and a type that has a low v and a high b. Indeed, our characterization of incentive compatibility implies that the 'flattest' mechanism the seller can offer, and thus the finest screening over b, is achieved by a take it or leave it offer for a half of the company.

In what follows we provide further details on how the result in Proposition 3 is obtained. It is convenient to use the following definitions:

$$\pi_0(v) \equiv \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db,$$

for all v, and

$$\pi(v) = \pi_0(v) - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)}. \tag{4}$$

Then, according to Proposition 1,

$$\pi = -U(\underline{b}, \underline{v}) + \int_{\underline{v}}^{\bar{v}} \pi(v) f(v) dv.$$

The term $\pi_0(v)$ is the standard representation of the seller's revenue in a one dimensional setting. It can be interpreted as the revenue the seller would obtain if he knew the v component of the buyer's type and had to screen only the private benefit component, b. There is an additional component of the seller's payoff in the multi-dimensional setting, namely $-P(v,\underline{b})Q(v,\underline{b})\frac{1-F(v)}{f(v)}$. This component is present due to screening along the v dimension. In a single-dimensional setting, an analogous term is taken care of by setting the lowest type's payoff to some constant. In our case, a single type's payoff can also be set to a constant, $(\underline{v},\underline{b})$, which leaves a ray $\{(v,\underline{b}) : v \in (\underline{v},\overline{v}]\}$ whose payoffs are linked to the type $(\underline{v},\underline{b})$. The additional cost due to screening of those types is reflected by $P(v,\underline{b})Q(v,\underline{b})\frac{1-F(v)}{f(v)}$. Notice that this cost needs to be paid only if the type $(v,\underline{b})$ receives a controlling stake of the company, as his payoff is otherwise the same as the payoff of the type $(\underline{v},\underline{b})$.

The idea of the proof of Proposition 3 can now be described as follows. Function α from V to B maximizes $\pi_0(v)$ pointwise. The proof shows that for each incentive-compatible mechanism $(\tilde{Q},\tilde{T})$ and its exclusion boundary $\tilde{\beta}$, there exists another boundary β^* corresponding to a take-it-or-leave-it offer for half of the company such that $\beta^*(v)$ is between $\alpha(v)$ and $\tilde{\beta}(v)$ for every v. The single-crossing assumption implies that $\pi_0^*(v) \geq \tilde{\pi}_0(v)$ for all v. Indeed, if, for example, $\alpha(\hat{v}) \leq \beta^*(\hat{v}) \leq \tilde{\beta}(\hat{v})$ for some $\hat{v}$, then neither $\tilde{\beta}$ nor β^* assign a controlling stake to types with the post-takeover value $\hat{v}$ and a negative virtual valuation. However, $\tilde{\beta}$ might exclude some types with a positive virtual valuation that β^* does not. Moreover, by construction β^* has a smaller or equal cost of screening along the ray $\{(v,\underline{b}) : v \in (\underline{v},\overline{v}]\}$ than does $\tilde{\beta}$. Therefore, $\pi^*(v) \geq \tilde{\pi}(v)$. The existence of such a β^* is possible due to the assumption that α has a slope greater than or equal to -0.5 everywhere in the interior of $V \times B$, while $\tilde{\beta}$ has a slope in $[-1,-0.5]$, from the incentive compatibility of the mechanism.

Below, we give a simple sufficient condition on distributions under which Assumption 1 is satisfied and therefore Proposition 3 applies.

Corollary 1. *Suppose v and b are distributed independently, with distributions F and G, respectively. Suppose G has a continuously differentiable density g on $[\underline{b},\bar{b}]$, such that $g(b) > 0$ and $g'(b) \geq 0$ for all $b \in [\underline{b},\bar{b}]$. Then a take-it-or-leave-it offer for half of the firm is an optimal mechanism for the seller.*

Proof. Since $g'(b) \geq 0$ for all b,

$$\phi_b(v,b) = 2 + \frac{g'(b)}{g^2(b)}\,(1 - G(b)) > 0.$$

This implies that $\phi(v,\cdot)$ crosses 0 at most once and from below.

Define a function $\hat{\alpha} : [\underline{v}, \bar{v}] \to \mathbb{R}$ by

$$v + \hat{\alpha}(v) - \frac{1 - G(\hat{\alpha}(v))}{g(\hat{\alpha}(v))} = 0$$

for every $v \in [\underline{v}, \bar{v}]$. Now define a function α by truncating $\hat{\alpha}$ to $[\underline{b}, \bar{b}]$.[12]

For any v such that $\alpha(v) \in \{\underline{b}, \bar{b}\}$ and α is differentiable, $\frac{d\alpha}{dv}(v) = 0$. For any v such that $\alpha(v) \in (\underline{b}, \bar{b})$, differentiating

$$v + \alpha(v) - \frac{1 - G(\alpha(v))}{g(\alpha(v))} = 0$$

with respect to v yields

$$\frac{d\alpha(v)}{dv} = -\frac{g(\alpha(v))}{2g(\alpha(v)) + (v + \alpha(v))g'(\alpha(v))} \geq -\frac{1}{2}.$$

Since α, g and g' are continuous, the derivative $\frac{d\alpha}{dv}$ is also bounded from above, rendering α Lipschitz continuous. Thereby Assumption 1 is satisfied and Proposition 3 applies. □

Next we provide a parametric example to demonstrate under what conditions Assumption 1 is likely to be satisfied.

Example 1. Suppose that the post-takeover value is distributed with some atomless distribution on $[0, \bar{v}]$. The private benefits, conditional on v, are distributed uniformly on the interval $[kv, 1 + kv]$ where k is a constant. Then

$$\begin{aligned}\phi(v, b) &= v + b - (1 - G(b; v))/g(b; v) \\ &= v + b - (1 + kv - b) \\ &= v(1 - k) + 2b - 1.\end{aligned}$$

α, which is defined through $\phi(v, \alpha(v))$, is therefore

$$\alpha(v) = \frac{1}{2} - \frac{1-k}{2}v,$$

for all $v \in [0, \bar{v}]$. For $k \geq 0$ Assumption 1 is satisfied, Proposition 3 applies, and therefore a take-it-or-leave-it offer for half of the company is optimal.

Slightly more can be said when $k = 0$ and therefore $d\alpha/dv = -1/2$. As long as $\bar{v} \leq 1$, or $1/2 - 1/2\bar{v} \geq 0$, the boundary α never hits the lower edge of $[0, \bar{v}] \times [0, 1]$ box and therefore the offer for half of the company at the price $1/2$ implements the optimal mechanism for the seller. When $\bar{v} > 1$ an optimal mechanism will still be an offer for a half of the company, however the exclusion boundary of the optimal mechanism will not coincide with the α.

[12] If $\hat{\alpha}(v) > \bar{b}$, then $\alpha(v) = \bar{b}$. If $\hat{\alpha}(v) < \underline{b}$, then $\alpha(v) = \underline{b}$.

The above example suggests that Assumption 1 is more likely to be satisfied, and therefore an offer for half of the company optimal, when there is positive correlation between the post-takeover value and the private benefits.

In the following Proposition, on the other hand, we show that if the zero virtual valuation curve is very steep, and if it does not hit the lower boundary, then a take-it-or-leave-it offer for the whole company is optimal for the seller.

Proposition 4. *Let α be absolutely continuous and piecewise differentiable with $\alpha'(v) \leq -1$ at each point of differentiability $(v, \alpha(v))$ in the interior of $V \times B$, and $\alpha(\bar{v}) > \underline{b}$. Then a take-it-or-leave-it offer for the whole firm is an optimal mechanism for the seller.*

The intuition for Proposition 4 is as follows: Ideally, the seller would use a mechanism whose exclusion boundary coincides with the zero virtual valuation curve α. However, the slope of the exclusion boundary of an incentive-compatible mechanism is bounded below by -1. Therefore, the seller will benefit by using a mechanism whose exclusion boundary has slope -1, i.e., a take-it-or-leave-it offer for the whole firm. In other words, this mechanism screens most finely across cash flows v. The assumption that $\alpha(\bar{v}) > \underline{b}$ guarantees that the optimal take-it or leave-it offer for the whole firm will exclude all types with a private benefit $\underline{b}$.

Example 2. Here we revisit the environment studied in Example 1. Proposition 4 states that an offer for the whole company is optimal when $\alpha'(v) \leq -1$ and $\alpha(\bar{v}) > \underline{b}$. This will be the case in the setting of the example when $k \leq -1$ and $0.5 - \frac{1-k}{2}\bar{v} \geq k\bar{v}$, or

$$1 - \bar{v} \geq k\bar{v}.$$

The above inequality is always satisfied for $k \leq -1$.

The example suggests that an offer for the whole company is more likely to be optimal when v and b are strongly negatively related in the sense that the conditional mean of b is decreasing fast with v. The above example is slightly biased, because for all $k \leq -1$, $\alpha(\bar{v}) > \underline{b}$ is satisfied. This will not be the case in general. The latter condition tends to be more likely satisfied when $\bar{v}$ is smaller, or with the other words, when there is little uncertainty about $\bar{v}$.

Example 3. Let b and v be independent, where b is distributed according to an exponential distribution on $[0, \infty)$ with a hazard rate λ, i.e.,

$$G(b; v) = 1 - e^{-\frac{b}{\lambda}}.$$

The cash flow v is distributed according to some atomless distribution F on $[0, \bar{v}]$. It is easy to verify that

$$\alpha(v) = \lambda - v.$$

$\alpha'(v) = -1$, therefore Proposition 4 applies as long as $\alpha(\bar{v}) > 0$, or $\bar{v} < \lambda$. In that case an offer for the whole company is optimal.

Some caution is required when trying to adapt the approach used in the single-dimensional case to our setting. In a single-dimensional environment, under single crossing, one solves a relaxed problem and then shows that the mechanism that serves only types with nonnegative virtual valuations can be implemented in an incentive-compatible and individually rational mechanism. One could try to do the same in our case. Function α serves that purpose. However, two difficulties arise. First, α can be of a form that cannot arise as an exclusion boundary of an incentive-compatible mechanism. For example, if α is linear with a slope $-1/4$, no IC mechanism exists whose boundary is α, due to Proposition 2. Second, even if α is such that it could arise as an exclusion boundary in an IC mechanism, it might not solve the seller's problem. The hiccup is that α maximizes π_0 point by point rather than π. However, if α takes the form that could arise from an IC mechanism and $\alpha(\bar{v}) > \underline{b}$, then the corresponding mechanism has $P(v, \underline{b}) = 0$ for all v. In this case the solution to the relaxed problem indeed yields a solution of the original problem. We demonstrate this in the next proposition and in the example that follows it.

Assumption 2. α is regular and $\alpha(\bar{v}) > \underline{b}$.

Proposition 5. *If Assumption 2 is satisfied, then the optimal mechanism's exclusion boundary is $\alpha(v)$. Moreover, the seller's expected profits in this mechanism are identical to his expected profits if he employs the optimal mechanism after observing v.*

Proposition 5 asserts that the seller's expected profits are equal to the maximum expected profits had he known the common-value component. It highlights that the ability of the seller to claim a fraction of the post-transaction profits gives him the ability to perfectly screen the common-value component of the buyer's information. In such a mechanism, the buyer retains informational rents only on his private benefits.

In light of Propositions 1 and 2, Proposition 5 follows straightforwardly. If the zero virtual valuation curve is regular and does not hit the lower boundary, then excluding only the types who have negative virtual valuations yields an implementable exclusion boundary. Clearly the seller cannot do better than this. Inspection of the seller's profit function given by equation (3) shows that setting $P(v, \underline{b}) = 0$ and setting $P(v, b) = 1$ only when $\phi(v, b) \geq 0$ yields the maximum profit for the seller ignoring the incentive compatibility conditions implied by the exclusion boundary.

The following example demonstrates an instance where revenue is maximized by an offer for a fraction of the company strictly between 0.5 and 1.

Example 4. Let v be distributed on $[0, 2/3]$. The details of the distribution of v are not important. Nature first chooses v and then b from a uniform distribution on $[0, 1 - v/2]$. Therefore,

$$g(b; v) = \frac{1}{1 - v/2},$$

and

$$G(b; v) = b\frac{1}{1 - v/2}.$$

The adjusted virtual valuation is then easily computed to be

$$\phi(v, b) = \frac{3}{2}v + 2b - 1.$$

And finally,

$$\alpha(v) = \frac{1}{2} - \frac{3}{4}v.$$

The optimal mechanism can now be implemented as a take-it-or-leave-it offer for 3/4 of the company at the price of 1/2.

4 Discussion

In related papers Cornelli and Li (1997) and Cornelli and Felli (2012) analyzed an environment in which there are both post-takeover values as well as private benefits, but the two are linearly related.[13] We show how our tools can be applied to such a setting and characterize optimal mechanisms under general conditions. This provides a further insight into when it is optimal to sell only half and when to sell the whole firm. A byproduct of these results is that we uncover some errors in the two above mentioned papers.

Suppose that nature draws v from a distribution F, with density f, on the interval $[0, \bar{v}]$. The buyer's private benefits are a linear function of the post-takeover value of

[13] While most of the analysis here concerns the environment with one buyer Cornelli and Li (1997) and Cornelli and Felli (2012) allow for several buyers.

the firm:

$$b = B - \gamma v,$$

for some constants B and $\gamma \geq 0$. To simplify the analysis we assume $B \geq \gamma \bar{v}$.

Applying standard revenue equivalence arguments one can show that in any incentive compatible mechanism the payoff of the v type buyer from reporting truthfully is

$$U(v) = \int_0^v P(s)[Q(s) - \gamma]ds + U(0),$$

and the seller's payoff is

$$\int_{\hat{v}}^{\bar{v}} [B + v(1-\gamma) - (Q(v) - \gamma)\frac{1 - F(v)}{f(v)}]P(v)f(v)dv + \int_0^{\hat{v}} [B + v(1-\gamma) + (Q(v) - \gamma)\frac{F(v)}{f(v)}]P(v)f(v)dv - U(\hat{v}), \quad (5)$$

where $\hat{v}$ is any type in $[0, \bar{v}]$ and $U(\hat{v})$ is its utility.[14] The seller's payoff is decreasing in Q for $v > \hat{v}$ and increasing for $v < \hat{v}$, for v with $P(v) = 1$, as long as P and $U(\hat{v})$ are held constant.

The following example puts equation (5) to use. Suppose that $B = 8, \bar{v} = 10$, and $\gamma = 0.8$. Consider an offer to the buyer for 60% of the company at price 7. At this price types with v above 5 do not buy, and types with $v < 5$ do; see Figure 3.[15] In particular, $U(0) = 1$ and $U(\bar{v}) = 0$. One can therefore fix $\hat{v} = \bar{v}$ and increase Q while holding P and $U(\bar{v})$ fixed. This corresponds to pivoting the exclusion boundary around the point $(v, b) = (5, 4)$ towards the line of types. Pivoting can be continued until the two lines coincide. Interestingly, if one starts with an offer that has an exclusion boundary steeper than the line of types, pivoting the exclusion boundary towards the line of types also increases the payoff. In that case, $\hat{v} = 0$ and the seller's payoff is decreasing in Q for a fixed P and $U(0)$. Therefore an optimal offer is the one whose exclusion boundary coincides with the line of types. In the above case, this is an offer for 80% of the company at price 8. Notice that this offer extracts the full surplus.

More generally, if $\gamma \in [0.5, 1]$, then an offer for a fraction γ of the company at the price B leaves all the types with payoff zero and the seller with the full surplus. The reason is that all the types on the line value the fraction γ of the company equally, while on the reminder of the company, $1 - \gamma$, the seller automatically obtains the full surplus after the buyer gains the control. Similar reasoning as above can be applied to show that an offer for the whole company is optimal when $\gamma > 1$ and an offer for 50% of the company is optimal when $\gamma \leq 0.5$. Notice that the form of the optimal offer—that

[14] One might be tempted to fix the value of the type $\hat{v} = 0$. This might, however, cause confusion. The seller's valuation consists of both v and b. Therefore, the type with $\hat{v} = 0$, might not value the object the least.

[15] Although types are here distributed on a line, it is useful to think of incentive compatible mechanisms in terms of their exclusion boundaries (as in the previous sections) in the space $[0, \bar{v}] \times [0, B]$. The exclusion boundary corresponding to the discussed mechanism is $0.6v + b = 7$. The type on the exclusion boundary gets payoff zero, while the types below it do not accept the offer.

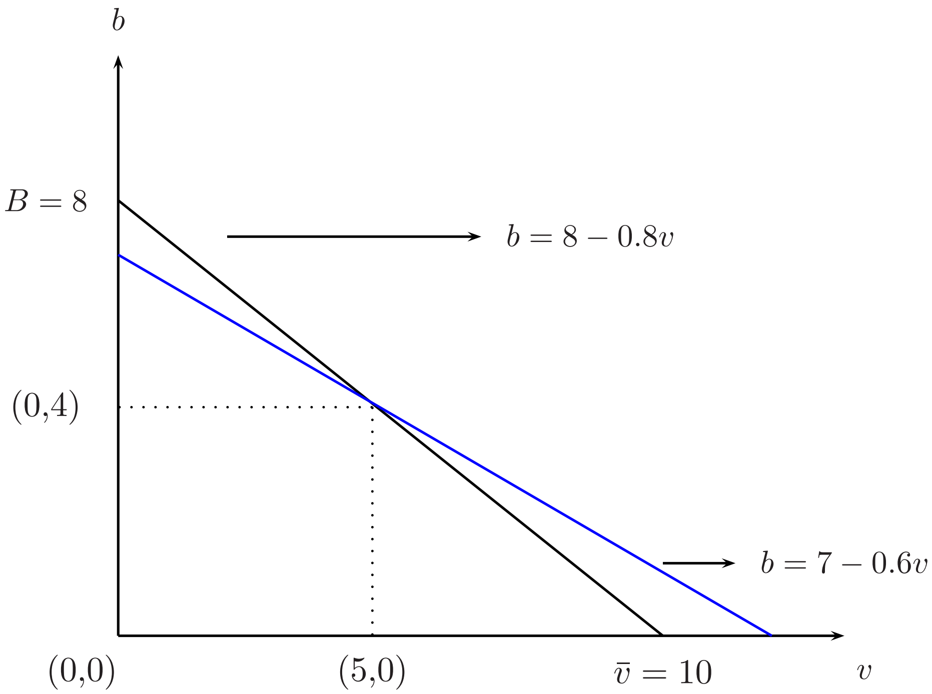

Figure 3: The line $b = 7 - 0.6v$ is the exclusion boundary of the contract that sells 60% of the company at a price of 7. The type space is the line segment in the positive quadrangle of the line $b = 8 - 0.8v$.

an optimal mechanism is an offer for a fixed share of the company and what this share is—is independent of the distribution of v. However, the optimal price at which the offer for a given share of the company is made will depend on the said distribution.

For $\gamma \in [0.5, 1]$ the result immediately extends to the case of several buyers. The following mechanism extracts all the surplus: buyers report their v and the agent with the highest v obtains fraction γ of the company at the price B. In this case the seller obtains the payoff $(1-\gamma)v + B$ when selling to the agent with the post-takeover value v. Therefore, it is optimal for him to sell to the buyer with the highest v. At the same time, the buyers obtain payoff zero regardless of whether they receive the share γ or not. Therefore they are indifferent and thus willing to report their true v.

The above environment is analyzed in Cornelli and Li (1997).[16] They show that when $\gamma < 0.5$ (γ in our setting corresponds to β in theirs) the seller optimally sells half of the firm; Proposition 3 in their paper. However, complex interplay of assumptions needed for their propositions 1 and 2 hinders the direct comparisons of those results with the ones obtained here. In fact, their propositions 1 and 2, which focus on the case $\gamma \geq 0.5$, do not exhaust all the combinations of γ and distributions over v. The supposition of their Proposition 1 is, for example, satisfied when $\lambda = 0.5$, v is distributed uniformly, and γ is between 0.75 and 1. Their result then states that the seller should sell half of the company. Above we have, however, shown that the seller can do better and indeed extract the full surplus by selling the fraction γ.[17] The same example with $\gamma \in [0.5, 0.75)$ can be used to show that their Proposition 2 is wrong.

Cornelli and Felli (2012) analyzed a similar environment with a distinction that they restricted their analysis to finding optimal mechanisms for sale of a fixed proportion of the company. In their model, the amount of the sold shares, when the control is transfered, is not allowed to depend on the reported type. They find in Proposition 1, part B.a, that the seller might want to auction precisely fraction γ ($-\beta$ in their notation) of the shares under some condition. However, they failed to observe that the seller in such a case extracts the full surplus. Their footnotes 5 and 12 claims that one will be able to do even better by more general mechanisms. On the other hand, part B.b of Proposition 1 in their paper is erroneous. The result claims that if the virtual valuation is decreasing, the optimal offer is for the minimal controlling stake of the company. Same example as above ($\lambda = 0.5$, $\underline{\alpha} = 0.5$, $\gamma \in (0.75, 1]$, v distributed uniformly), yields a decreasing virtual valuations. But in that case, since $\gamma \in [0.5, 1]$, one can extract the full surplus by making an offer for the fraction γ of the company, as we have shown above. Indeed, when $\gamma \in [0.5, 1]$, one can extract the full surplus irrespective of how v is distributed.

[16] They consider more general objective functions. Our objective for the seller corresponds to $\lambda = 0.5$ in their environment. The requirement in our paper that the buyer acquires 50% of the shares to gain the control corresponds to $\underline{\alpha} = 0.5$ in their paper.

[17] The authors seemed to have missed the fact that the seller's payoff is decreasing in Q (α in their notation) only as long as one fixes $\hat{v} = 0$ and keeps $U(0)$ ($U(0, 0)$ in their notation) and P constant.

5 Conclusion

The market for corporate control is different from many other exchange markets in which consumers trade with producers. Shares of a company carry both cash-flow rights and voting rights. Therefore, transferring control requires a minimum transfer of 50% of the shares. The seller may sell all the shares or alternatively claim a minority portion of the profits that result from the transfer. In this paper, we analyze the optimal mechanism for the sale of such a firm to a buyer whose private information is two-dimensional. We find that, unlike in the case where the uncertainty is unidimensional, the optimal mechanism may not be a take-it-or-leave-it offer. In general it is a menu of tuples of fractions of shares and cash transfers.

Among other things, we identify sufficient conditions on the joint distribution of v and b for which the optimal mechanism takes one of the following forms:

1. A take-it-or-leave-it offer for the smallest fraction of the company that facilitates the transfer of control.
2. A take-it-or-leave-it offer for *all* the shares of the company.

We also identify a sufficient condition for the seller to extract the full value, v, per share so that the buyer earns information rents only on the private benefits of control. The main insight is that the seller can reduce the information rents the buyer enjoys on the benefits of control through the quantity of shares retained.

Although our model is a stylized model, our analysis applies also to the case of a large block holder with control rights of a firm who is negotiating with a potential buyer for the sale of the control rights. If the fraction of shares the block holder needs to sell for the transfer of control is less than his total holdings, then a slight modification of our analysis applies. Our results imply that, in private negotiations for block trades, the block holder can use the fraction of shares he transfers to better screen the buyer's type.

Finally, our analysis is undertaken in the model with a single buyer. The case of multiple buyers will require a more elaborate approach and is therefore left for future research. Another interesting venue for the future research would be an environment in which the value of the firm under the seller's management and his private benefits are privately known, which would convert the environment into a nontrivial informed principal problem.[18]

[18] For a recent take on informed principal problems see Mylovanov and Trger (2014).

A Proofs

Proof of Proposition 1. Let (Q, T) be an incentive-compatible mechanism. We begin by showing that the seller's payoff from (Q, T) can be written as

$$\pi = -\int_{\underline{v}}^{\bar{v}} U(v, \underline{b}) f(v) dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db \right\} f(v) dv. \tag{6}$$

Indeed,

$$\begin{aligned}
\pi &= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} [T(v, b) + vP(v, b)[1 - Q(v, b)]] \, g(b; v) db \right\} f(v) dv \\
&= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} [-U(v, b) + (b + v) P(v, b)] \, g(b; v) db \right\} f(v) dv \\
&= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[-U(v, \underline{b}) - \int_{\underline{b}}^{b} P(v, x) dx + (b + v) P(v, b) \right] g(b; v) db \right\} f(v) dv \\
&= -\int_{\underline{v}}^{\bar{v}} U(v, \underline{b}) f(v) dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db \right\} f(v) dv,
\end{aligned}$$

where the second equality follows by the definition of U, the third from Lemma 1, and the fourth from interchanging the order of integration.

Next, we rewrite the seller's profit by using screening along v. For that purpose we use the equality, which is obtained using standard analysis as in Myerson (1981),

$$U(v, \underline{b}) = U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{v} P(x, \underline{b}) Q(x, \underline{b}) dx \tag{7}$$

for all v.

Using (6), the seller's payoff can be rewritten as

$$\begin{aligned}
\pi &= -\int_{\underline{v}}^{\bar{v}} U(v, \underline{b}) f(v) dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db \right\} f(v) dv \\
&= -U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)} \right\} f(v) dv,
\end{aligned}$$

where the second equality follows by plugging (7) into (6) and integrating by parts. $\square$

Proof of Lemma 2. Let (Q, T) be incentive compatible. First we show a preliminary result, which will be of use later in the proof.

Claim: If $Q(v, b) \geq 0.5$ for some (v, b), then $Q(v', b') \geq 0.5$ for all $(v', b') \geq (v, b)$. Proof of the Claim: Let (v, b) be such that $Q(v, b) \geq 0.5$; if no such (v, b) exists, we are done. Let $(v',b') \geq (v,b)$ and $(v',b') \neq (v,b)$. Incentive compatibility implies

$$u(Q(v,b),v,b) - u(Q(v',b'),v,b) \geq u(Q(v,b),v',b') - u(Q(v',b'),v',b');$$

equivalently,

$$u(Q(v,b),v,b) - u(Q(v,b),v',b') \geq u(Q(v',b'),v,b) - u(Q(v',b'),v',b').$$

Since $Q(v,b) \geq 0.5$ and consequently $P(v,b) = 1$,

$$Q(v,b)[v - v'] + b - b' \geq u(Q(v',b'),v,b) - u(Q(v',b'),v',b').$$

If $Q(v',b') < 0.5$, the right-hand side equals 0, while the left-hand side is negative, resulting in a contradiction. Thus, $P(v',b') = 1$ and $Q(v',b') \geq 0.5$, thereby concluding the proof of the claim.

Returning to the original Lemma, let $v \in [\underline{v}, \bar{v}]$ and $\beta(v)$ be defined as in the statement of the Lemma. The claim above implies that $P(v,b') = 1$ for every $b' > \beta(v)$, and $P(v,b') = 0$ for every $b' < \beta(v)$.

Suppose $v > v'$ and $\beta(v) > \beta(v')$. Then there exists a $\hat{b} \in (\beta(v'), \beta(v))$ such that $Q(v,\hat{b}) < 0.5$ and $Q(v',\hat{b}) \geq 0.5$, which contradicts the above Claim. Therefore, β is nonincreasing. □

Lemma 3. *Let (Q,T) be an incentive-compatible mechanism, and let (v,b) be such that $Q(v,b) \geq 0.5$. If $P(v',b') = 0$, then*

$$-Q(v,b)v' + (Q(v,b)v + b) \geq b'.$$

Proof. As above, incentive compatibility implies

$$u(Q(v,b),v,b) - u(Q(v,b),v',b') \geq u(Q(v',b'),v,b) - u(Q(v',b'),v',b'),$$

which then yields

$$Q(v,b)[v - v'] + b - b' \geq 0,$$

or equivalently

$$-Q(v,b)v' + (Q(v,b)v + b) \geq b'.$$

□

Lemma 3 implies the following. Let $(v, \beta(v)) \in V \times B$. Then if (v',b') is such that $P(v',b') = 0$, it has to be that (v',b') is below the linear function with the slope $-Q(v,\beta(v))$, which runs through $(v,\beta(v))$.

Proof of Proposition 2. *If β is implementable, then it is regular.*

Let (Q, T) be an incentive-compatible mechanism whose exclusion boundary is β, and let S be the set of types (v, b) for which $Q(v, b) \geq 0.5$. For any type $t = (v, b) \in S$, let $L(t) := \{(v', b') : Q(v, b)v' + b' < Q(v, b)v + b\}$. Consider the set $L := \cap_{t \in S} L(t)$. For any type $(v, b) \in L$, $P(v, b) < 0.5$ because the sets L and S are disjoint. For any set X, let $cl(X)$ denote the closure of X. We will show that for any $(v, b) \notin cl(L)$, $P(v, b) = 1$. Clearly, if $L = \emptyset$, or if $cl(L) = V \times B$, then the claim is true. So, we suppose that $L \neq \emptyset$ and $cl(L) \neq V \times B$.

Notice that L is a convex set because it is an intersection of convex sets. Clearly, its closure is also convex. Pick a type $t' = (v', b') \notin cl(L)$. Then, we claim there exists a type (v, b) such that $Q(v, b)v' + b' > Q(v, b)v + b$. By way of a contradiction, suppose not. Since L is convex, for any type $t'' = (v'', b'') \in L$, there is a $\lambda \in (0, 1)$ such that the type $\lambda t' + (1 - \lambda)t'' \notin cl(L)$. But this is a contradiction because for any $(v, b) \in S$, $Q(v, b)(\lambda v' + (1 - \lambda)v'') + \lambda b' + (1 - \lambda)b'' < Q(v, b)v + b$.

As there is a type $(v, b) \in S$ such that $Q(v, b)v' + b' > Q(v, b)v + b$, and since $P(v, b) = 1$, by Lemma 3 (in the Appendix), $P(v', b') = 1$. Therefore, if S is the set of types who get control of the firm, then $cl(L) \cup S = V \times B$, and L and S are disjoint.

Since the exclusion boundary of β separates the types who get control and those who do not, β has to be the boundary of L. Since L is convex, its boundary β is concave in the interval in which $\beta(v) > \underline{b}$. We have already shown that β is nonincreasing; therefore, there is a number $c \in [\underline{v}, \bar{v}]$ such that β is concave on $[\underline{v}, c)$ and $\beta(v) = \underline{b}$ for $v > c$. Because β is concave on $[\underline{v}, c)$, it is continuous, and almost everywhere differentiable on this interval. To complete the proof of continuity on the whole domain, we need to show that if $c < \bar{v}$, then $\lim_{v \to c} \beta(v) = \underline{b}$. Assume on the way to a contradiction that this is not true. Then $c < \bar{v}$ and there is an $\epsilon > 0$ such that $\beta(v) > \underline{b} + \epsilon$ for every $v < c$ because β is nonincreasing. But then the type $(c - \epsilon/2, \underline{b} + \epsilon)$ would prefer to mimic a type (v', b') that satisfies $v' \in (c, c + \epsilon/2)$ and $b' \in (\underline{b}, \underline{b} + \varepsilon)$ for some $\varepsilon < c + \epsilon/2 - v'$, which yields the desired contradiction.

Since β is continuous and nonincreasing, it is differentiable almost everywhere. Let $v \in (a, c)$ be a point of differentiability of β. Pick a $v' < v$. Since $(v', \beta(v')) \in cl(L)$, $Q(v, b)v + b(v) \geq Q(v, b)v' + \beta(v')$ for every $b > \beta(v)$. Because $Q(v, b) \in [0.5, 1]$ for every $b > \beta(v)$, it has to be that $\beta(v') - \beta(v) \leq v - v'$. Since this is true for every $v' < v$, it has to be that $\beta'(v) \geq -1$. Similarly, by picking $v' > v$, we argue that $\beta(v) - \beta(v') \geq 0.5(v' - v)$, establishing that $\beta'(v) \geq -0.5$.

We have thus proven that β is regular.

If β is regular, then it is implementable.

Let β be a regular boundary. Let (a, c) be the largest interval in which $\beta(v) > \underline{b}$. Because β is concave on $v \in [\underline{v}, c]$, it is differentiable almost everywhere on the same interval. This, together with the assumption that β is a regular boundary, implies $\beta'(v) \in [-1, -0.5]$ almost everywhere on $v \in [\underline{v}, c]$. Let $\beta'(v^-)$ denote the left-hand derivative of β at v. Because β is concave in $[\underline{v}, c]$, $\beta'(v^-)$ exists at every point $v \in (\underline{v}, c]$ and is nonincreasing in v. Note that $\beta'(v^-) = \beta'(v)$ whenever $\beta'(v)$ exists. Because

$\beta'(v) \in [-1, -0.5]$ almost everywhere in the interval $[a, c]$, $\beta'(v^-) \in [-1, -0.5]$ everywhere in the interval $[a, c)$. Moreover, $-\beta'(v^-)v' + b' \leq -\beta'(v^-)v + b$ for any $b' \leq \beta(v')$ for all $v \in (\underline{v}, c)$, because β is concave in that interval.

Consider now the following mechanism (Q, T). For $v \leq a$, $Q(v, b) = 0$, $T(v, b) = 0$ for every $b \in [\underline{b}, \bar{b}]$. For $v \geq c$, $Q(v, b) = -\beta'(c^-)$ and $T(v, b) = -\beta'(c^-)c + \underline{b}$. For $v \in (a, c)$, $Q(v, b) = T(v, b) = 0$ for every $b < \beta(v)$, and $Q(v, b) = \beta'(v^-)$, $T(v, b) = Q(v, b)v + \beta(v)$ for every $b \geq \beta(v)$. After noticing that for any fixed v the mechanism can be interpreted as a take it or leave it offer over b for some quantity, it is easy to check that (Q, T) satisfies all of the incentive compatibility constraints. Moreover, by construction β is the exclusion boundary of (Q, T).

We now show that $\beta'(v) = -Q(v, \beta(v))$ at all points $v \in [a, c]$, where β is differentiable and $Q(v, b) \geq 0.5$.

Fix an incentive-compatible mechanism (Q, T) and let β be its exclusion boundary. Pick a type $(v, \beta(v))$ for which $Q(v, \beta(v)) \geq 0.5$ and the derivative of β at v exists. Similar to the first part of the proof, let L be the set of types that are excluded from control of the firm. This set is convex, and hence the boundary β is concave. Therefore, for any type $(v', b') \in L$, $Q(v, \beta(v))v + \beta(v) \geq Q(v, \beta(v))v' + b'$. Since L is convex, and since $\beta'(v)$ exists, it has to be that $\beta'(v) = -Q(v, \beta(v))$, because the convex set L is supported with a line (hyperplane) that passes through $(v, \beta(v))$, and the slope of the line is equal to the derivative of the boundary. □

Lemma 4. *Suppose Assumption 1 is satisfied. Let $(\tilde{Q}, \tilde{T})$ be an incentive-compatible mechanism, and $\tilde{\beta}$ its corresponding exclusion boundary defined by $\tilde{\beta}(v) = \inf\{b : \tilde{P}(v, b) = 1\}$ for all $v \in [\underline{v}, \bar{v}]$. If v is such that $\tilde{\beta}(v) < \alpha(v)$, then $\tilde{\beta}(v') \leq \alpha(v')$ for all $v' > v$.*

Proof. β is absolutely continuous because, it is continuous, almost everywhere differentiable on $[\underline{v}, c]$ and its derivative is bounded whenever it exists. Therefore, we can write $\beta(v) = \beta(\underline{v}) + \int_{\underline{v}}^{v} \beta'(x)dx$. Similarly, $\alpha(v) = \alpha(\underline{v}) + \int_{\underline{v}}^{v} \alpha'(x)dx$. Let $\hat{v}$ be such that $\tilde{\beta}(\hat{v}) < \alpha(\hat{v})$. Then $\alpha(v) - \tilde{\beta}(v) = \alpha(\hat{v}) - \tilde{\beta}(\hat{v}) + \int_{\hat{v}}^{v}[\alpha'(x) - \tilde{\beta}'(x)]dx$. Let $\tilde{v} \equiv \min\{v' \geq v : \tilde{\beta}(v') = \underline{b}\}$. On the interval $[v, \tilde{v})$, $\tilde{\beta}'(x) \in [-1, -0.5]$ and $\alpha'(x) \geq -0.5$; therefore, $\alpha(v) - \tilde{\beta}(v) = \alpha(\hat{v}) - \tilde{\beta}(\hat{v}) + \int_{\hat{v}}^{v}[\alpha'(x) - \tilde{\beta}'(x)]dx \geq 0$. On the other hand, incentive compatibility of $(\tilde{Q}, \tilde{T})$ implies $\tilde{\beta}(v) = \underline{b}$ for all $v \in [\tilde{v}, \bar{v}]$. Namely, if $\tilde{\beta}(v) = \underline{b}$, then $\tilde{\beta}(v') = \underline{b}$ for all $v' \geq v$. Since both α and $\tilde{\beta}$ take values in $[\underline{b}, \bar{b}]$, it cannot be the case that $\alpha(v) < \tilde{\beta}(v)$ for $[\tilde{v}, \bar{v}]$, which concludes the proof. □

Proof of Proposition 3. The seller is maximizing

$$\pi = -U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)} \right\} f(v) dv$$

over incentive-compatible and individually rational mechanisms.

Let (Q,T) be some incentive-compatible and individually rational mechanism. Define

$$\pi(v) \equiv \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] P(v,b) g(b;v) db - P(v,\underline{b}) Q(v,\underline{b}) \frac{1 - F(v)}{f(v)}$$

for all v. Proposition 1 implies $\pi = -U(\underline{b},\underline{v}) + \int_{\underline{v}}^{\bar{v}} \pi(v) f(v) dv$.

Let $(\tilde{Q},\tilde{T})$ be an arbitrary IC and IR mechanism with an exclusion boundary $\tilde{\beta}$. We will show there exists an alternative mechanism that takes the form of a take-it-or-leave-it offer for half of the company and yields to the seller at least as high a profit as $(\tilde{Q},\tilde{T})$ does.

Due to Lemma 4, in the Appendix, the analysis can be divided into three cases. In the first, β is never below α; in the second β crosses α from above in the interior of $[\underline{v},\bar{v}] \times [\underline{b},\bar{b}]$; and in the third, β is never above α.

Case 1. $\tilde{\beta}(v) \geq \alpha(v)$ for all v.

This is the case where $\tilde{\beta}$ is never below α. Two additional cases are to be considered here, depending on whether there exists a $v \in [\underline{v},\bar{v}]$ such that $\tilde{\beta}(v) = \underline{b}$. We first consider the former case, the latter is elaborated on at the end of Case 1.

Let $v_\alpha = \inf\{v : \alpha(v) = \underline{b}\}$ and $\tilde{v} = \inf\{v : \tilde{\beta}(v) = \underline{b}\}$. In other words, v_α is the smallest v at which $\alpha(v)$ hits $\underline{b}$. Similarly, $\tilde{v}$ is the lowest v at which $\tilde{\beta}$ hits $\underline{b}$. Given the assumption of Case 1, $v_\alpha \leq \tilde{v}$.[19]

We now describe the alternative mechanism, (Q^*,T^*), which is a take-it-or-leave-it offer for half of the company at the price $\tilde{v}/2 + \underline{b}$. The boundary β^* corresponding to this mechanism is given by

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \underline{b} + \frac{\tilde{v}-v}{2} > \bar{b} \\ \underline{b} + \frac{\tilde{v}-v}{2} & \text{if } \underline{b} \leq \underline{b} + \frac{\tilde{v}-v}{2} \leq \bar{b} \\ \underline{b}, & \text{if } \underline{b} + \frac{\tilde{v}-v}{2} < \underline{b}, \end{cases}$$

for all v. We can now write (Q^*,T^*) as $Q^*(v,b) = 0.5 * 1_{[(v,b) \geq (v,\beta^*(v))]}$, and $T^*(v,b) = (v^*/2 + \underline{b}) * 1_{[(v,b) \geq (v,\beta^*(v))]}$; therefore $P^*(v,b) = 1_{[(v,b) \geq (v,\beta^*(v))]}$. Clearly this mechanism is incentive-compatible and individually rational. In the following development we will show that (Q^*,T^*) yields at least as high a payoff to the seller as $(\tilde{Q},\tilde{T})$.

Let $v^* = \inf\{v : \beta^*(v) = \underline{b}\}$. Then $v^* = \tilde{v}$. Notice that $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ for all $v \in [\underline{v},\bar{v}]$. The first inequality holds because $v_\alpha \leq v^*$ and β^* is at least as steep (and decreasing) as α at any v such that $\beta^* \in (\underline{b},\bar{b})$. More precisely, at any v such that $\beta(v) \in (\underline{b},\bar{b})$, $(\beta^*)'(v) = -0.5$, while $\alpha'(v) \geq -0.5$ for all v. The inequality is then proven the same way as Lemma 4. The other inequality, $\beta^* \leq \tilde{\beta}$, follows from $v^* = \tilde{v}$ and the fact that $\tilde{\beta}$ is at least as steep (and decreasing) as β^*.

[19] Given that we are considering the case where there exists a v such that $\tilde{\beta}(v) = \underline{b}$, both v_α and $\tilde{v}$ are well-defined.

In what follows we compare the seller's profit from mechanisms $(\tilde{Q}, \tilde{T})$, denoted by $\tilde{\pi}$, with the profit from (Q^*, T^*), denoted by π^*. In fact, we will show $\pi^*(v) \geq \tilde{\pi}(v)$, which together with the fact that type $(\underline{v}, \underline{b})$ gets utility 0 in (Q^*, T^*) yields the desired result.

For $v \in [\underline{v}, \tilde{v})$,

$$\begin{aligned}
\pi^*(v) - \tilde{\pi}(v) &= \int_{\beta^*(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v) dx - \int_{\tilde{\beta}(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v) dx \\
&= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v) dx \\
&\geq 0,
\end{aligned}$$

where the first equality uses the fact that $\tilde{P}(v, \underline{b}) = P^*(v, \underline{b}) = 0$ on the specified range of v. The inequality follows from $\alpha \leq \beta^* \leq \tilde{\beta}$ and the fact that $v + b - \frac{1-G(b;v)}{g(b;v)}$ is nonnegative for $b \geq \alpha(v)$.

For $v \in (\tilde{v}, \bar{v}]$, $\tilde{\beta}(v) = \beta^*(v) = \underline{b}$; therefore,

$$\begin{aligned}
\pi^*(v) - \tilde{\pi}(v) &= [\tilde{Q}(v, \underline{b}) - Q^*(v, \underline{b})] \frac{1 - F(v)}{f(v)} \\
&= [\tilde{Q}(v, \underline{b}) - 1/2] \frac{1 - F(v)}{f(v)} \\
&\geq 0
\end{aligned}$$

for all v, where the last inequality follows from $\tilde{\beta}(v) = \underline{b}$ for $v \in (\tilde{v}, \bar{v}]$; therefore, $\tilde{Q}(v, \underline{b}) \geq 1/2$ for $v \in (\tilde{v}, \bar{v}]$. This concludes the analysis of the first subcase of Case 1.

We are left to consider the case where $\tilde{\beta} > \underline{b}$ for all v. In this case one can show that a take-it-or-leave-it offer for half of the company at the price $\bar{v}/2 + \tilde{\beta}(\bar{v})$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. The proof is very similar to the proof of the above considered case and is therefore omitted.

Case 2. There exists a v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Case 2 covers the environments in which, roughly speaking, $\tilde{\beta}$ crosses α in the interior of $[\underline{v}, \bar{v}] \times [\underline{b}, \bar{b}]$.

Fix some v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Let (Q^*, T^*) be a direct mechanism corresponding to the take-it-or-leave-it offer for half of the firm at the price of $\alpha(v_c) + v_c/2$. The corresponding β^* is

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \bar{b} < \alpha(v_c) + [v_c - v]/2 \\ \alpha(v_c) + [v_c - v]/2, & \text{if } \underline{b} \leq \alpha(v_c) + [v_c - v]/2 \leq \bar{b} \\ \underline{b}, & \text{if } \alpha(v_c) + [v_c - v]/2 < \underline{b}, \end{cases}$$

for all v.

Due to the continuity properties of $\alpha, \tilde{\beta}, \beta^*$, and their slopes, $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$

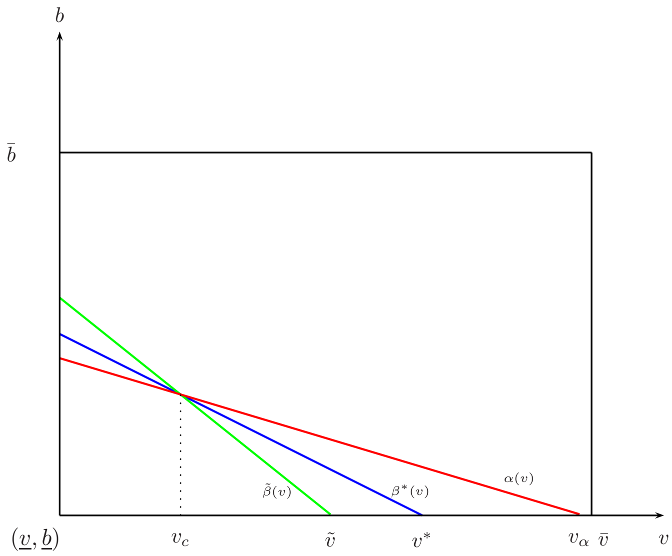

Figure 4: This figure depicts the alternative mechanism in Case 2. The slope of the exclusion boundary of the alternative mechanism is -0.5. Note that β^* is closer to the zero virtual valuation curve α and hits the $\underline{b}$ at $v^* \geq \tilde{v}$.

for $v \leq v_c$ and $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v > v_c$. These inequalities can be proven following the reasoning of the proof of Lemma 4 (see also Figure 4 for a depiction).

As in Case 1, we will argue that $\pi^*(v) \geq \tilde{\pi}(v)$ for all v, which together with the fact that $U(\underline{v}, \underline{b}) = 0$ in (Q^*, T^*) implies $\pi^* \geq \tilde{\pi}$. Let $\tilde{v} = \inf\{v : \tilde{\beta}(v) = \underline{b}\}$ and $v^* = \inf\{v : \beta^*(v) = \underline{b}\}$. The fact that $\underline{b} < \beta^*(v_c) = \tilde{\beta}(v_c) < \bar{b}$ together with the fact that $\tilde{\beta}$ is at least as steep as β^* imply $\tilde{v} \leq v^*$. That is, for each v such that $\beta^*(v) \in (\underline{b}, \bar{b})$, $(\beta^*)'(v) = -0.5$, while for each v such that $\tilde{\beta}(v) \in (\underline{b}, \bar{b})$, $\tilde{\beta}'(v) \in [-1, -0.5]$.

For $v \leq v_c$, $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$; therefore, $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. Now,

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx \\ &\geq 0,\end{aligned}$$

where the inequality follows because on this range of v, $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$, and $v + b - \frac{1-G(b;v)}{g(b;v)} \geq 0$ for $b \geq \alpha(v)$.

For v such that $v_c \leq v < v^*$, $\alpha(v) \geq \beta^*(v) \geq \tilde{\beta}(v) > 0$. Therefore,

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= - \int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx + \tilde{P}(v, \underline{b}) \tilde{Q}(v, \underline{b}) \frac{1 - F(v)}{f(v)} \\ &\geq 0,\end{aligned}$$

where the last inequality follows from $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$ for $b \leq \alpha(v)$. The last term is 0 for $v < \tilde{v}$ and strictly positive for $v \in (\tilde{v}, v^*)$.

Finally, for v such that $v \geq v^*$,

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= [\tilde{Q}(v, \underline{b}) - Q^*(v, \underline{b})]\frac{1 - F(v)}{f(v)} \\ &= [\tilde{Q}(v, \underline{b}) - 1/2]\frac{1 - F(v)}{f(v)} \\ &\geq 0,\end{aligned}$$

where $\tilde{Q}(v, \underline{b}) \geq 1/2$ follows from $\tilde{\beta}(v) = \underline{b}$ on the relevant range. This concludes the proof of Case 2.

Case 3. $\alpha(v) \geq \tilde{\beta}(v)$ for all v.

The analysis here divides into two additional cases depending on whether $\tilde{\beta}(\underline{v}) = \bar{b}$ or $\tilde{\beta}(\underline{v}) < \bar{b}$. We consider the first case here and the second case is simpler and similar. Let $\tilde{v}_u = \sup\{v : \tilde{\beta}(v) = \bar{b}\}$. Let (Q^*, T^*) be a take-it-or-leave-it offer for half of the firm at the price $\bar{b} + \frac{1}{2}\tilde{v}_u$. The corresponding boundary is given by

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \bar{b} \leq [v - \tilde{v}_u]/2 + \bar{b} \\ [v - \tilde{v}_u]/2 + \bar{b}, & \text{if } \underline{b} \leq [v - \tilde{v}_u]/2 + \bar{b} \leq \bar{b} \\ \underline{b}, & \text{if } [v - \tilde{v}_u]/2 + \bar{b} < \underline{b}. \end{cases}$$

By the definition of β^*, β^* and $\tilde{\beta}$ coincide on $[\underline{v}, \tilde{v}_u]$. On $(\tilde{v}_u, \bar{v}]$, $\tilde{\beta}(v) \leq \beta^*(v)$ because $\tilde{\beta}$ is at least as steep as β^* for all v at which $\tilde{\beta} > \underline{b}$. Likewise, $\beta^*(v) \leq \alpha(v)$ for $[\underline{v}, \tilde{v}_u]$. Therefore,

$$\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$$

for all v. Define v^* as in Case 1.

For $v < v^*$,

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)}\right] g(b; v)dx + \tilde{P}(v, 0)\tilde{Q}(v, 0)\frac{1 - F(v)}{f(v)} \\ &\geq 0\end{aligned}$$

because for b such that $\tilde{\beta}(v) \leq b \leq \beta^*(v) \leq \alpha(v)$, $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$.

For $v \geq v^*$,

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= [\tilde{Q}(v, \underline{b}) - Q^*(v, \underline{b})]\frac{1 - F(v)}{f(v)} \\ &= [\tilde{Q}(v, \underline{b}) - 1/2]\frac{1 - F(v)}{f(v)} \\ &\geq 0,\end{aligned}$$

using the same argument as in the previous cases, which concludes the proof. □

Proof of Proposition 4. The proof is similar to the proof of Proposition 3; therefore, we will be somewhat less formal here. Let $(\tilde{Q}, \tilde{T})$ be an incentive-compatible and individually rational mechanism with an exclusion boundary $\tilde{\beta}$. We argue that there exists a take-it-or-leave-it offer for the whole company that does at least as well for the seller. Using the same argument as in the proof of Proposition 3, with the difference that here α is steeper than $\tilde{\beta}$ in the interior of the valuation space, we can split the analysis into three cases: $\tilde{\beta}$ is never bellow α, $\tilde{\beta}$ crosses α from bellow or $\tilde{\beta}$ is never above α.

Case 1. $\tilde{\beta}(v) \geq \alpha(v)$ for all v. If $\alpha(v) < \bar{b}$ for all v, then the mechanism (Q^*, T^*) corresponding to the take-it-or-leave-it offer for the whole company at the price $\underline{v} + \alpha(\underline{v})$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. The boundary β^* corresponding to (Q^*, T^*) is given by

$$\beta^*(v) = \begin{cases} \alpha(\underline{v}) + \underline{v} - v, & \text{if } \alpha(\underline{v}) + \underline{v} - v \geq \underline{b}, \\ \underline{b}, & \text{if } \alpha(\underline{v}) + \underline{v} - v < \underline{b}. \end{cases}$$

It is easy to verify that the boundary β^* corresponding to (Q^*, T^*) is between α and $\tilde{\beta}$ for each v: i.e., $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$.

In what follows we compare the seller's profit from the mechanism $(\tilde{Q}, \tilde{T})$, denoted by $\tilde{\pi}$, with the profit from (Q^*, T^*), denoted by π^*. For the definition of $\pi(v)$ see (4). In fact we will show that $\pi^*(v) \geq \tilde{\pi}(v)$ which together with the fact that type $(\underline{v}, \underline{b})$ gets utility 0 in (Q^*, T^*) yields the desired result.

Now,

$$\begin{aligned} \pi^*(v) - \tilde{\pi}(v) &= \int_{\beta^*(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v)dx - \int_{\tilde{\beta}(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v)dx \\ &= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v)dx \\ &\geq 0, \end{aligned}$$

where the first equality uses the fact that $\tilde{P}(v, \underline{b}) = P^*(v, \underline{b}) = 0$ for all v, which is due to $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$. The inequality follows from $\alpha \leq \beta^* \leq \tilde{\beta}$ and the fact that $v + b - \frac{1-G(b;v)}{g(b;v)}$ is nonnegative for $b \geq \alpha(v)$.

If instead $\alpha(\underline{v}) = \bar{b}$, then let $v^\alpha = \sup\{v : \alpha(v) = \bar{b}\}$. A take-it-or-leave-it offer for the whole company at the price $v^\alpha + \bar{b}$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. This is easily verified using the same reasoning as in the $\alpha(\underline{v}) < \bar{b}$ case.

Case 2. There exists a v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$, and for $v \leq v_c$ we have $\tilde{\beta}(v) \leq \alpha(v)$; for $v \geq v_c$ we have $\alpha(v) \leq \tilde{\beta}(v)$. Case 2 covers the environments in which, roughly speaking, $\tilde{\beta}$ crosses α from below in the interior of $[\underline{v}, \bar{v}] \times [\underline{b}, \bar{b}]$.

Fix some v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Let (Q^*, T^*) be a direct mechanism corresponding to the take-it-or-leave-it offer for the whole firm at the price of $\alpha(v_c)+v_c$. The corresponding β^* is

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \bar{b} < \alpha(v_c) + v_c - v \\ \alpha(v_c) + v_c - v, & \text{if } \underline{b} \leq \alpha(v_c) + v_c - v \leq \bar{b} \\ \underline{b}, & \text{if } \alpha(v_c) + v_c - v < \underline{b} \end{cases}$$

for all v.

The continuity properties of $\alpha, \tilde{\beta}, \beta^*$ and their slopes imply $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v \leq v_c$ and $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ for $v > v_c$. These inequalities can be proven following the reasoning of the proof of Lemma 4.

As in Case 1, we will argue that $\pi^*(v) \geq \tilde{\pi}(v)$ for all v, which together with the fact that $U(\underline{v}, \underline{b}) = 0$ in (Q^*, T^*) implies $\pi^* \geq \tilde{\pi}$. For $v \leq v_c$,

$$\begin{aligned} \pi^*(v) - \tilde{\pi}(v) &= -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx \\ &\geq 0, \end{aligned}$$

where the first equality is the consequence of $\underline{b} < \tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v \leq v_c$, and therefore $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. The inequality follows because on this range of v, $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$, and $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$ for $b \leq \alpha(v)$.

For $v > v_c$,

$$\begin{aligned} \pi^*(v) - \tilde{\pi}(v) &= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx \\ &\geq 0, \end{aligned}$$

where the equality holds due to $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ and consequently $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. Notice that this is an instance where the assumption $\alpha(\bar{v}) > \underline{b}$ is used. The inequality follows from $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$ for $b \leq \alpha(v)$.

Case 3. $\tilde{\beta}(v) \leq \alpha(v)$ for all v. We will argue that the take-it-or-leave-it offer for the whole company at the price of $\bar{v} + \alpha(\bar{v})$ does at least as well for the seller as $(\tilde{Q}, \tilde{T})$. The boundary corresponding to the take-it-or-leave-it offer is

$$\beta^*(v) = \begin{cases} \alpha(\bar{v}) + \bar{v} - v, & \text{if } \alpha(\bar{v}) + \bar{v} - v \leq \bar{b} \\ \bar{b}, & \text{if } \alpha(\bar{v}) + \bar{v} - v > \bar{b}, \end{cases}$$

for $v \in [\underline{v}, \bar{v}]$.

Since $\beta^*(\bar{v}) = \alpha(\bar{v})$, α is at least as steep as β^*, and both are nonincreasing, $\beta^*(v) \leq \alpha(v)$ for all v. A more formal argument can be made along the lines of the argument used in the Proof of Proposition 3. On the other hand, $\beta^*(v) = \alpha(v) \geq \tilde{\beta}(v)$, β^* being at least as steep in the interior of the type space as $\tilde{\beta}$, and both β^* and $\tilde{\beta}$ being nonincreasing yields $\tilde{\beta}(v) \leq \beta^*(v)$ for all v.

Let $\pi(v)$ be as in (4). Then

$$\begin{aligned}\pi^*(v) - \tilde{\pi}(v) &= -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b;v)}{g(b;v)}\right] g(b;v)dx + \tilde{P}(v,0)\tilde{Q}(v,0)\frac{1-F(v)}{f(v)} \\ &\geq 0,\end{aligned}$$

where the first line follows due to $P^*(v, \underline{b}) = Q^*(v, \underline{b}) = 0$ and the inequality due to $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$ for $b \in [\tilde{\beta}(v), \beta^*(v)]$ since $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$. This is another instance where the assumption $\alpha(\bar{v}) > \underline{b}$ is used. It enables us to construct the take-it-or-leave-it offer for the whole firm (Q^*, T^*) which does at least as well as the original mechanism $(\tilde{Q}, \tilde{T})$ in such a way that $P^*(v, \underline{b}) = Q^*(v, \underline{b}) = 0$ for all v. □

Proof of Proposition 5. Under Assumption 2, the exclusion boundary that coincides with α is implementable. Therefore, all that remains to be shown is that the seller cannot increase his profits by using any other incentive-compatible mechanism. This is straightforward because equation (3) shows that the seller's profits in any incentive-compatible mechanism are bounded above by the profits he obtains by setting $P(v, \underline{b}) = 0$, and setting $P(v, b) = 1$ only when $\phi(v, b) \geq 0$. The mechanism that implements the exclusion boundary α achieves this upper bound.

Clearly the optimal mechanism that the seller would choose if he knew v would be the one that allocates the good only if $\phi(v, b) \geq 0$. Therefore, his profits when he does not know v are identical to his profits if he knew v. □

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