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# Lecture 1: Fundamentals of Mean-Variance Analysis

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Portfolio Management

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# Overview

- Generalities on the investment management process
- How to correctly measure portfolio performance
- The characteristics of the opportunity set
- Portfolio combinations
- Measuring linear co-movements: correlations
- The effects of diversification: systematic risk
- Mean-variance and efficient frontiers: logical meaning
- The case of no borrowing and lending and two risky assets
- Generalizations to the case of $N$ risky assets
- Two-fund separation result
- Extension to unlimited borrowing and lending
- Limited borrowing and lending
- Short-sale constraints

# Generalities

Investment (asset) management begins with an analysis of the investment objectives of the entity whose funds are managed

- The purpose of this course is to describe the activities and investment vehicles associated with investment management
  - Investment management also referred to as ptf./money management
- Our immediate goals are to achieve an understanding of:

  ① How investment objectives are determined

  ② The investment vehicles to which an investor can allocate funds (not necessarily these are securitized), the asset menu

  ③ The investment strategies that can be employed by an investor to realize a specified investment objective

  ④ The best way to construct a ptf., given an investment strategy

  ⑤ The techniques for evaluating performance
- The allocating entities may consist of either individual investors or institutional investors

# Generalities

- Institutional investors include:
  - Pension funds
  - Depository institutions (commercial banks, savings and loan associations, and credit unions)
  - Insurance companies (life companies, property and casualty companies, and health companies)
  - Regulated investment companies (mutual and closed-end funds)
  - Endowments and foundations
  - Treasury department of corporations, municipal governments, and government agencies
- We can classify institutional investors into two categories: those that must meet contractually specified liabilities vs. those that don't
- In the 1st category we have institutions with "liability-driven objectives" vs. those in the 2nd with "non-liability driven objectives"
  - An institutional investor is concerned with both the amount and timing of liabilities because its assets must produce the cash flow to meet any payments it has promised to make

# Generalities

Defining clear objectives delimits with precision a sensible asset menu, the strategies and techniques of portfolio selection, and eventually the methodologies of performance assessment

- To form an investment policy, many factors are considered: client constraints, regulatory constraints, tax and accounting issues
  - Examples of client-imposed constraints are restrictions that specify the types of securities in which to invest and concentration limits on how much invested in a particular asset class or in a particular issuer
  - When a benchmark is established, there may be a restriction as to the degree to which the manager may deviate from some key characteristics of that benchmark (passive management)
  - Regulatory constraints also affect the asset classes that are permissible and impose concentration limits on investments
- In making the asset allocation decision, consideration must be given to any risk-based capital requirements
- Sometimes, certain institutional investors are exempt from income taxation if they invest in certain ways or asset classes

# Generalities

## Portfolio strategies may be active or passive

- Portfolio strategies can be classified as either active or passive
- An active portfolio strategy uses available information and forecasting techniques to seek a better performance than a portfolio that is simply diversified broadly
  - Essential to all active strategies are expectations about the factors that have been found to influence the performance of an asset class
- A passive portfolio strategy involves minimal expectational input, and instead relies on diversification to match the performance of some market index
  - Passive strategies assume that prices impound all information
- Which should be selected? The answer depends on (1) the client's or money manager's view of how "price-efficient" the market is; (2) the client's risk tolerance; and (3) a client's liabilities
- Once a portfolio strategy is selected, the next task is to construct the portfolio (i.e., select the specific assets to be included)

# Generalities

- An efficient portfolio is one that provides the highest expected return for a given level of risk, or equivalently, the lowest risk for a given expected return
  - Three key inputs are needed: future expected return, variance of asset returns, and correlation of asset returns
  - Sometimes more complex inputs will be needed, in the limit the dynamics of the joint conditional density of the returns of the asset in the menu of selection
- Finally, performance measurement involves the calculation of the return realized by a portfolio manager over some time interval
- Performance evaluation is concerned with three issues:

① Whether the portfolio manager added value by outperforming the established benchmark

② Identifying how the manager achieved the calculated return

③ Assessing whether the portfolio manager achieved superior performance (i.e., added value) by skill or by luck

# How to correctly measure realized performance

The arithmetic average return is the average value of the withdrawals that can be made at the end of each sub-period while keeping the portfolio's initial market value intact

- When it comes to measure the overall performance of a manager, 3 methodologies are used in practice to calculate the average of sub-period returns
  - ① Arithmetic average rate of return
  - ② Time-weighted rate of return (geometric rate of return)
  - ③ Dollar-weighted return
- The arithmetic average (mean) rate of return is an un-weighted average of the sub-period returns:

$$R_A = \frac{R_{P1} + R_{P2} + \cdots + R_{PN}}{N}$$

$R_A$ = the arithmetic average rate of return
$R_{Pk}$ = the portfolio return for subperiod $k$, where $k = 1, \ldots, N$
$N$ = the number of subperiods in the evaluation period

  - E.g., if the ptf. returns were –10%, 20%, and 5% in months July, August, and September, the arithmetic average monthly return is 5%

# How to correctly measure realized performance

- However, there is a problem: suppose a portfolio's initial market value is $280 million, and the market values at the end of the next two months are $560 million and $280 million, respectively
- Assume no client distributions or contributions for either month
- Then the subperiod return for the first month ($R_{P1}$) is 100%, and the subperiod return for the second month ($R_{P2}$) is –50%; the arithmetic average rate of return is then 25%. Not a bad return!
- But think about this number. The portfolio's initial market value was $280 million. The return over this 2-month evaluation period is zero!

- Thus it is improper to interpret the arithmetic average rate of return as a measure of the average return over an evaluation period
- The proper interpretation is: average value of the withdrawals (expressed as a fraction of initial value) that can be made at the end of each period while keeping the portfolio's initial value intact
- The average monthly return of 25% means that 100% of the portfolio's initial value ($280 million) can be withdrawn at the end of the first month, and 50% must be added at the end of the second

# How to correctly measure realized performance

In general, the arithmetic average rate of return will exceed the geometric rate of return

- The time-weighted rate of return measures the compounded rate of growth of the portfolio's market value during the evaluation period, assuming that all cash distributions are reinvested in the portfolio
- Commonly referred to as the geometric mean return:

$$R_T = [(1 + R_{P1})(1 + R_{P2}) \ldots (1 + R_{PN})]^{1/N} - 1$$

  - E.g., in the earlier example, $[(2.00)(0.50)]^{1/2} - 1 = 0\%$
- In general, the arithmetic and time-weighted average returns will give different values for the portfolio return
- This is because, in computing the arithmetic average rate of return, the amount invested is assumed to be maintained (through additions or withdrawals) at the portfolio's initial market value
- The time-weighted return, in contrast, is the return on a portfolio that varies in size because of the assumption that all proceeds are reinvested

# How to correctly measure realized performance

**The magnitude of the difference btw. the two averages is smaller the less the variation in the subperiod returns over the period**

- The exception is in the special situation where all the subperiod returns are the same, in which case the averages are identical
- The dollar-weighted rate of return is computed by finding the discount rate that will make the present value of the cash flows from all the subperiods in the evaluation period plus the portfolio's terminal market value equal to the portfolio's initial market value
  - It is not necessary to know the portfolio's market value for each subperiod to determine the dollar-weighted rate of return
- The dollar-weighted rate of return is simply an internal rate of return calculation:

$$V_0 = \frac{C_1}{(1+R_D)} + \frac{C_2}{(1+R_D)^2} + \cdots + \frac{C_N + V_N}{(1+R_D)^n}$$

$R_D$ = the dollar-weighted rate of return
$V_0$ = the portfolio's initial market value
$V_N$ = the portfolio's terminal market value
$C_k$ = the portfolio's cash flow (cash inflows minus cash outflows) for subperiod $k$, where $k = 1, 2, \ldots, N$

# How to correctly measure realized performance

## The dollar-weighted rate of return is an internal rate of return

- The dollar-weighted return and the time-weighted return will produce the same result if no withdrawals or contributions occur over the period and if all investment income is reinvested
- The problem with the dollar-weighted rate of return is that it is affected by factors beyond the control of the money manager
- Any contributions made by the client or withdrawals that the client requires will affect the calculated return
- This may make it difficult to compare the performance of two portfolio managers
- Subperiod returns are usually calculated for less than one year
- The subperiod returns are then annualized using:

$$\text{Annual return} = (1 + \text{Average period return})^{\text{Number of periods in year}} - 1$$

  - E.g., suppose the evaluation period is 3 years, and a monthly period return is calculated to be 2%. Then the annual return would be

$$\text{Annual return} = (1.02)^{12} - 1 = 26.8\%$$

# The characteristics of the opportunity set

## Asset payoffs under risk are described by return distributions

- The existence of risk means that the investor can no longer associate a single number or payoff with investing in any asset
- The payoff must be described by a set of outcomes and each of their associated probability, called a frequency or return distribution
- However, to work with densities is extremely complex
- Often specific attributes (moments) are used to summarize the key features of such distribution: a measure of central tendency, called the expected return, and a measure of risk or dispersion around the mean, called the standard deviation
- A frequency distribution is a list of all possible out-comes along with the prob. of each
- Usually we do not delineate all of the possibilities

| *State* | *Security A* | | *Security B* | |
|---|---|---|---|---|
| | Payoff | Prob. | Payoff | Prob. |
| *i* | 20 | 3/15 | 18 | 3/15 |
| *ii* | 18 | 5/15 | 18 | 5/15 |
| *iii* | 14 | 4/15 | 10 | 4/15 |
| *iv* | 10 | 2/15 | 5 | 2/15 |
| *v* | 6 | 1/15 | 5 | 1/15 |

# The characteristics of the opportunity set

## The common measure of location for returns is the expectation

- It takes at least two measures to capture the relevant information about a frequency function: one to measure the average value and one to measure the dispersion around the average value
- Using the summation notation, we have:

$$E[R_i] = \sum_{s=1}^{S} Prob(state = s) R_i(s)$$

  where Prob(*state*) is the probability of the *s*th state for asset *i*
- Certain properties of expected values are extremely useful
  - The expected value of the sum of two returns is equal to the sum of the expected value of each return, that is, $E[R_1 + R_2] = E[R_1] + E[R_2]$
  - The expected value of a constant "*C*" times a return is the constant times the expected return, that is, $E[CR_1] = CE[R_1]$
- Not only is it necessary to have a measure of the average return, it is also useful to have some measure of how much the outcomes differ from the average

# The characteristics of the opportunity set

## The common measure of dispersion for returns is the variance

- The average squared deviation is the variance

$$Var[R_i] = \sum_{s=1}^{S} Prob(state = s)[R_i(s) - E[R_i]]^2$$

  - Many utility functions can be expressed either exactly or approximately in terms of mean and variance
  - Furthermore, regardless of the investor's utility function, if returns are normally distributed, the mean and variance contain all relevant information about the distribution
  - The square root of the variance is called the standard deviation
  - In this case, even though security B gives a lower mean return, it is more risky than A

| *State* | *Security A* | | *Security B* | |
|---|---|---|---|---|
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| *i* | 20 | 3/15 | 18 | 3/15 |
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| *iii* | 14 | 4/15 | 10 | 4/15 |
| *iv* | 10 | 2/15 | 5 | 2/15 |
| *v* | 6 | 1/15 | 5 | 1/15 |
| Mean | 15.47 | | 13.27 | |
| Variance | 16.78 | | 28.46 | |

# The characteristics of the opportunity set

## Several measures of dispersion focus on downside risk only

- There are other measures of dispersion that could be used
  - The average absolute deviation
  - A semi-variance based only on deviations below the mean
  - The argument is that returns above the average return are desirable: the only returns that disturb an investor are those below average
- It is just one of a number of possible measures of downside risk
  - More generally, we can consider returns relative to other benchmarks, including a risk-free return or zero return
  - These generalized measures are referred to as lower partial moments
  - Other measure of downside risk is the so-called Value at Risk, which is widely used by banks to measure their exposure to adverse events
  - It measures the least expected loss (relative to zero, or relative to wealth) that will be expected with a certain probability
  - E.g., if 5% of the outcomes are below -30% and if the decision maker is concerned about how poor the outcomes are 5% of the time, then -30% is the 5% value at risk

# Portfolio combinations

## Several measures of dispersion focus on downside risk only

- Intuitively, these measures of downside risk are reasonable and some portfolio theory has been developed using them
- They are difficult to use when we move from single assets to ptfs
- In cases where the distribution of returns is symmetrical, the ordering of portfolios in mean variance space will be the same as the ordering of portfolios in mean semi-variance space or mean and any of the other measures of downside risk discussed above
  - If returns on an asset are symmetrical, the semi-variance is proportional to the variance; therefore in most of the portfolio literature the variance is used as a measure of dispersion
- Consider two assets. How can we decide which one we prefer?
- Intuitively one would think that most investors would prefer the one with the higher expected return if standard deviation was held constant
- In the following table, most investors would prefer asset 2 to 1

# Portfolio combinations

The risk of a combination of assets is very different from a simple average of the risk of individual assets

| Market Condition | Return[a] Asset 1 | Asset 2 | Asset 3 | Asset 5 | Rainfall | Return[a] Asset 4 |
|---|---|---|---|---|---|---|
| Good | 15 | 16 | 1 | 16 | Plentiful | 16 |
| Average | 9 | 10 | 10 | 10 | Average | 10 |
| Poor | 3 | 4 | 19 | 4 | Poor | 4 |
| Mean return | 9 | 10 | 10 | 10 | | 10 |
| Variance | 24 | 24 | 54 | 24 | | 24 |
| Standard deviation | 4.9 | 4.9 | 7.35 | 4.90 | | 4.9 |

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

- Similarly, if expected return were held constant, investors would prefer the one with the lower variance
  - In the table, the investor would prefer asset 2 to asset 3
- However, the options open to an investor are not to simply pick between assets 1, 2, 3, 4, or 5 but also to consider combinations
- This makes sense because the risk of a combination of assets is very different from a simple average of the risk of individual assets

# Portfolio combinations

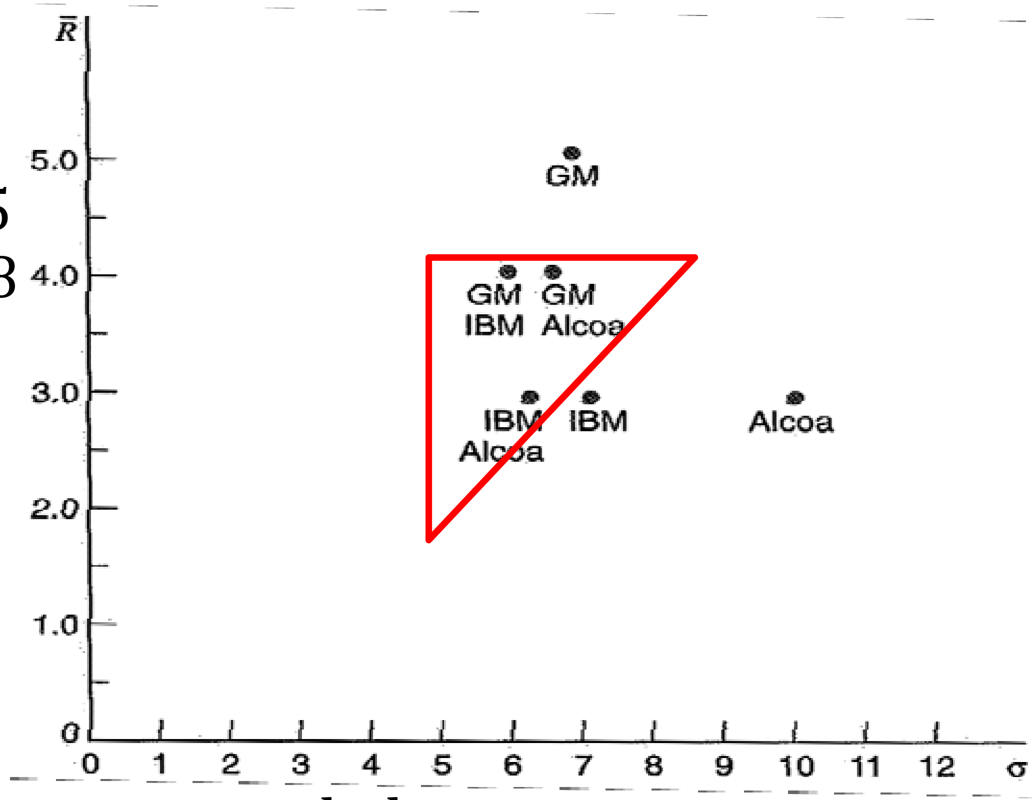
When returns of assets that are less than perfectly correlated, the resulting ptf. has less dispersion than either assets

| Condition of Market | Asset 2 | Asset 3 | Combination of Asset 2 (60%) and Asset 3 (40%) |
|---|---|---|---|
| Good | $1.16 | $1.01 | $1.10 |
| Average | 1.10 | 1.10 | 1.10 |
| Poor | 1.04 | 1.19 | 1.10 |

- Most dramatically, the variance of a combination of two assets may be less than the variance of either of the assets themselves
  - In the table, there is a combination of asset 2 and asset 3 that is less risky than either asset 2 or 3
  - When two assets have their good and poor returns at opposite times, an investor can always find some combination of these assets that yields the same return under all market conditions
  - Even when the returns on assets are independent such as the returns on assets 2 and 4, a portfolio of such assets can have less dispersion than either asset

# Portfolio combinations

- The figure shows a real-life combinations of 3 stocks
  - Correlations: IBM-Alcoa =0.05 GM-Alcoa=0.22; IBM-GM=0.48
  - 50-50 combinations shown
  - A ptf. 50% IBM and 50% Alcoa has the same return as each stock but less risk




- As mean and variance are used to summarize return distributions, we need formal ways to study how assets co-move
- The return on a portfolio of assets is simply a weighted average of the return on the individual assets: $$R_{Pj} = \sum_{i=1}^{N} (X_i R_{ij})$$

  $X_i$ is the fraction of the investor's funds invested in the *i*th asset
- Given properties of expectations, $$\bar{R}_P = E(R_P) = E\left(\sum_{i=1}^{N} X_i R_{ij}\right) = \sum_{i=1}^{N} (X_i \bar{R}_i)$$

# Portfolio combinations: return covariances

- The variance on a portfolio is a little more difficult to determine than the expected return; we start out with a two-asset example:

$$\sigma_P^2 = E\left(R_P - \bar{R}_P\right)^2 = E\left[X_1 R_{1j} + X_2 R_{2j} - \left(X_1 \bar{R}_1 + X_2 \bar{R}_2\right)\right]^2$$

$$= E\left[X_1\left(R_{1j} - \bar{R}_1\right) + X_2\left(R_{2j} - \bar{R}_2\right)\right]^2$$

$$= E\left[X_1^2\left(R_{1j} - \bar{R}_1\right)^2 + 2X_1X_2\left(R_{1j} - \bar{R}_1\right)\left(R_{2j} - \bar{R}_2\right) + X_2^2\left(R_{2j} - \bar{R}_2\right)^2\right]$$

- Applying our two rules on expected values, we have

$$\sigma_P^2 = X_1^2 E\left[\left(R_{1j} - \bar{R}_1\right)^2\right] + 2X_1X_2E\left[\left(R_{1j} - \bar{R}_1\right)\left(R_{2j} - \bar{R}_2\right)\right] + X_2^2 E\left[\left(R_{2j} - \bar{R}_2\right)^2\right]$$

$$= X_1^2\sigma_1^2 + 2X_1X_2E\left[\left(R_{1j} - \bar{R}_1\right)\left(R_{2j} - \bar{R}_2\right)\right] + X_2^2\sigma_2^2$$

- $E[(R_{1j} - \bar{R}_1)(R_{2j} - \bar{R}_2)]$ has a special name: the covariance, $\sigma_{12}$
- The covariance is the expected value of the product of two deviations: the deviations of the returns on security 1 from its mean and the deviations of security 2 from its mean
- As such it can be positive or negative

# Portfolio combinations: return covariances

## Covariance and correlations measure linear co-movements

- It will be large when the good outcomes for each stock occur together and when the bad outcomes for each stock occur together
- The covariance is a measure of how returns move together
- For many purposes it is useful to standardize the covariance: dividing the covariance between two assets by the product of the standard deviation of each asset produces a variable with the same properties as the covariance but with a range of -1 to + 1
- This measure is the correlation coefficient: $$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$$
- Covariances and correlations are traditionally collected in symmetric matrices
  - Note that when $\rho_{12} = -1$ a ptf. may be built with no risk
  - In general ptf. risk is less than individual securitities when $\rho_{12} = 0$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | | 24 (+1) | −36 (−1) | 0 (0) | 24 (+1) |
| 2 | | | −36 (−1) | 0 (0) | 24 (+1) |
| 3 | | | | 0 (0) | −36 (−1) |
| 4 | | | | | 0 (0) |
| 5 | | | | | |

# Aggregate portfolio variance

## A ptf. of $N$ uncorrelated assets has a zero variance as $N \rightarrow \infty$

- The formula for variance of a portfolio can be generalized to any generic number N of uncorrelated assets
  - The variance of each asset is multiplied by the square of the proportion invested in it: $\sum_{j=1}^{N}\left(X_j^2\sigma_j^2\right)$
  - With N assets the proportion invested in each asset is l/N, applying our formula yields:
$$\sigma_P^2 = \sum_{j=1}^{N}(1/N)^2\sigma_j^2 = 1/N\left[\sum_{j=1}^{N}\frac{\sigma_j^2}{N}\right]$$
  - Thus the formula reduces to $\sigma_P^2 = 1/N\bar{\sigma}_j^2$, where "sigma-bar" represents the average variance of the stocks in the portfolio
  - As N gets larger and larger, the variance of the portfolio gets smaller and smaller and the variance of the portfolio approaches zero
  - This is a general result: if we have enough uncorrelated assets, the variance of a portfolio of these assets approaches zero
  - In general, we are not so fortunate as in most markets the correlation coefficient and the covariance between assets is positive
  - In these markets the risk on the portfolio cannot be made to go to zero

# Aggregate portfolio variance

While as N $\rightarrow \infty$ the contribution to variance goes to zero, the contribution to covariance has a finite limit

- The general formula is:

$$\sigma_P^2 = \sum_{j=1}^{N} \left( X_j^2 \sigma_j^2 \right) + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \left( X_j X_k \sigma_{jk} \right)$$

  - Once again, consider equal investment in N assets:

$$\sigma_P^2 = \sum_{j=1}^{N} (1/N)^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (1/N)(1/N) \sigma_{jk}$$

  - Factoring out l/N from the first summation and (N - l)/N from the second yields:

$$\sigma_P^2 = (1/N) \sum_{j=1}^{N} \left[ \frac{\sigma_j^2}{N} \right] + \frac{(N-1)}{N} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \left[ \frac{\sigma_{jk}}{N(N-1)} \right]$$

  - Both of the terms in the brackets are averages: the second term is the summation of covariances divided by the number of covariances
  - Replacing the summations by averages: $\sigma_P^2 = \frac{1}{N} \bar{\sigma}_j^2 + \frac{N-1}{N} \bar{\sigma}_{jk}$
  - The contribution to the portfolio variance of the variance of the individual securities goes to zero as N gets very large

# The effects of diversification: systematic risk

## The contribution to total risk caused by covariance is systematic

- The contribution of the covariance terms approaches the average covariance as N→∞
- The individual risk of securities can be diversified away, but the contribution to total risk caused by the covariance terms cannot be diversified away
- The table illustrates how this relationship looks when dealing with U.S. equities
  - The average variance and average covariance were calculated using monthly data for all stocks listed on the New York Stock Exchange
- As more and more securities are added, the average variance on the portfolio declines until it approaches the average covariance
  - The figure illustrate this same relationship for common equities in a number of countries

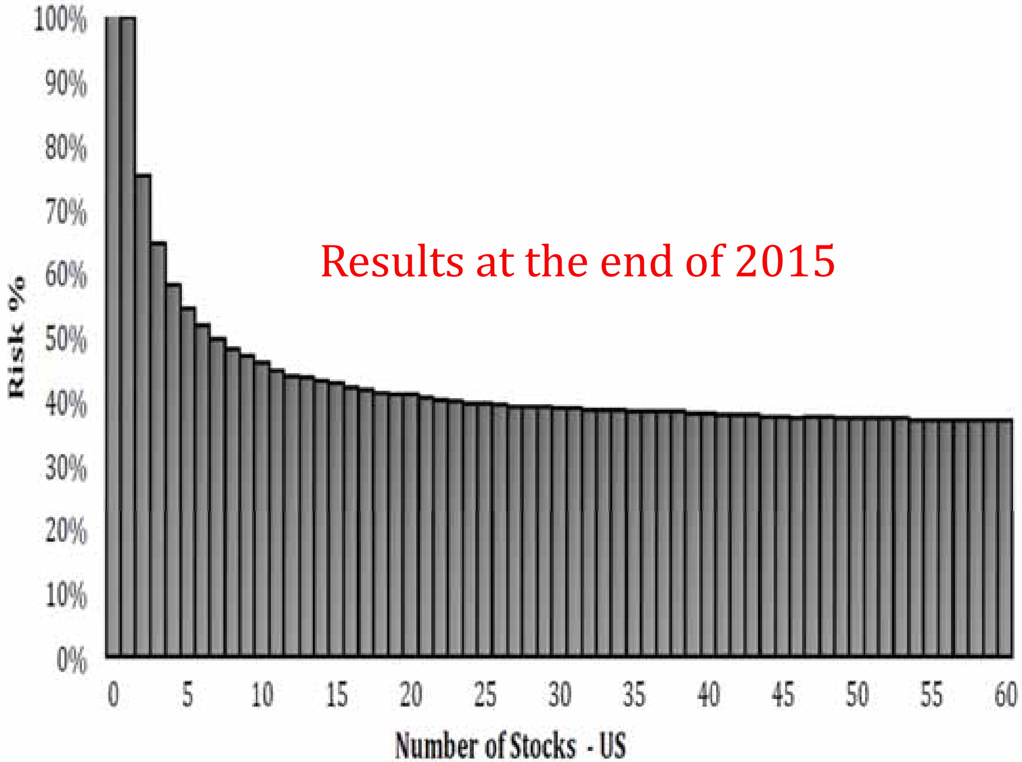
| Number of Securities | Expected Portfolio Varianc |
|---|---|
| 1 | 46.619 |
| 2 | 26.839 |
| 4 | 16.948 |
| 6 | 13.651 |
| 8 | 12.003 |
| 10 | 11.014 |
| 12 | 10.354 |
| 14 | 9.883 |
| 16 | 9.530 |
| 18 | 9.256 |
| 20 | 9.036 |
| 25 | 8.640 |
| 30 | 8.376 |
| 35 | 8.188 |
| 40 | 8.047 |
| 45 | 7.937 |
| 50 | 7.849 |
| 75 | 7.585 |
| 100 | 7.453 |
| 125 | 7.374 |
| 150 | 7.321 |
| 175 | 7.284 |
| 200 | 7.255 |
| 250 | 7.216 |
| 300 | 7.190 |
| 350 | 7.171 |
| 400 | 7.157 |
| 450 | 7.146 |
| 500 | 7.137 |
| 600 | 7.124 |
| 700 | 7.114 |
| 800 | 7.107 |
| 900 | 7.102 |
| 1000 | 7.097 |
| Infinity | 7.058 |

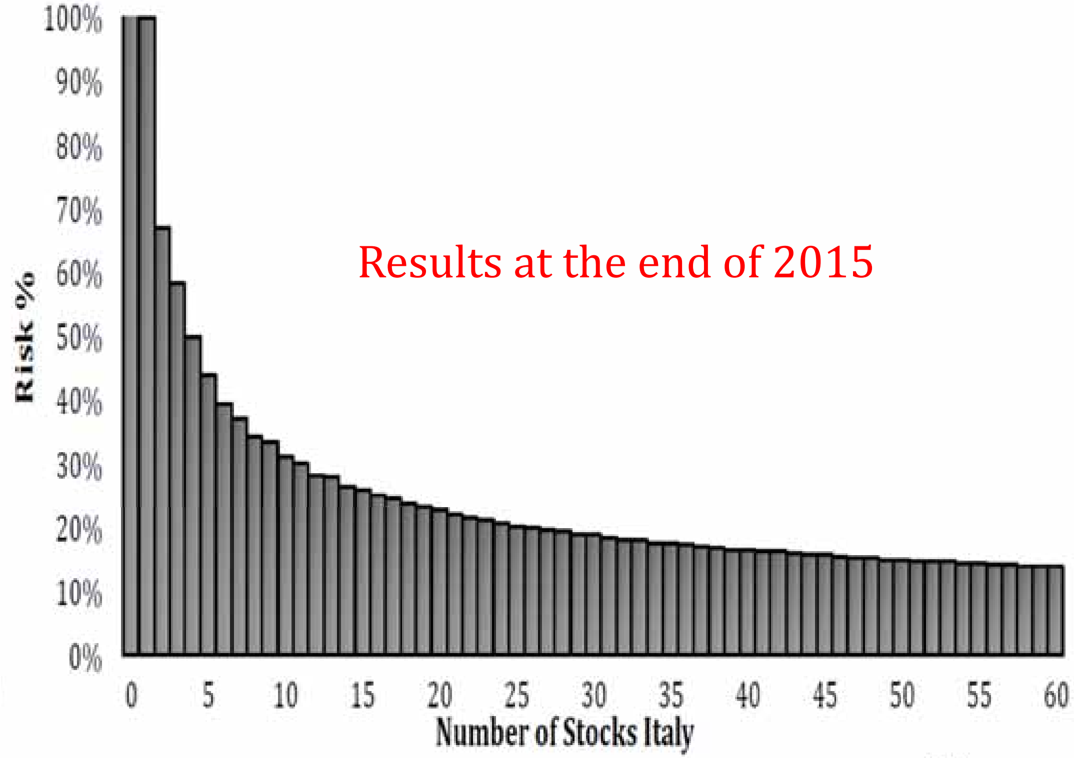
# The effects of diversification: systematic risk

Percentage of the Risk on an Individual Security that Can Be Eliminated by Holding a Random Portfolio of Stocks within Selected National Markets and among National Markets [13]

| | |
|---|---|
| United States | 73 |
| U.K. | 65.5 |
| France | 67.3 |
| Germany | 56.2 |
| Italy | 60.0 |
| Belgium | 80.0 |
| Switzerland | 56.0 |
| Netherlands | 76.1 |
| International stocks | 89.3 |

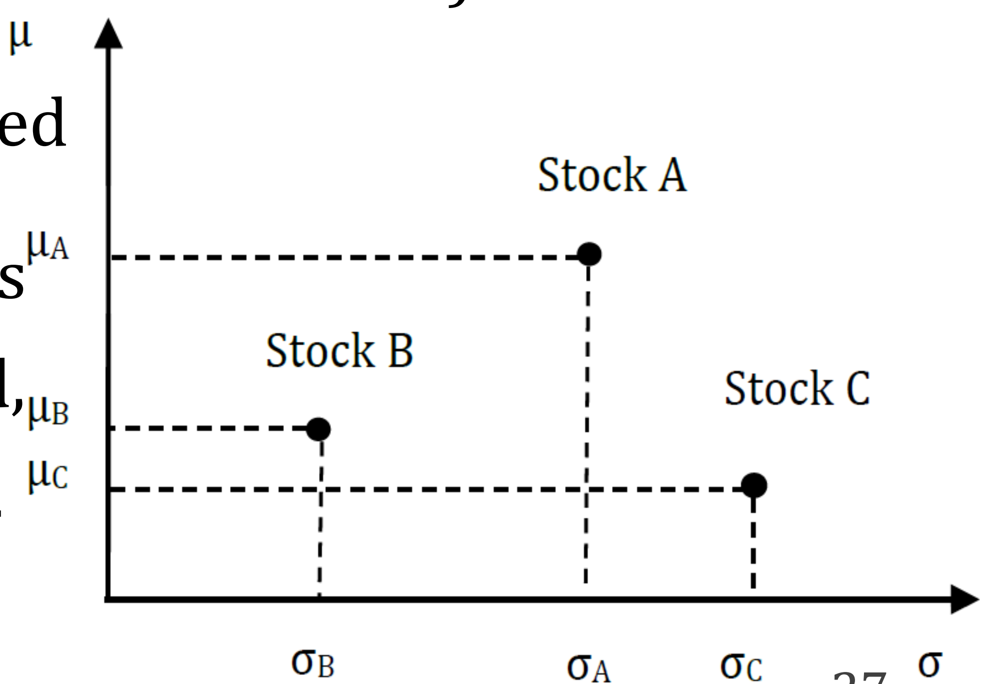
Results from the 1970s



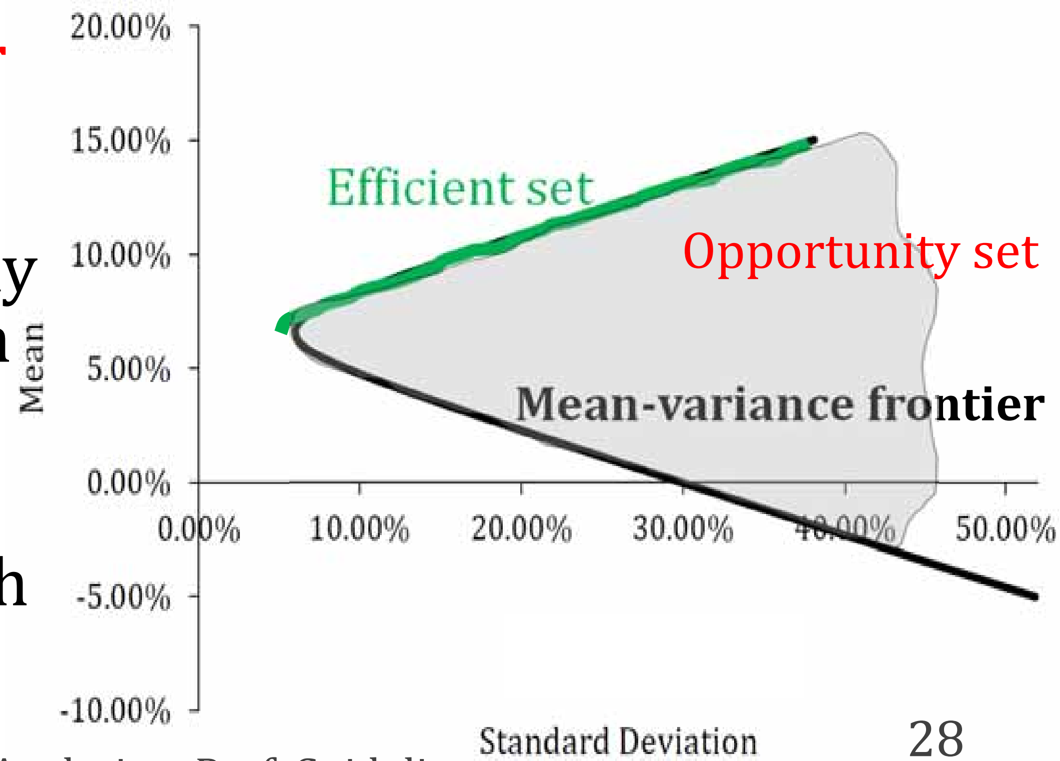
# Key Concepts of Mean-Variance Analysis/1

- We review the development of the celebrated mean-variance framework introduced by Markowitz in the 1950s
- Initially at least, risky assets only, no borrowing or lending
- Assume that for some reasons, the joint distribution of asset returns is completely characterized by their means, variances, and covariances
- We represent each asset (and portfolio of assets) in a two-dimensional diagram, where expected portfolio return is plotted on the vertical axis and standard deviation is on the horizontal axis
- Not all securities may be selected, e.g., stock C is dominated by the remaining two stocks in terms of MV dominance



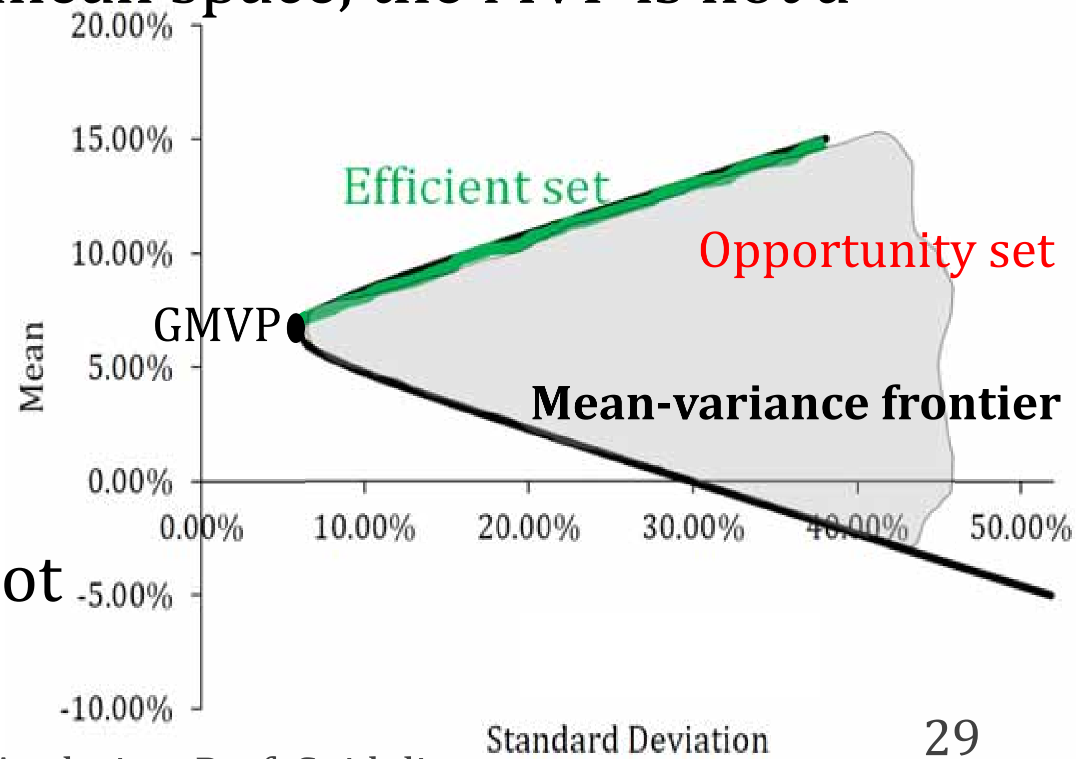
# Key Concepts of Mean-Variance Analysis/2

- According to MV criterion a portfolio is efficient if and only if there is no other portfolio that allows the investor to achieve the same expected return with a lower level of risk or a higher level of expected return with the same level of risk
- Three key notions: i) the opportunity set (feasible region), which includes all the portfolios (both efficient and inefficient) that the investor is able to build given securities in the asset menu
- (ii) the mean-variance frontier (aka minimum variance frontier, MVF) subset of the opportunity set containing only the portfolio(s) with minimum variance for any target level of expected returns
- (iii) the efficient frontier, which only includes efficient ptfs



# Key Concepts of Mean-Variance Analysis/3

- Because it is possible that a portfolio exists which has a higher return than another portfolio with the same level of risk, only portfolios that have a higher expected return than the global minimum variance portfolio (GMVP) are efficient
- The preferences of the investor(s) for risk are not relevant to the determination of the efficient frontier
- In the classical standard dev.-mean space, the MVF is not a function, but a «right-rotated hyperbola”
- The GMPV is Global Minimum Variance Ptf. and it is of high interest because «separates» the efficient set from the MVF
- The structure of GMVP does not depend on expected returns



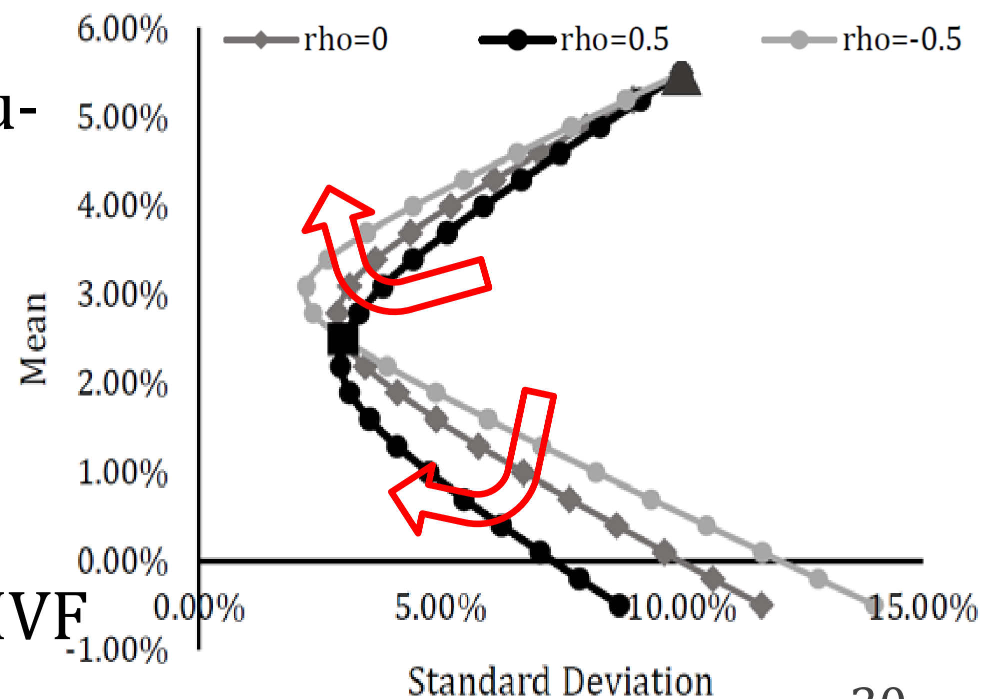
# Key Concepts of Mean-Variance Analysis/4

- The GMVP and the entire MVF depend strongly on the correlation structure of security returns: the lower are the correlations (on average), the more the efficient set moves up and to the left, improving the risk-expected return trade-off
- The position and shape of the MVF reflects the diversification opportunities that a given asset menu offers
- Even though, MVF ptfs are solutions of a complex quadratic programming program, in the absence of constraints, their stru-cture is relatively simple:

$$\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu}$$

$$\mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})]$$

$$\mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})]$$




- Combinations of MVF ptfs. are MVF

# Key Concepts of Mean-Variance Analysis/5

- It is sufficient to know two points (portfolios) on the mean-variance frontier to generate all the others
- A two-fund separation result holds: all MV-optimizers can be satisfied by holding a combination of only two mutual funds (provided these are MV efficient), regardless of their preferences
- Their heterogeneous preferences will only impact the way in which they combine the two funds that they choose to hold
- As an implication, when there are $N > 2$ securities, the primitive assets need not to lie on the MVF
- Arguably, also risk-free assets exist, securities with zero variance of their returns and zero correlation with other assets
- Resorting to unlimited borrowing and lending and the risk-free rate changes the locus on which a rational investor performs her portfolio decisions

# Key Concepts of Mean-Variance Analysis/6

- The presence of riskless assets creates capital transformation lines (CTL) and investors select efficient pfs. on the MVF such that they end up selecting their optimum on the steepest CTL
- When investors have homogeneous beliefs on means, variances, and correlations, in the absence of frictions, all investors will hold an identical tangency portfolio that maximizes the Sharpe ratio of the steepest CTL
- Such steepest CTL is called the Capital Market Line
- While the share of wealth an investor lends or borrows at the risk-free rate depends on the investor's preference for risk, the risky portfolio should be the same for all the investors
- This is special case of two-fund separation result stated above
- When lending and borrowing is only possible at different rates, it is no longer possible to determinate a tangency portfolio and the efficient set fails to be linear, the steepest CLT

# The Efficient Frontier with Two Risky Assets

- Assume no borrowing or lending at the risk-free rate
- Re-cap of a few basic algebraic relationships that exploit the fact that with two risky assets, $\omega_B = 1 - \omega_A$
  - See textbook for detailed derivations $\mu_P = \omega_A \mu_A + (1 - \omega_A)\mu_B$
  - Portfolio mean & variance: $\sigma_P^2 = \omega_A^2 \sigma_A^2 + (1 - \omega_A)^2 \sigma_B^2 + 2\omega_A(1 - \omega_A)\sigma_{AB}$
  - Using the definitions of correlation and of standard deviation:

$$\sigma_P = \sqrt{\omega_A^2 \sigma_A^2 + (1 - \omega_A)^2 \sigma_B^2 + 2\omega_A(1 - \omega_A)\rho_{AB}\sigma_A\sigma_B}$$

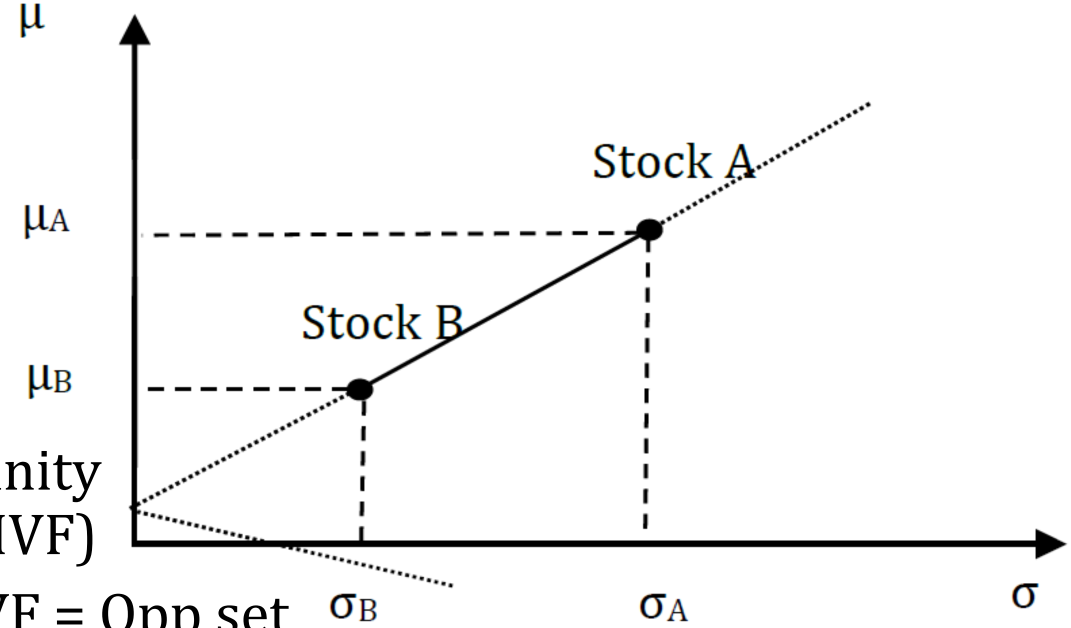
  - Solve mean equation for $\omega_A$ and plug the result into st. dev. equation ⇒ a system of 2 equations in 2 unknowns
  - The system has in general a unique solution ⇒ the opportunity set is a curve and it coincides with the mean-variance frontier (there is only one possible level of risk for a given level of return)
  - The shape of set depends on the correlation between the 2 securities
- Three possible cases: (i) $\rho_{AB}$ = + 1; (ii) $\rho_{AB}$ = -1; (iii) $\rho_{AB} \in (0,1)$
- Case (i): $\rho_{AB} = +1$: the expression for $\sigma^2_P$ becomes a perfect square sum and this simplifies the algebra

# The Efficient Frontier with Two Risky Assets

- After algebra (see textbook), we have:

$$\mu_P = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\mu_A + \left(1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\,(\mu_A - \mu_B)$$

  the equation of a straight line, with slope $(\sigma_P - \sigma_A)/(\sigma_A - \sigma_B)$
  - In the picture, dashed lines == ptfs. require short selling
  - Without short sales, the least risky stock == GMVP
  - With short sales, the GMVP has zero risk
  - In this special case, the opportunity set = mean-variance frontier (MVF)
  - With no short sales, EffSet = MVF = Opp set




- Case (ii): $\rho_{AB}$ = -1: the expression for $\sigma^2_P$ becomes a perfect square difference and this simplifies the algebra (see textbook) to yield:

$$\sigma_P = \omega_A\sigma_A - (1-\omega_A)\sigma_B \quad \text{or to} \quad \sigma_P = -\omega_A\sigma_A + (1-\omega_A)\sigma_B$$

- Yet, each of the equations only holds when the RHS is positive

# The Efficient Frontier with Two Risky Assets

- The opportunity set is a straight line, but its slope depends on which of the equations above holds
- If the first equation applies, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\,(\mu_A - \mu_B)$$

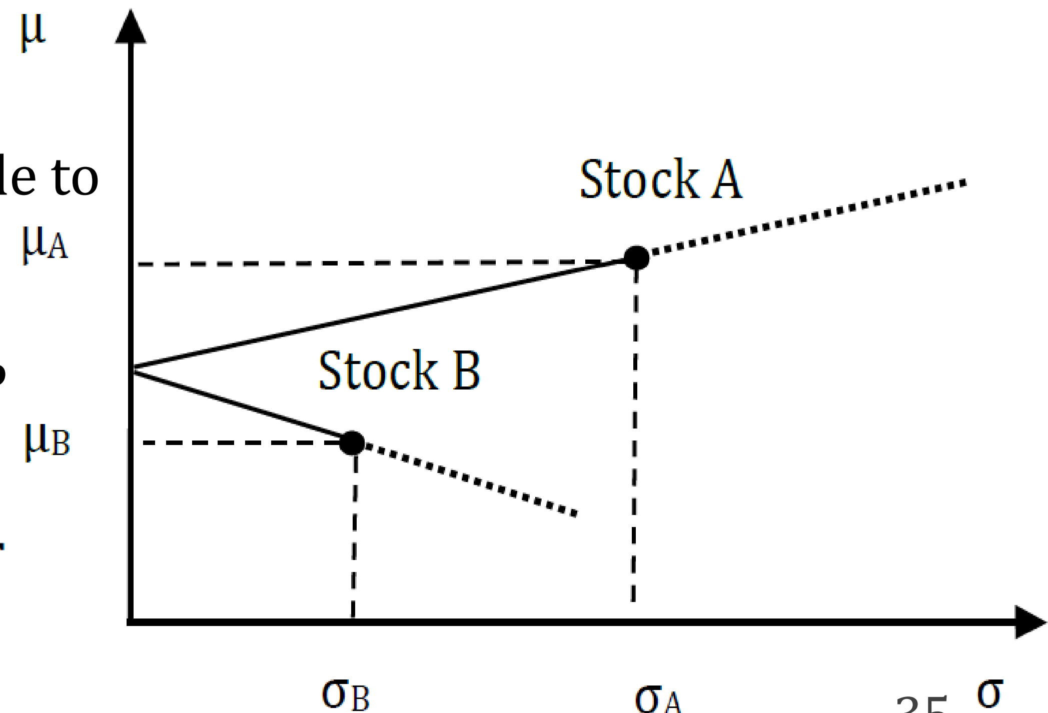
while if the second equation holds, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\,(\mu_A - \mu_B)$$

- In the picture, dashed lines == ptfs. require short selling
- Even without short sales, possible to find a combination that has zero variance, i.e., it is risk-free
- Such a riskless portfolio is GMVP
- The expression for such a ptf. is:

$$\omega_A^{GMVP}\sigma_A - (1 - \omega_A^{GMVP})\sigma_B = 0 \text{ or}$$

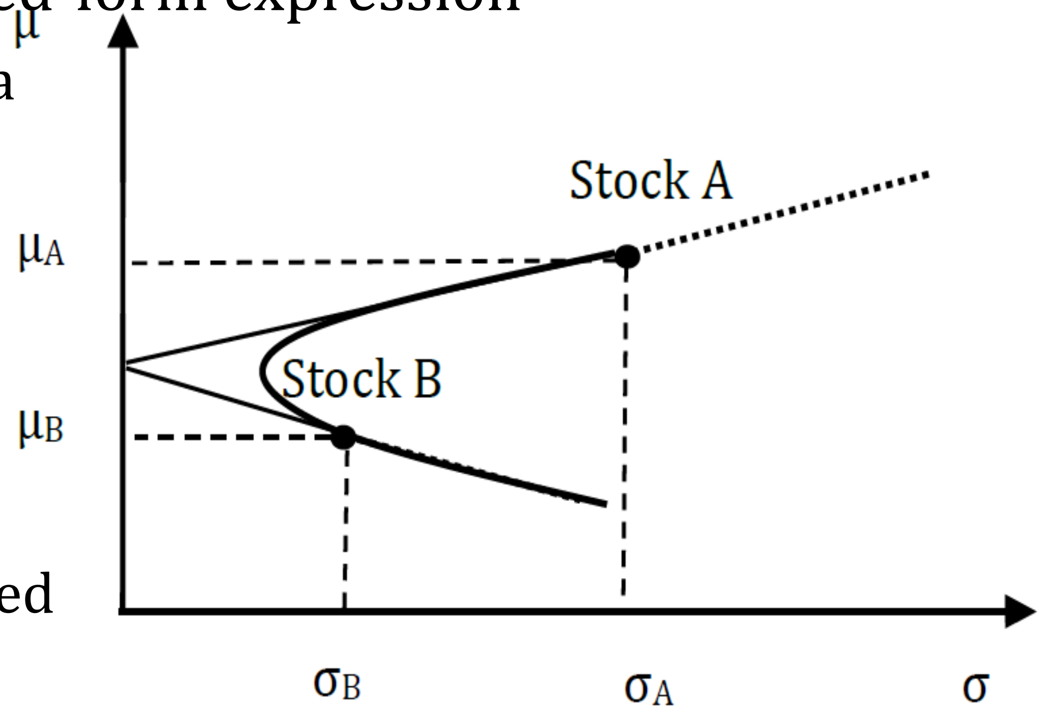
$$-\omega_A^{GMVP}\sigma_A + (1 - \omega_A^{GMVP})\sigma_B = 0$$



# The Efficient Frontier with Two Risky Assets

$$\Longrightarrow \quad \omega_A^{GMVP} = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

- Case (iii): $\rho_{AB} \in (0,1)$: In this case, although tricks exist to trace it out, the MVF does not have a closed-form expression
  - The MVF is non-linear, a parabola (i.e., a quadratic function) in the variance-mean space
  - Or a a (branch of) hyperbola in standard deviation-mean space
  - In such a space, the MVF is not a function, it is just a «correspondence», a "right-rotated hyperbola"

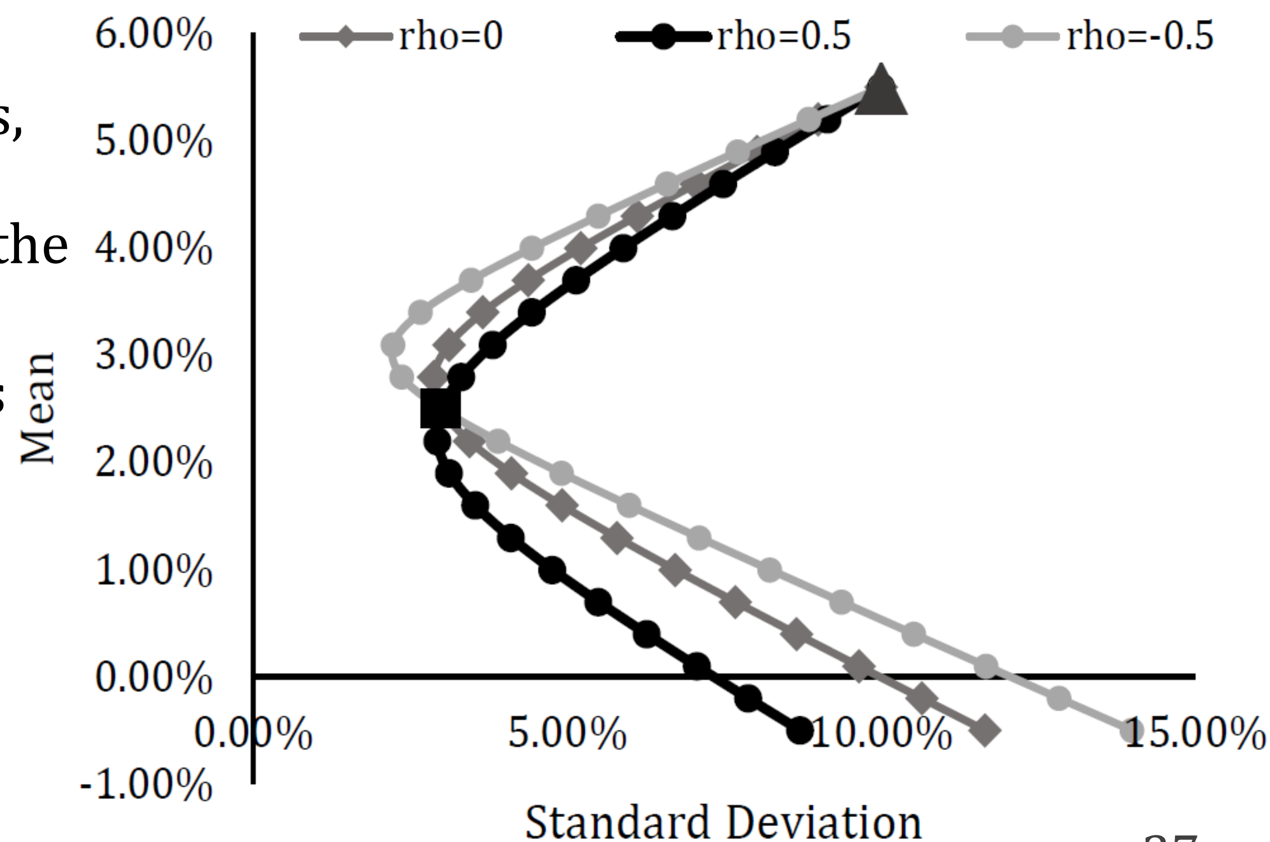



  - The efficient set == a portion of the MVF, the branch of the "rotated hyperbola" that lies above (and includes) the GMVP
  - To distinguish the efficient set from the MVF we have to find the GMVP:

$$\frac{\partial \sigma_P^2}{\partial \omega_A} = 2\omega_A\sigma_A^2 - 2(1-\omega_A)\sigma_B^2 + 2(1-2\omega_A)\rho_{AB}\sigma_A\sigma_B \Longrightarrow \omega_A^{GMVP} = \frac{\sigma_B^2 - \rho_{A,B}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B}$$

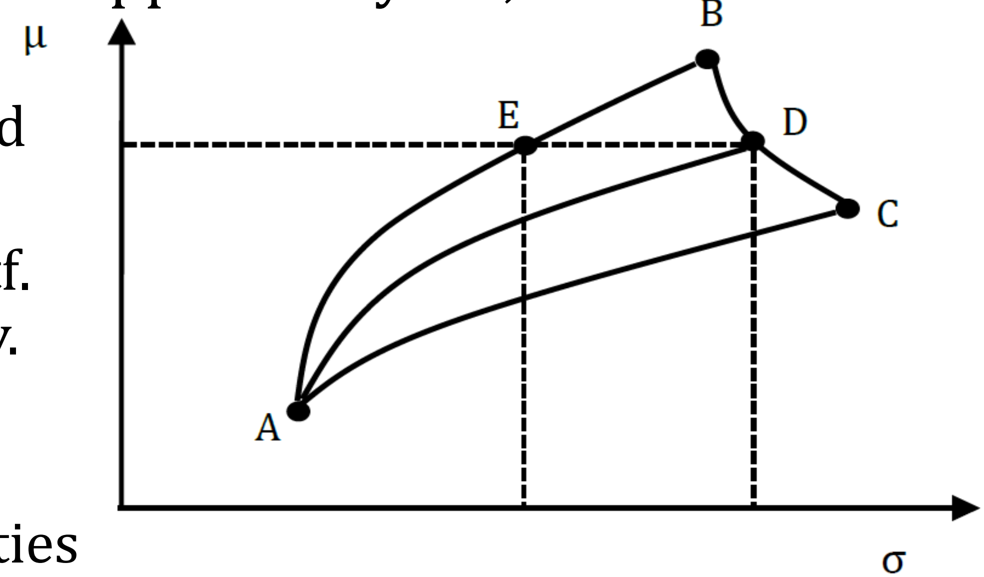
# One Numerical Example

- For instance, consider stock A with $\mu_A = 5.5\%$ and $\sigma_A = 10\%$, and stock B with $\mu_B = 2.5\%$ and $\sigma_B = 3\%$
- Draw MVF in Excel for $\rho_{A,B} = 0$, $\rho_{A,B} = 0.5$, and $\rho_{A,B} = -0.5$
- See textbook for calculations and details and book's website for exercises in Excel related to this case
  - When the $\rho_{A,B} < 0$, it is possible to form ptfs. that have a lower risk than each of the 2 assets
  - Clearly, as $\rho_{A,B}$ declines, risk characterizing the GMVP moves towards the left, inward
  - The entire MVF rotates upward, less risk may be borne for identical expected ptf. return
  - Note tha the GMVP often needs to include short positions



# The Case of N Risky Assets

- Usually investors choose among a large number of risky securities
  - E.g., allocation among the 500 stocks in the S&P 500
- Extend our framework to the general case, with *N* risky assets
- The MVF no longer coincides with the opportunity set, which now becomes a region and not a line




  - Ptf. D, a combination of assets B and C, is not MV efficient
  - It gives the same mean return as ptf. E but implies a higher standard dev. and a risk-averse investor would never hold portfolio D
  - To exclude all the inefficient securities and ptfs., as first step the investor needs to trace out the MVF, i.e., select ptfs. with minimum variance (std. dev.) for each level of μ
  - Only interested in the upper bound of the feasible region
- We solve the following quadratic programming problem:

$$\min_{\{\omega\}} \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \quad Subject\ to \quad \boldsymbol{\iota}'\boldsymbol{\omega} = 1, \quad \boldsymbol{\mu}'\boldsymbol{\omega} = \bar{\mu}$$

Nx1 vector of 1s

Target mean

# The Case of N Risky Assets

- For the time being, no short-sale restrictions have been imposed
  - To solve the program, assume that no pair or general combination of asset returns are linearly dependent
  - $\Rightarrow \Sigma$ is nonsingular and invertible; in fact, $\Sigma$ is (semi-)positive definite
- Under these conditions, it is a constrained minimization problem that can be solved through the use of Lagrangian multiplier method
- See your textbook for algebra and details
- If one defines $A \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$, $B \equiv \boldsymbol{\mu}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $C \equiv (\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})$, and $D = BC - A^2$ then the unique solution to the problem, $\boldsymbol{\omega}^*$, is:

$$\boxed{\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu}} \qquad \mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})],\ \mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota})]$$

i.e., any combination of MVF ptf. weights gives another MVF ptf.

  - Consider two MVF ptfs. $P_1$ and $P_2$ with mean $\mu_{P1}$ and $\mu_{P2}$, and assume that $P_3$ is a generic portfolio on the MVF: always possible to find a quantity $x$ such that $\mu_{P_3} = x\mu_{P_1} + (1-x)\mu_{P_2}$
  - Other MVF ptf: $\boldsymbol{\omega}_{P_3} = x\omega_{P_1} + (1-x)\omega_{P_2} = x(\mathbf{g} + \mathbf{h}\mu_{P_1}) + (1-x)(\mathbf{g} + \mathbf{h}\mu_{P_2})$

$$= \mathbf{g} + \mathbf{h}(x\mu_{P_1} + (1-x)\mu_{P_2}) = \mathbf{g} + \mathbf{h}\mu_{P_3},$$

# Two-Fund Separation

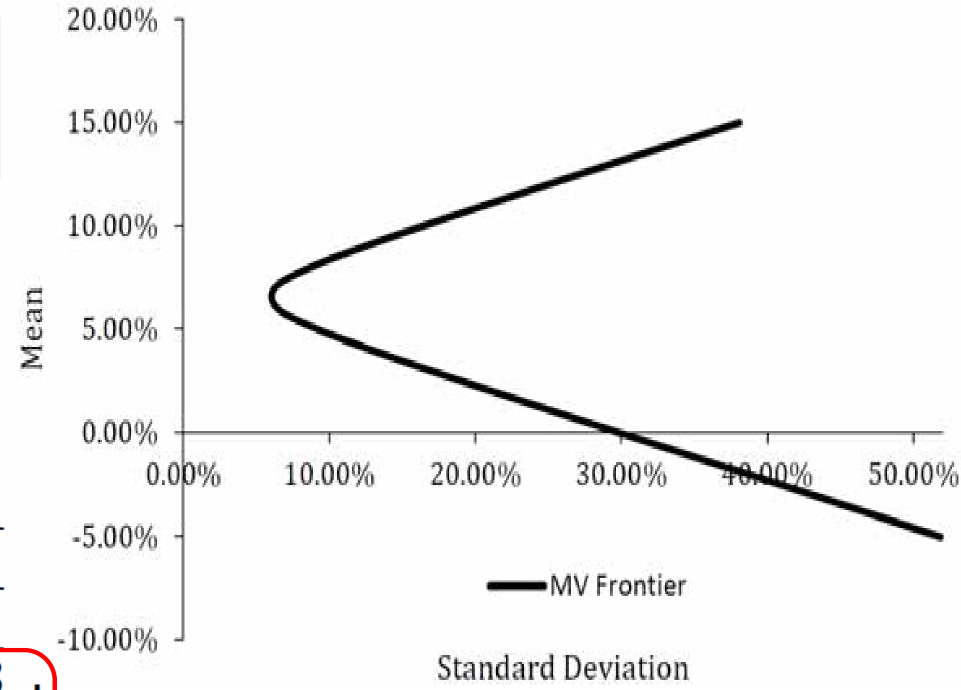
It is sufficient to know two points (portfolios) on the mean-variance frontier to generate all the others

- All MV-optimizers are satisfied by holding a combination of two mutual funds (provided they are MV efficient), regardless of preferences
- Their heterogeneous preferences will only impact the way in which they combine the two funds that they choose to hold
- In equilibrium, if all investors are rational MV optimizers, the market portfolio, being a convex combination of the optimal portfolios of all the investor, has to be an efficient set portfolio
- As for the shape of MVF when $N$ assets are available, this is a rotated hyperbola as in case of 2 assets: $$\sigma_P^2 = \frac{1}{D}[C(\mu_P)^2 - 2A\mu_P + B]$$
  - Equation of a parabola with vertex in $((1/C)^{1/2}, A/C)$, which also represents the global minimum variance portfolio
  - The textbook shows that GMV weights are: $\boldsymbol{\omega}_{GMVP} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{C} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}},$

# One Strategic Asset Allocation Example

- Consider three assets – U.S. Treasury, corporate bonds, and equity – characterized by the mean vector and the variance-covariance matrix:

$$\mu = \begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} 0.005 & 0.004 & -0.002 \\ 0.004 & 0.008 & 0.003 \\ -0.002 & 0.003 & 0.025 \end{bmatrix}$$




- The textbook guides you to perform calculations of A, B, C, D using Excel:

$$\mathbf{g} = \frac{1}{D}[B(\mathbf{\Sigma}^{-1}\boldsymbol{\iota}) - A(\mathbf{\Sigma}^{-1}\boldsymbol{\mu})]$$

$$= \frac{1}{14.21}\left\{1.26 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 6{,}00 \\ 7{,}50 \\ 9{,}00 \end{bmatrix}\right\} = \begin{bmatrix} 4.63 \\ -3.27 \\ -0.37 \end{bmatrix}$$

$$\mathbf{h} = \frac{1}{D}[C(\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) - A(\mathbf{\Sigma}^{-1}\boldsymbol{\iota})] = \frac{1}{14.21}\left\{282.10 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix}\begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} - 18.5 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} -57.78 \\ 48.89 \\ 8.89 \end{bmatrix}$$

# Unlimited, Riskless Borrowing and Lending

- So far, we have ignored the existence of a risk-free asset == a security with return $R^f$ known with certainty and zero variance and zero covariance with all risky assets
  - Buying such a riskless asset == lending at a risk-free rate to issuer
  - Assume investor is able to leverage at riskless rate
  - There is no limit to the amount that the investor can borrow or lend at the riskless rate (we shall remove this assumption later)
- Fictional experiment in which the possibility to borrow and lend at $R^f$ is offered to investor who already allocated among N risky assets
- X is the fraction of wealth in an efficient frontier, risky portfolio (A) characterized $\mu_A$ and $\sigma_A$, respectively; a share 1 - X is invested in the riskless asset, to obtain mean and standard deviation:

$$\mu_P = X\mu_A + (1 - X)R^f = R^f + X(\mu_A - R^f)$$

$$\sigma_P = \sqrt{\sigma_P{}^2} = \sqrt{X^2\sigma_A{}^2} = X\sigma_A$$

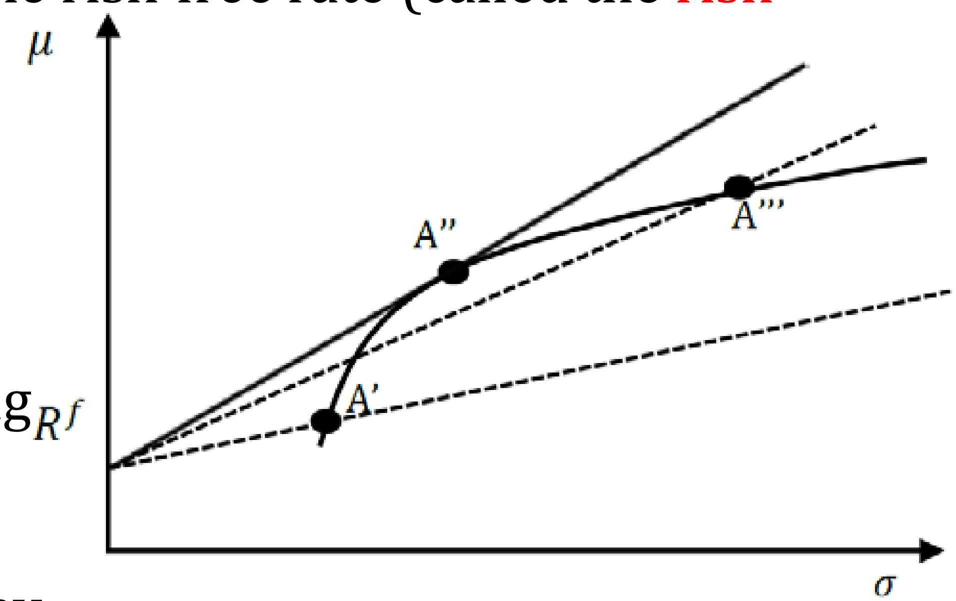
- Solving from X in the first equation and plugging into the second:

$$\mu_P = R_F + \frac{\sigma_P}{\sigma_A}(\mu_A - R^f) = R^f + \frac{(\mu_A - R^f)}{\sigma_A}\sigma_P$$

# Unlimited, Riskless Borrowing and Lending

The capital transformation line measures at what rate unit risk (st. dev.) can be transformed into average excess return (risk premium)

- The equation of a straight line with intercept $R^f$ and slope $(\mu_A - R^f)/\sigma_A$
- This line is sometimes referred to as capital transformation line
- The term $(\mu_A - R^f)/\sigma_A$ is called Sharpe ratio (SR), the total reward for taking a certain amount of risk, represented by the st. dev.
  - SR is the mean return in excess of the risk-free rate (called the risk premium) per unit of volatility
  - The plot shows 3 transformation lines for 3 choices of the risky benchmark A (A', A'', and A''') on the efficient frontier
  - Points to the left of A involve lending at the risk-free rate while the ones to the right involve borrowing
  - As investors prefer more to less, they will welcome a "rotation" of the straight line passing through $R^f$ as far as possible in a counterclockwise direction, until tangency

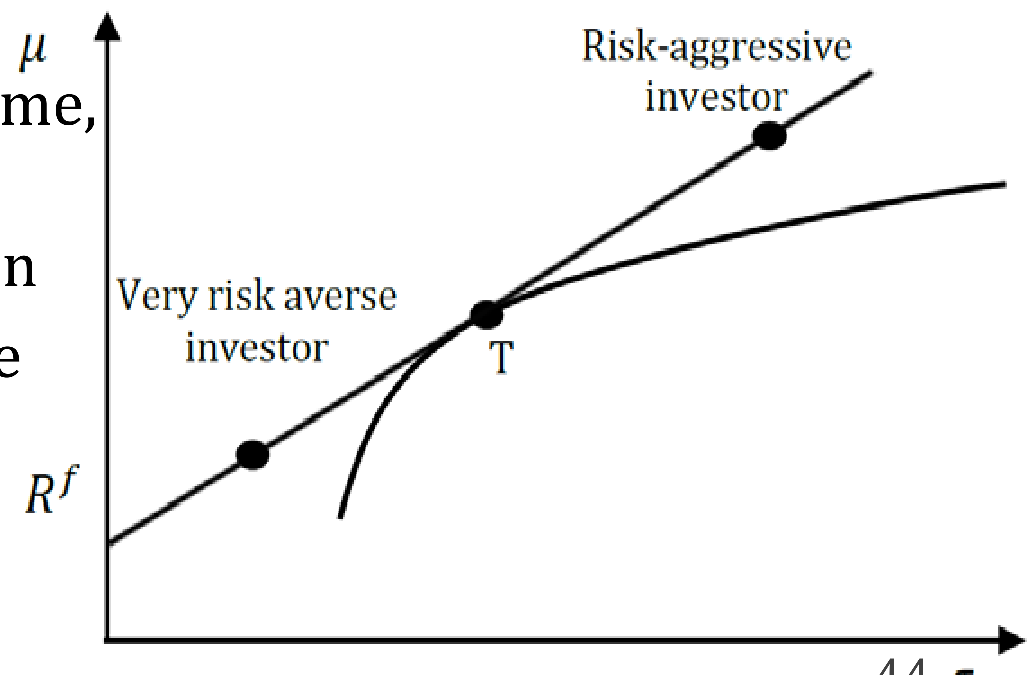


# The Tangency Portfolio and the Capital Market Line

Under no frictions and homogeneous beliefs, there exists a tangency ptf. that maximizes the slope of the transformation line

- Assuming beliefs are homogeneous, there are no frictions or taxes, and that individuals face the same $R^f$ and identical asset menus, all rational, non-satiated investors hold the same tangency portfolio
- It is combined with a certain share of risk-free lending or borrowing
- While the share of wealth an investor lends or borrows at $R^f$ depends on the investor's preference for risk, the risky portfolio should be the same for all the investors
- The steepest CTL gets a special name, the Capital Market Line (CML)
- Special case of two-fund separation
- To determine the tangency ptf. one needs to solve:

$$\max_{\{\omega\}} \frac{(\boldsymbol{\omega}'\boldsymbol{\mu} - R^f)}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{\frac{1}{2}}} \quad \textit{subject to} \quad \boldsymbol{\omega}'\boldsymbol{\iota} = 1$$



# The Tangency Portfolio and the Capital Market Line

- The textbook explains how the problem may be written as a simple unconstrained max problem that we can solved by solving the FOCs: $\max_{\{\omega\}} \frac{\boldsymbol{\omega}'(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{1/2}}$

- The resulting vector of optimal ptf. weights is: $\boldsymbol{\omega}_T = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{A - CR^f}$
- Using the same data as in the strategic asset allocation example on three assets – U.S. Treasury, corporate bonds, and equity – we have:

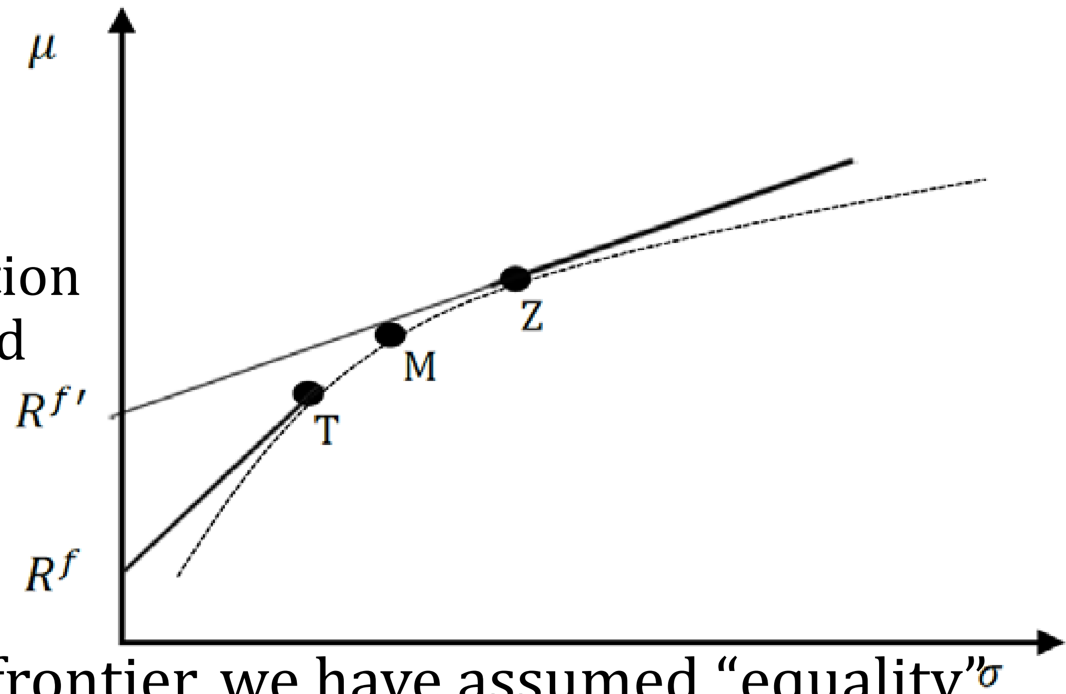
$$\boldsymbol{\omega}_T = \frac{1}{18.5 - 282.11 \cdot 2.5\%} \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \begin{bmatrix} 3.50\% \\ 5.00\% \\ 6.50\% \end{bmatrix} = \begin{bmatrix} 0.59 \\ 0.16 \\ 0.25 \end{bmatrix}$$

- Textbook gives indications on how to use Microsoft Excel'sl Solver®
  - The Solver will iteratively change the values of the cells that contain the weights until the value of the Sharpe ratio is maximized
  - We shall analyze the use of the Solver soon and in your homeworks
- Up to this point, we have assumed that the investor can borrow money at the same riskless rate at which she can lend
- More reasonable assumption: the investor is able to borrow money, but at a higher rate than the one of the risk free (long) investment

# Unlimited, Riskless Borrowing and Lending

When lending and borrowing is possible at different rates, it is no longer possible to determine a single tangency portfolio

- The figure shows how the CML is modified when borrowing is only possible at a rate $R^{f}{}' > R^{f}$
- There are now two CTLs, both tangent to the efficient frontier
  - All the points falling on the portion of the efficient frontier delimited by T (below) and Z (above) will be efficient even though these do not fall on the straight, CML-type line




- While constructing the efficient frontier, we have assumed "equality" constraints (e.g., ptf weights summing to one), but no "inequality" constraints (e.g., positive portfolio weights)
- Inequality constraints complicate the solution techniques
- However, unlimited short-selling assumption is often unrealistic (see margin accounts)

# Short-Selling Constraints

- When short-selling is not allowed, portfolio weights should be positive, i.e. the constraint $\boldsymbol{\omega} \geq 0$ (to be interpreted in an element-by-element basis) has to be imposed
  - When $\boldsymbol{\omega}$ has to be positive, the unconstrained maximum may be at a value of that is not feasible
  - Therefore, it is necessary to impose the Kuhn-Tucker conditions
  - The textbook gives a heuristic introduction to what these are
  - Fortunately, Microsoft's Excel Solver® offers the possibility to solve the problem numerically, by-passing these complex analytical details
- Consider again our earlier strategic asset allocation example and let's set $\bar{\mu} = 9\%$
  - In the absence of constraints, the solution is $\boldsymbol{\omega}_T = \begin{bmatrix} -56.66\% \\ 113.33\% \\ 43.33\% \end{bmatrix}$
  - This makes sense because the second asset is characterized by a large Sharpe ratio and hence must be exploited to yield a high mean return by leveraging the first security
  - Selling -57% of the first security is a major hurdle
  - Under nonnegativity constraints we obtain: $\boldsymbol{\omega}_T^{constrain} = \begin{bmatrix} 0\% \\ 0\% \\ 100\% \end{bmatrix}$

# Summary and conclusions

- We have seen that there are precise ways, based on moments, to summarize the distribution of future asset returns
- Our next task is to show how the characteristics on combinations of securities can be used to define the opportunity set of investments from which the investor must make a choice
- In the process, we have discovered the principle of diversification
- By mixing a large number of assets, we can push their individual variance-driven contribution to the total portfolio variance to zero
- However, even as the number of assets $N \to \infty$, their average contribution to total risk (ptf. variance) cannot be pushed to zero if the assets are correlated on average
- The portion of risk that cannot be simply diversified away by increasing the number of assets in portfolio is called systematic or, indeed non-diversifiable, risk
- One has to bear in mind that in the process choices were made; e.g., risk has been measured as total variance and not as downside risk