Università Commerciale Luigi Bocconi

Lecture 2: Essential Concepts in Time Series Analysis

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20192– Financial Econometrics

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Overview

- Time Series: When Can We Focus on the First Two Moments Only?
- Strict vs. Weak Stationarity
- White noise processes
- The sample autocorrelation function vs. the population ACF
- The sample partial autocorrelation function vs. the population PACF
- Box-Pierce-Ljung test for sample ACF
- The sample partial autocorrelation function vs. the population PACF

Time Series

- A time series consists of a sequence of random variables, y_1, y_2, ..., y_T, also known as a stochastic process $\{y_t\}_{t=1}^{T}$, of which we only observe the empirical realizations
  - An observed time series $\{y_t\}_{t=1}^{T}$ (technically, a sub-sequence because limited to a finite sample) of the realized values of a family of random variables $\{Y_t\}_{t=-\infty}^{+\infty}$ defined on an appropriate probability space
  - See difference between sample ($\{y_t\}_{t=1}^{T}$) and population ($\{Y_t\}_{t=-\infty}^{+\infty}$)

- A time series model for the observations $\{y_t\}_{t=1}^{T}$ is a specification of the joint distribution of the set of random variables of which the sampled data are a realization
  - We often exploit the linearity of the process to specify only the first- and second-order moments of the joint distribution, i.e., the mean, variances and covariances of $\{Y_t\}_{t=-\infty}^{+\infty}$

Linear Processes

Definition **(Linear process)** A time series $\{y_t\}$ is said to be a linear process if it has the representation

$$y_t = \mu + \sum_{j=-\infty}^{\infty} \phi_j z_{t-j},$$

for all t, where μ is a constant, $\{\phi_j\}$ is a sequence of constant coefficients where $\phi_0 = 1$ and $\sum_{j=-\infty}^{\infty} |\phi_j| < \infty$, and $\{z_t\}$ is a sequence of independent and identically distributed (IID) random variables with a defined distribution function. In particular, we assume that the distribution of z_t is continuous, with $E[z_t] = 0$ and $\text{var}(z_t) = \sigma_z^2$. Noticeably, if $\sigma_z^2 \sum_{i=1}^{\infty} \phi_i^2 < \infty$, then y_t is weakly stationary, with the meaning that we shall see below.

- If a time series process is linear, modelling its conditional mean and variance is sufficient in a mean-squared error sense

Strict Stationarity

- To use past realizations of a variable of interest to forecast its future values, it is necessary for the stochastic process that has originated the observations to be stationary
- Loosely speaking, a process is said to be stationary if its statistical properties do not change over time

Definition **(Strict stationarity)** A process is **strictly stationary** if the joint distribution of the variable associated to any sub-sequence of times $t_1, t_2, \ldots, t_n$ is the same as the joint distribution of the sequence of all times $t_{1+k}, t_{2+k}, \ldots, t_{n+k}$ (where k is an arbitrary time shift). In other words, a strictly stationary time series $\{y_t\}$ has the following properties:

- the random variables y_t are identically distributed;
- the two random vectors $[y_t, y_{t+k}]'$ and $[y_1, y_{1+k}]'$ have the same joint distribution for any t and k.

Weak (Covariance) Stationarity

- In many applications, a weaker form of stationarity generally provides a useful sufficient condition

Definition **(Weak stationarity)** A stochastic process $\{y_t\}$ is **weakly stationary** (or, alternatively, **covariance stationary**) if it has time invariant first and second moments, i.e., if for any choice of $t = 1, 2, \ldots, \infty$, the following conditions hold:

$$\mu_y \equiv E(y_t), \text{ with } |\mu_y| < \infty$$

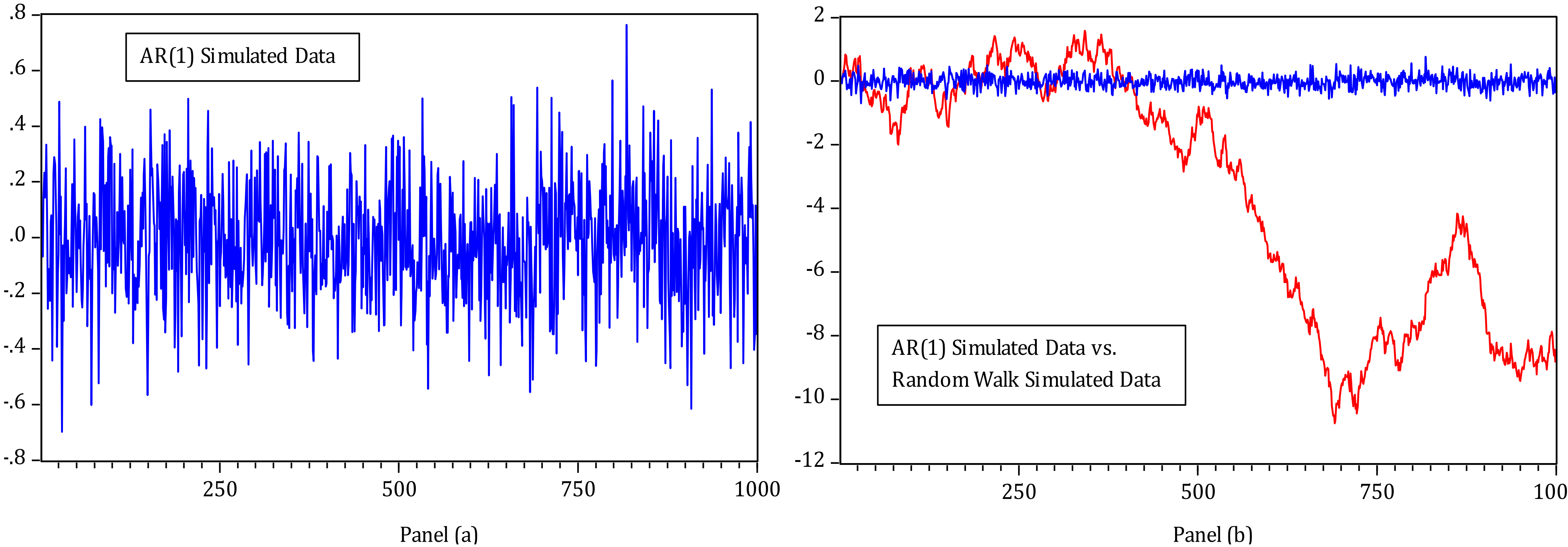
$$\sigma_y^2 \equiv E\left[(y_t - \mu_y)(y_t - \mu_y)\right] = E\left[(y_t - \mu_y)^2\right] < \infty$$

(Autocovariance function) $$\gamma_h \equiv E\left[(y_t - \mu_y)(y_{t-h} - \mu_y)\right] \ \forall h, \text{ with } |\gamma_h| < \infty.$$

where $h = \ldots, -3, -2, -1, 1, 2, 3, \ldots$.

- $\rho_h \equiv \gamma_h / \gamma_0$ (where γ_0 is the variance) is called autocorrelation function (ACF), for h = …, -2, -1, 1, 2, ….
  - Often more meaningful than ACVF because it is expressed as pure numbers that fall in [-1, 1]

An Example of Stationary Series

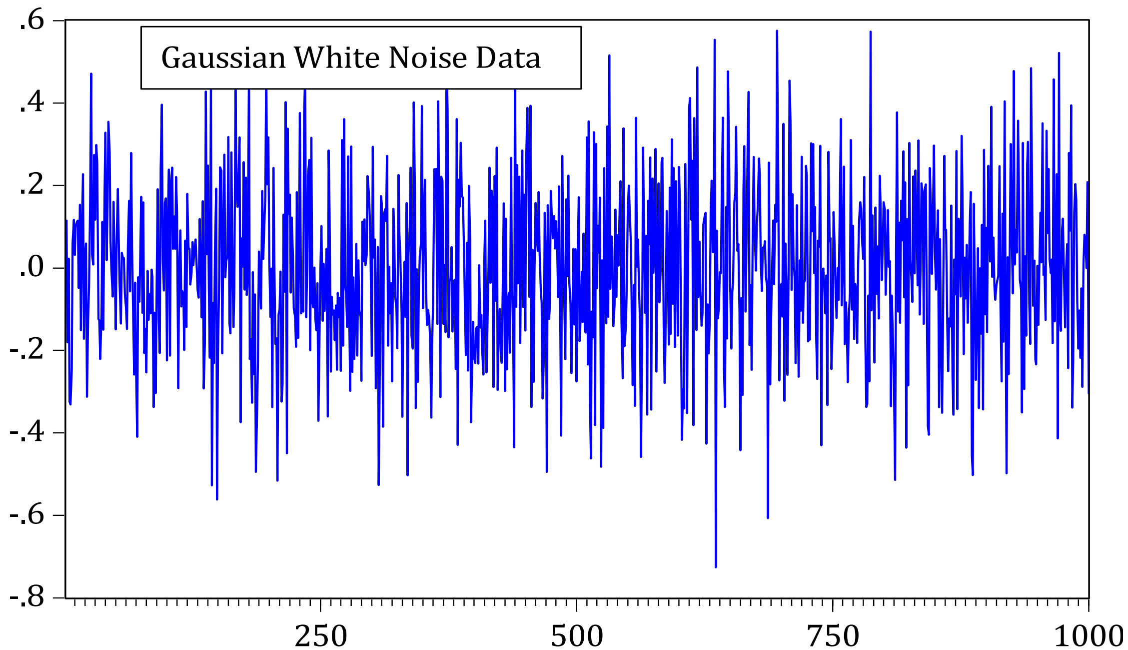


- A time series generated by a stationary process fluctuates around a constant mean, because its memory of past shocks decays over time
  - The data plotted in panel (a) are 1,000 realizations of a first-order autoregressive (henceforth, AR) process of the type $y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$, with $\phi_0 = 0$ and $\phi_1 = 0.2$
  - In panel (b) we have a nonstationary random walk, $y_t = y_{t-1} + \epsilon_t$
  - We shall describe these models later

White Noise Process

- A fundamental class of stationary processes is the fundamental building block of all (covariance) stationary processes: white noise

> **Definition (White Noise)** A white noise (WN) process is a sequence of random variables $\{z_t\}$ with mean equal to zero, constant variance equal to σ^2, and zero autocovariances (and autocorrelations) except at lag zero. If $\{z_t\}$ is normally distributed, we shall speak of a Gaussian white noise.



Sample Autocorrelation Function

- Stationary AR and white noise processes may sometimes be hard to tell apart – what tools are available to identify them?
- The sample ACF reflects important information about the linear dependence of a series at different times

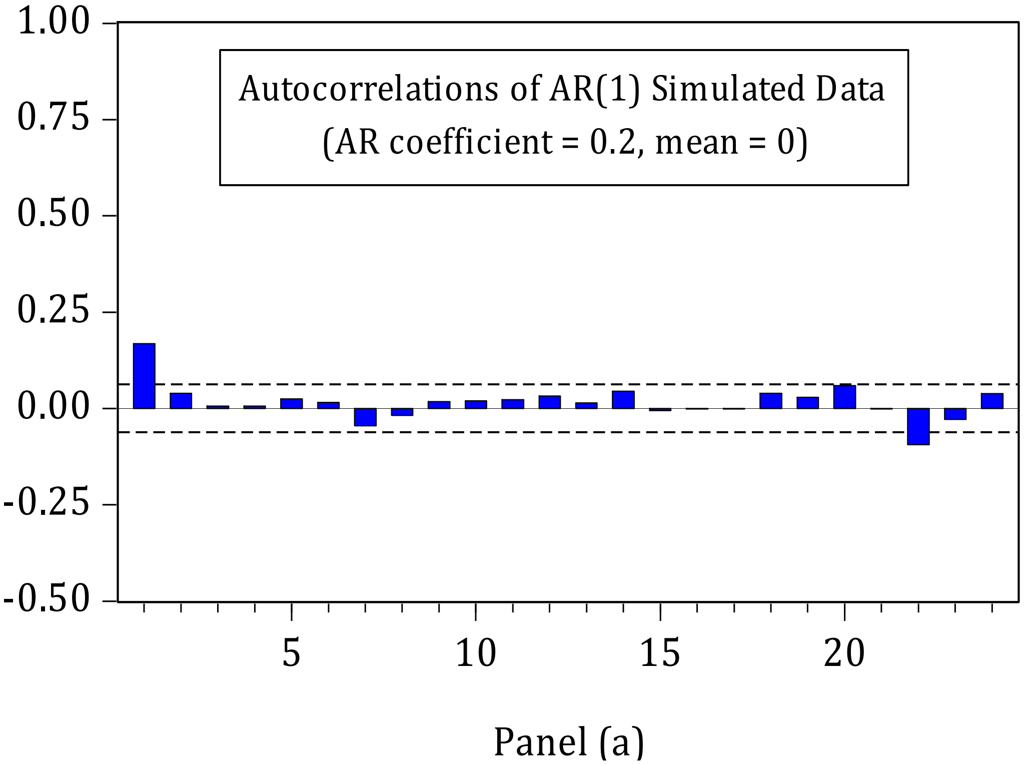
Definition (Sample autocorrelations) Given a sample of T observations of the variable y_t, $y_1, y_2, \ldots, y_T$, the estimated or sample autocorrelation function $\hat{\rho}_h$ (where h is a positive integer) is computed as

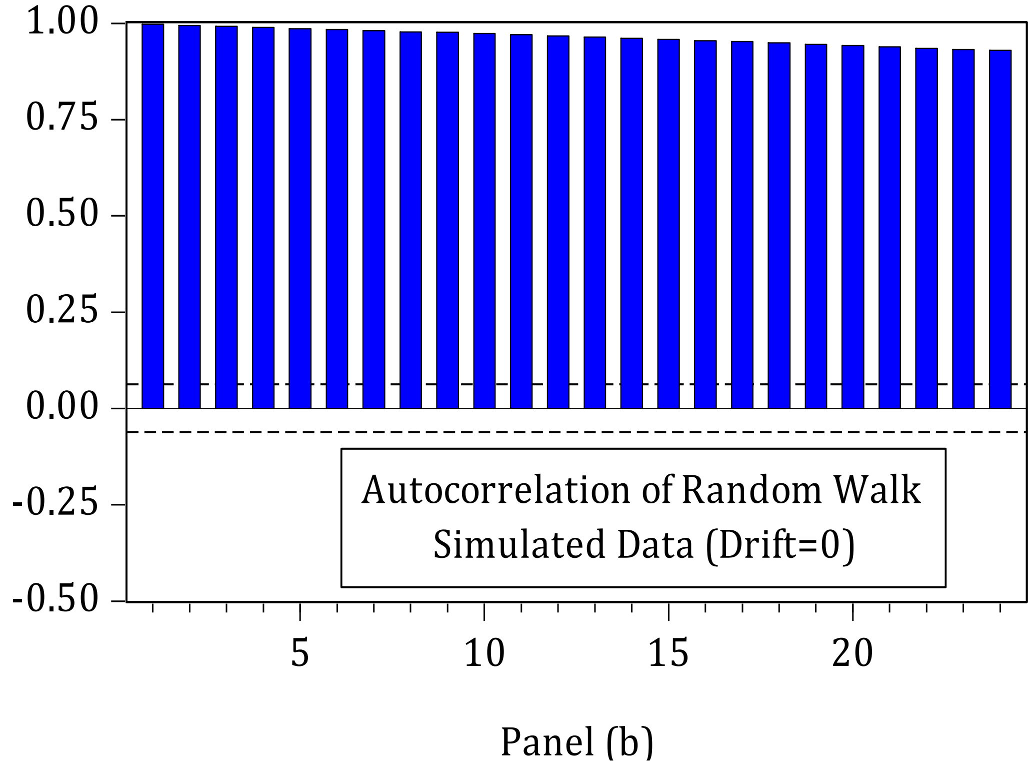
$$\hat{\rho}_h = \frac{\sum_{t=h+1}^{T}(y_t - \hat{\mu})(y_{t-h} - \hat{\mu})}{\sum_{t=1}^{T}(y_t - \hat{\mu})^2} = \frac{\hat{\gamma}_h}{\hat{\gamma}_0},$$

where $\hat{\mu}$ is the sample mean computed as $\hat{\mu} = T^{-1}\sum_{t=1}^{T} y_t$.

- If $\{Y_t\}_{t=-\infty}^{+\infty}$ is an i.i.d. process with finite variance, then for a large sample, the estimated autocorrelations will be asymptotically normally distributed with mean zero and variance $1/T$

Sample Autocorrelation Function





- The dashed lines correspond to approximate (asymptotic) 95% confidence intervals built as $\pm 1.96/\sqrt{T}$
- The SACF in panel (a) shows that a stationary process quickly "forgets" information from a distant past
- The theoretical ACF for a random walk process shall be exactly one at all lags but because SACF is a downward biased estimates of the true and unobserved ACF, the sample coefficients are less than 1

Ljung-Box Test for SACF

- It is also possible to jointly test whether several (say, M) con-secutive autocorrelation coefficients are equal to zero:

$$H_o: \rho_1 = \rho_2 = \cdots = \rho_M = 0 \quad vs. H_a: \exists\ some\ j\ s.t.\ \rho_j \neq 0$$

- Box and Pierce (1970) and Ljung and Box (1978) deve-loped a well-known port-manteau test based on the Q- or LB-statistic

$$\boxed{Q^*(M) = T(T+2)\sum_{h=1}^{M}\frac{\hat{\rho}_h^2}{T-h} \sim \chi_M^2}$$

Prob
0.000
0.000
0.000
0.000
0.000
0.000
0.000
0.000
0.000

Serial Correlation Structure of Simulated AR(1) Data

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.168	0.168	28.433	0.000
		2	0.039	0.010	29.925	0.000
		3	0.006	-0.002	29.961	0.000
		4	0.006	0.005	29.997	0.000
		5	0.025	0.024	30.646	0.000
		6	0.015	0.007	30.873	0.000
		7	-0.045	-0.052	32.956	0.000
		8	-0.018	-0.003	33.282	0.000
		9	0.018	0.024	33.594	0.000
		10	0.020	0.013	34.001	0.000
		11	0.023	0.016	34.531	0.000
		12	0.032	0.028	35.566	0.000
		13	0.014	0.005	35.764	0.001
		14	0.045	0.038	37.802	0.001
		15	-0.005	-0.022	37.828	0.001
		16	-0.001	0.002	37.829	0.002
		17	-0.001	-0.000	37.829	0.003
		18	0.040	0.042	39.502	0.002
		19	0.029	0.016	40.336	0.003
		20	0.059	0.051	43.884	0.002
		21	-0.001	-0.018	43.884	0.002
		22	-0.094	-0.099	52.881	0.000
		23	-0.028	-0.001	53.672	0.000
		24	0.038	0.046	55.161	0.000

Panel (a)

Serial Correlation Structure of Simulated White Noise Data

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.022	0.022	0.4673	0.494
		2	-0.010	-0.010	0.5598	0.756
		3	0.051	0.051	3.1776	0.365
		4	0.027	0.025	3.9294	0.416
		5	-0.056	-0.056	7.1017	0.213
		6	0.033	0.034	8.2260	0.222
		7	0.004	-0.001	8.2428	0.312
		8	0.006	0.011	8.2735	0.407
		9	-0.011	-0.012	8.4026	0.494
		10	0.028	0.024	9.2163	0.512
		11	-0.033	-0.032	10.308	0.503
		12	-0.051	-0.050	12.961	0.372
		13	-0.031	-0.031	13.937	0.378
		14	0.008	0.008	13.996	0.450
		15	0.059	0.069	17.483	0.291
		16	0.010	0.008	17.584	0.349
		17	-0.013	-0.015	17.752	0.405
		18	0.020	0.013	18.143	0.446
		19	0.014	0.013	18.333	0.500
		20	0.051	0.059	21.024	0.396
		21	-0.009	-0.014	21.100	0.453
		22	-0.037	-0.041	22.539	0.428
		23	-0.033	-0.039	23.667	0.422
		24	-0.037	-0.042	25.038	0.404

Panel (b)

Prob
0.494
0.756
0.365
0.416
0.213
0.222
0.312
0.407
0.494

Sample Partial Autocorrelation Function

- The partial autocorrelation between y_t and y_{t-h} is the autocorrelation between the two random variables in the time series, conditional on $y_{t-1}, y_{t-2}, \ldots, y_{t-h+1}$
- Or, the ACF measured after netting out the portion of the variability linearly explained already by the lags between y_{t-1} and y_{t-h+1}e
  - o The sample estimate of the partial autocorrelation at lag h is obtained as the ordinary least square estimator of $\boxed{\phi_h}$ in an autoregressive model: $y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} \ldots + \phi_h y_{t-h} + \epsilon_t$

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Panel (b)