Università Commerciale
Luigi Bocconi

Lecture 4: Forecasting with option implied information

Prof. Massimo Guidolin

Advanced Financial Econometrics III

Winter/Spring 2019

Overview

- A two-step approach
- Black-Scholes single-factor model
- Heston's two-factor square root stochastic volatility model
- Model-free implied volatility
- Option-implied correlations
- Model-free option-implied skewness and kurtosis
- Model-implied, parametric forecasts of skewness and kurtosis
- Model-free forecasts of densities
- From risk-neutral to physical forecasts

The key point

- Derivative prices contain information useful to forecast any twice differentiable function of the future underlying price
- Focus on European-style options, especially equity index

- Derivative prices contain useful information on the conditional density of future underlying asset returns
- A derivative contract is an asset whose future payoff depends on the uncertain realization of the price of an underlying asset
  - Futures and forward contracts, swaps (e.g., CDS and variance swaps), collateralized debt obligations (CDOs) and basket options, European style call and put options, American style and exotic options, etc.
  - Several of these classes of derivatives exist for many different types of underlying assets, such as commodities, equities, and equity indexes
  - However, some derivative contracts such as forwards and futures are linear in the return on the underlying security, and therefore their payoffs are too simple to contain useful reliable information
  - Other securities, such as exotic options, have path-dependent payoffs, which may make information extraction cumbersome

A two-step approach

- Forecasting with option-implied information proceeds in two steps:

  ① Derivative prices are used to extract a relevant aspect of the option-implied distribution of the underlying asset
  - E.g., ATM implied volatility (IV), closest to 30-day to maturity

  ② An econometric model is used to relate this option-implied information to the forecasting object of interest
  - E.g., realized 30-day variance is regressed on IV inferred from observed option prices 30 days before
- In this lecture, brief review of the methods used in the first step
- The two most commonly used models for option valuation are the Black and Scholes (1973, JPE) and Heston (1993, RFS) models
- Black and Scholes ⇒ constant volatility geometric Brownian motion

$$dS = rSdt + \sigma Sdz$$

  where *r* is a constant risk-free, σ is volatility, and *dz* a normal shock
- The future log price is normally distributed and option price for a European call with maturity T and strike price X is

Black-Scholes, single-factor model

$$C^{BS}(T, X, S_0, r; \sigma) = S_0 N(d) - X \exp(-rT) N\left(d - \sigma\sqrt{T}\right)$$

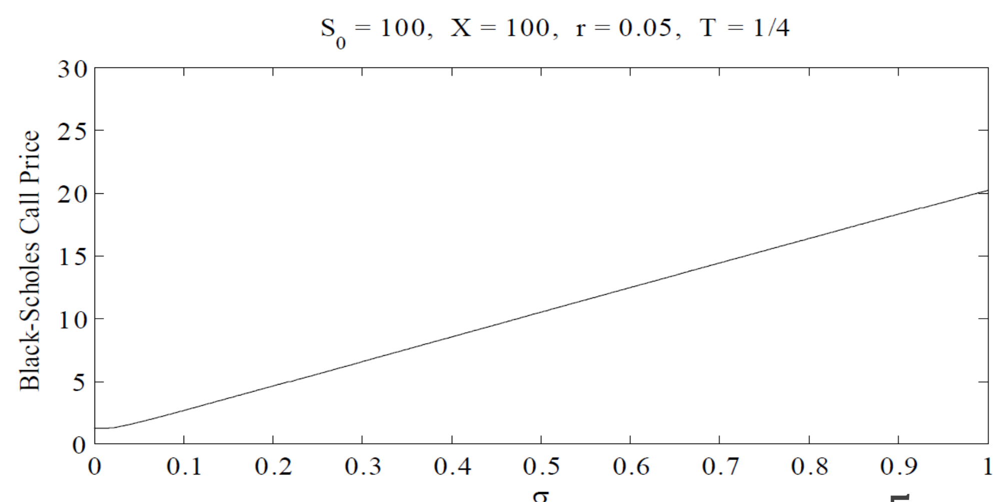
$$d = \frac{\ln(S_0/X) + T\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}} \qquad P_0 + S_0 = C_0 + X\exp(-rT)$$

(put-call parity)

- The formula has just one unobserved parameter, namely volatility that can be backed out for any given option with market price as:

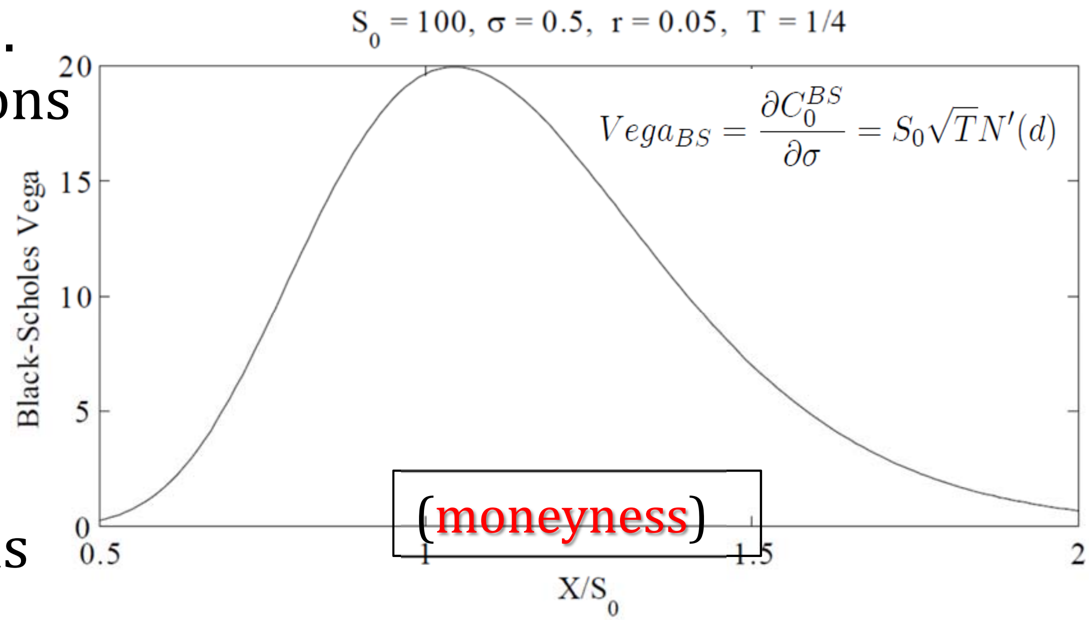
$$C_0^{Mkt} = C^{BS}(T, X, S_0, r; BSIV)$$

- The resulting option-specific volatility is called BSIV
- Although the BS formula is clearly non-linear, for ATM options, the relationship between volatility and option price is virtually linear
  - In general the relationship btw volatility and option prices is increasing and monotone
  - Solving for BSIV is quick even if it must be done numerically
  - Vega captures the sensitivity of the price w.r.t. changes in σ



Black-Scholes, single-factor model

- For equity index options, BSIV as an adjusted R^2 of 62% and summarizes all other information
  - The sensitivity of the price vs. σ is the highest for ATM options
  - In the following table we show a predictive regression of realized volatility on BSIV
  - RV_M, RV_W, RV_D are monthly, weekly, and daily realized vol
  - RV is from intraperiod returns and C nets jumps out

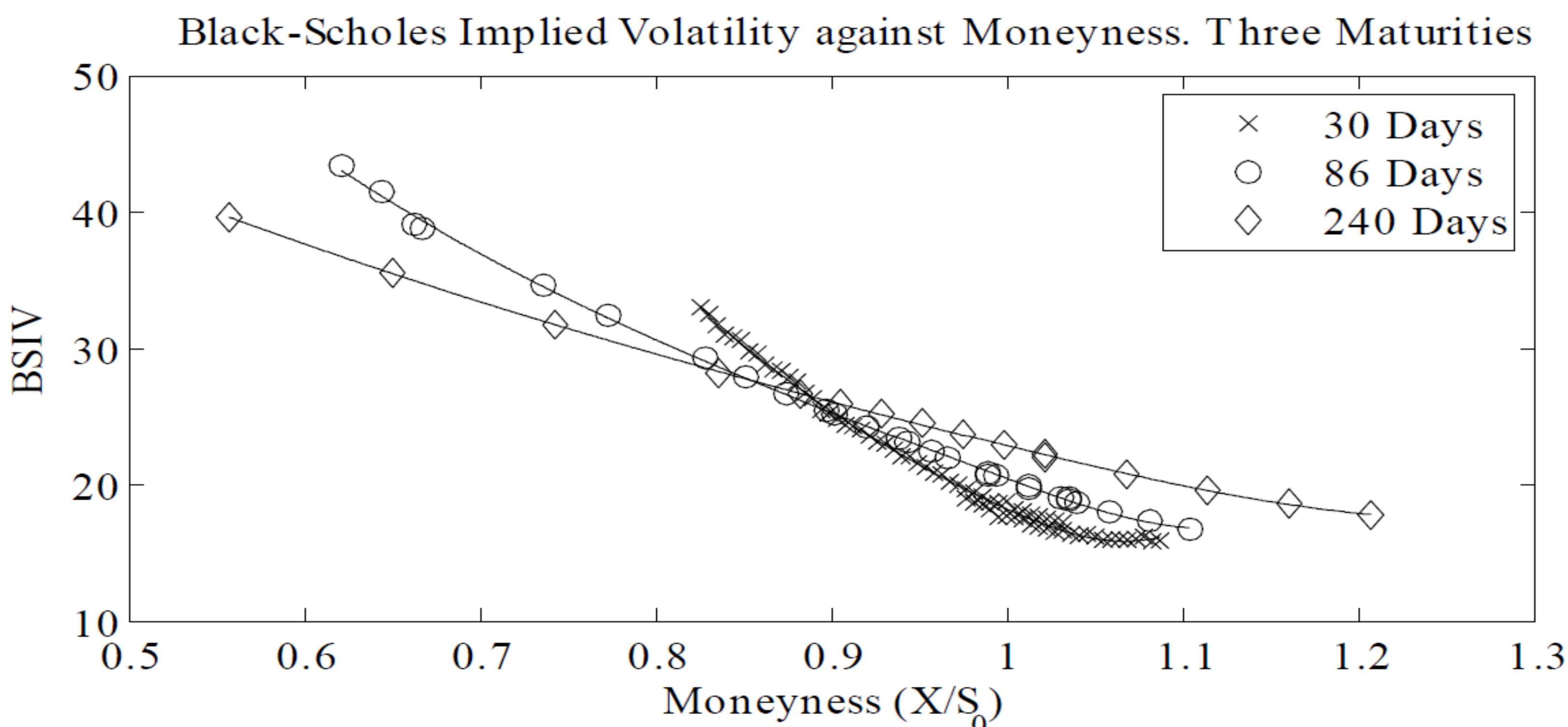


Panel B: S&P 500 data

Constant	RV_M	RV_W	RV_D	C_M	C_W	C_D	$BSIV$	Adj. R^2
0.0053 (0.0025)	0.6240 (0.1132)	−0.3340 (0.1039)	0.6765 (0.1007)	—	—	—	—	53.0
0.0037 (0.0023)	—	—	—	0.1568 (0.1327)	0.0407 (0.1353)	0.9646 (0.1088)	—	61.9
−0.0050 (0.0027)	—	—	—	—	—	—	1.0585 (0.0667)	62.1
−0.0052 (0.0027)	0.0378 (0.1311)	−0.1617 (0.0943)	0.3177 (0.1026)	—	—	—	0.9513 (0.1391)	64.0
−0.0051 (0.0027)	—	—	—	−0.1511 (0.1336)	0.0633 (0.1237)	0.6016 (0.1194)	0.7952 (0.1447)	68.2

Black-Scholes, single-factor model

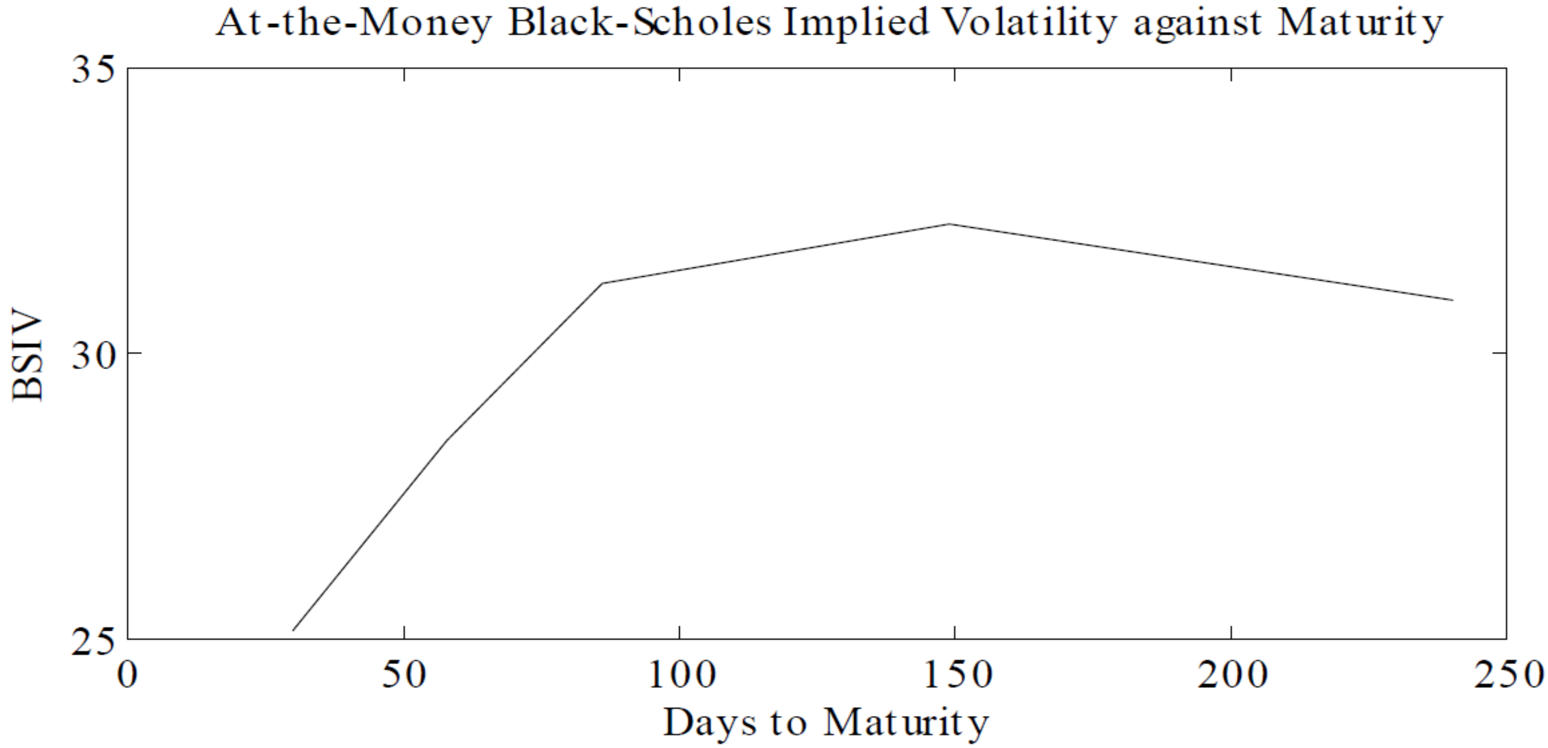
- Nonconstant patterns in BSIV vs. moneyness ⇒ misspecification



- The simple BSIV forecast is able to compete with some of the most sophisticated historical return-based forecasts
- Index-option BSIVs display a distinct downward sloping pattern commonly known as the smirk or the skew
- This is evidence that the BS model which relies on the normal distribution is misspecified

Black-Scholes, single-factor model

- Nonconstant patterns in BSIV vs. maturity ⇒ misspecification



- Deep out-of-the-money put options ($X/S_0 << 1$) are more expensive than the normal-based Black-Scholes model would suggest
- Only a distribution with a fatter left tail (that is negative skewness) would be able to generate these much higher prices for OTM puts
- BSIV for ATM ($X/S_0 = 1$) tends to be larger for long-maturity than short-maturity options

Heston, two-factor square root process

- Heston's model makes variance stochastic and square root

- For variances to change over time, we need a richer setup than the Black-Scholes models
- Most famous model that provides this result is Heston (1993, RFS), who assumes that the underlying follows a square-root process:

$$dS = rSdt + \sqrt{V}Sdz_1$$

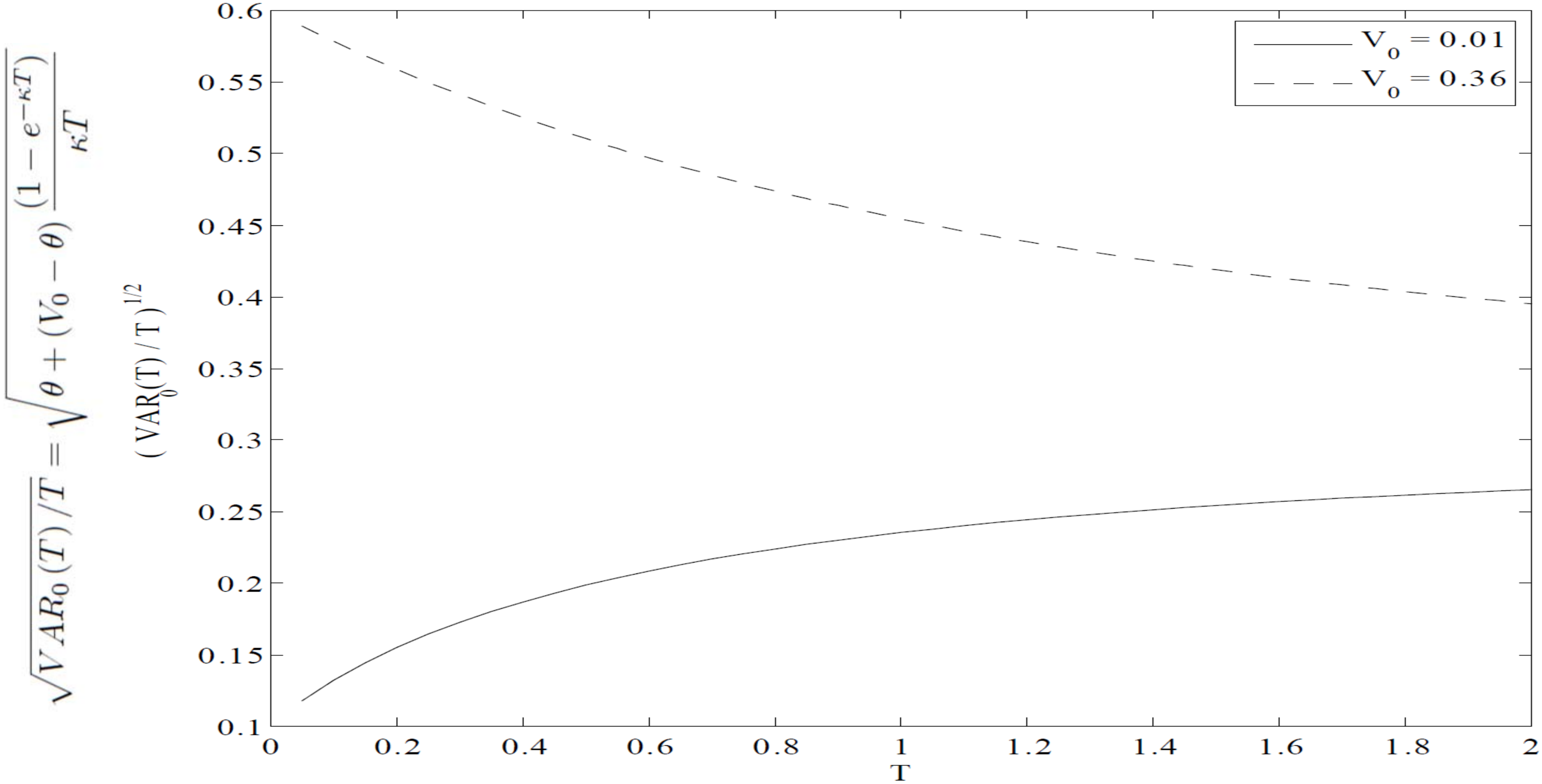
$$dV = \kappa(\theta - V)dt + \sigma_V\sqrt{V}dz_2$$

where the two innovations are correlated with parameter ρ
- At time zero, the variance forecast for horizon T can be obtained as

$$VAR_0(T) \equiv E_0\left[\int_0^T V_t dt\right] = \theta T + (V_0 - \theta)\frac{\left(1 - e^{-\kappa T}\right)}{\kappa}$$

- The mean-reversion parameter κ determines the extent to which the difference between current spot volatility and long run volatility, $(V_0 - \theta)$, affects the horizon T forecast
- Whereas the BS only has one parameter, Heston has 4 parameters

Heston, two-factor square root process



- Bakshi, Cao, and Chen (1997, JF) re-estimate the model daily treating V_0 as a fifth parameter to be estimated
- What if the model assumed to forecast volatility from option prices turns out to be misspecified?
- The answer is tragic: nothing good can be expected of the forecasts

Model-free volatility estimation and forecasting

- Luckily a few methods to achieve model-free volatility estimation are possible
- When investors can trade continuously, interest rates are constant, and the underlying futures price is a continuous semi-martingale, Carr and Madan (1998) and Britten-Jones and Neuberger (2000, JF) show that the expected value of future realized variance is:

$$E_0\left[\int_0^T (dS_t/S_t)^2\, dt\right] = 2\int_0^\infty \frac{C_0^F(T,X) - \max(F_0 - X, 0)}{X^2}\, dX$$

  - Jiang and Tian (2005, RFS) generalize this result and show that it holds even if the price process contains jumps

$$VAR_0(T) = 2\int_0^\infty \frac{C_0\left(T, e^{-rT}X\right) - \max(S_0 - X, 0)}{X^2}\, dX$$

  - In practice, a finite range, X_{max} - X_{min}, of discrete strikes are available and Jiang and Tian consider using the trapezoidal integration rule

$$VAR_0(T) \approx \sum_{i=1}^{m}\left\{\frac{\left[C_0^F(T,X_i) - \max(F_0 - X_i, 0)\right]}{X_i^2} + \frac{\left[C_0(T,X_{i-1}) - \max(F_0 - X_{i-1}, 0)\right]}{X_{i-1}^2}\right\}\Delta X$$

$$\Delta X = (X_{\max} - X_{\min})/m$$

Empirical evidence

- Overall, the evidence indicates that option-implied volatility is a biased predictor of the future volatility of the underlying asset
- Yet, most studies find that it contains useful information over traditional predictors based on historical prices
  - Option IV by itself often outperforms historical volatility
- BSIV is predictable and helps forecast volatility, but because arbitrage profits are impossible under transaction costs, predictability is consistent with EMH (see Goncalves and Guidolin, 2006, JoB)
- There is recent, strong evidence that the variance risk premium (VRP) can predict the equity risk premium
  - VRP is the difference between implied variance and realized variance
- Bakshi, Panayotov, and Skoulakis (2011, JFE) compute forward variance, the implied variance between two future dates, and find that it forecasts stocks, T-bills, and changes in real activity
- Feunou, Fontaine, Taamouti, and Tedongap (2013, RoF) find that the term structure of IVs can predict both equity risk and VRP

Option-implied correlations

- BSIV is useful in forecasting the volatility of individual stocks
- Implied volatility has also been used to predict future stock returns
- Information in options leads analyst recommendation changes
- The VIX is a priced risk factor with a negative price of risk, so that stocks with higher sensitivities to the innovation in VIX exhibit on average future lower returns
  - VIX is a weighted average of BSIVs
  - The CBOE computes VIX using OTM and ATM call and put options
  - It calculates the volatility for the two available maturities that are the nearest and second-nearest to 30 days.
- Certain derivatives contain very rich information on correlations between financial time series
- E.g., in currency markets $S_{\$/£} = S_{\$/¥} S_{¥/£} \Rightarrow R_{\$/£} = R_{\$/¥} + R_{¥/£}$

  where R denotes a continuously compounded return
- Therefore $VAR_{\$/£} = VAR_{\$/¥} + VAR_{¥/£} + 2COV(R_{\$/¥}, R_{¥/£})$

Option-implied correlations

- While implied correlations for currencies are derived from the triangular equality, in the case of stocks only an implied average correlation may be estimated

- The implied correlation is:

$$CORR(R_{\$/¥}, R_{¥/£}) = \frac{(VAR_{\$/£} - VAR_{\$/¥} - VAR_{¥/£})}{2VAR_{\$/¥}^{1/2} VAR_{¥/£}^{1/2}}$$

- Provided we have option-implied variance forecasts for 3 currencies, we can use this to get an implied correlation forecast
- Option-implied exchange rate correlations for the DM/GBP pair and the DM/JPY, and USD/DM/JPY pairs predict significantly better than historical correlations between the pairs
- There is a measure of average option-implied correlation between the stocks in an index, I,

$$\rho_{ICI} = \frac{VAR_I - \sum_{j=1}^{n} w_j^2 VAR_j}{2\sum_{j=1}^{n-1} \sum_{i>j}^{n} w_i w_j VAR_i^{1/2} VAR_j^{1/2}}$$

Weight of stock j

- Skintzi and Refenes (2005, JFM) use options on the DJIA index

Option-implied correlations

- Smirks (asymmetric smiles) in IVs indicate left-skewness in the density of underlying returns, while symmetric smiles point to excess kurtosis

- Implied correlation index is biased upward, but is a better predictor of future correlation than historical correlation
- Implied correlations may be used to estimate betas and the literature finds that option-implied betas predict realized betas well
- However, using option-implied information in portfolio allocation does not improve the Sharpe ratio or CER of the optimal portfolio
- We saw earlier that BS is unlikely to be correctly specified: the very option prices (IVs) contain robust evidence of asymmetries and fat tails in the predictive density of underlying asset returns
- Can we extract option-implied skewness and kurtosis?
- It is sensible to proceed with a model-free approach, called option replication approach, see Bakshi and Madan (2000, JFE)

Model-free option-implied skewness and kurtosis

- For any twice differentiable function of the future underlying price, there is a spanning portfolio made of bonds, stock, and European call and put options

- Bakshi, Carr, and Madan show that any twice continuously differentiable fnct, H(S_T), of terminal price S_T, can be replicated (spanned) by a unique position in the risk-free, stocks and European options

$$H(S_T) = \underbrace{\left[H(S_0) - H'(S_0) S_0\right]}_{\text{Units of risk-free bond}} + \underbrace{H'(S_0)}_{\text{Units of underlying}} S_T$$

$$+ \int_0^{S_0} H''(X) \max(X - S_T, 0)\, dX + \int_{S_0}^{\infty} H''(X) \max(S_T - X, 0)\, dX$$

  - H''(X)dX are units of OTM call and put options with strike price X
  - From a forecasting perspective, for any H(•), there is a portfolio of risk-free bonds, stocks, and options whose current aggregate market value provides an option-implied forecast of H(S_T)

$$E_0\left[e^{-rT} H(S_T)\right] = e^{-rT}\left[H(S_0) - H'(S_0) S_0\right] + S_0 H'(S_0)$$

$$+ \int_0^{S_0} H''(X) P_0(T, X)\, dX + \int_{S_0}^{\infty} H''(X) C_0(T, X)\, dX$$

Model-free option-implied skewness and kurtosis

- Under mild assumptions, the prices of OTM puts and calls can be used to infer risk-neutral volatility, skewness, and kurtosis

- Consider now higher moments of simple returns:

$$H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^2 \quad H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^3, \text{ and } H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^4$$

- We can use OTM European call and put prices to derive the quadratic, cubic, and quartic contracts as

$$M_{0,2}(T) \equiv E_0\left[e^{-rT}\left(\frac{S_T - S_0}{S_0}\right)^2\right] = \frac{2}{S_0^2}\left[\int_0^{S_0} P_0(T,X)\,dX + \int_{S_0}^{\infty} C_0(T,X)\,dX\right]$$

$$M_{0,3}(T) \equiv E_0\left[e^{-rT}\left(\frac{S_T - S_0}{S_0}\right)^3\right] = \frac{6}{S_0^2}\left[\int_0^{S_0}\left(\frac{X - S_0}{S_0}\right) P_0(T,X)\,dX + \int_{S_0}^{\infty}\left(\frac{X - S_0}{S_0}\right) C_0(T,X)\,dX\right]$$

$$M_{0,4}(T) \equiv E_0\left[e^{-rT}\left(\frac{S_T - S_0}{S_0}\right)^4\right] = \frac{12}{S_0^2}\left[\int_0^{S_0}\left(\frac{X - S_0}{S_0}\right)^2 P_0(T,X)\,dX + \int_{S_0}^{\infty}\left(\frac{X - S_0}{S_0}\right)^2 C_0(T,X)\,dX\right]$$

  - High option prices imply high volatility
  - High OTM put and low OTM call prices ⇒ negative skewness
  - High OTM call and put prices at extreme moneyness ⇒ high kurtosis

Model-free option-implied skewness and kurtosis

- Now compute option-implied volatility, skewness, and kurtosis:

$$VOL_0(T) \equiv [VAR_0(T)]^{1/2} = \left[e^{rT} M_{0,2} - M_{0,1}^2\right]^{1/2}$$

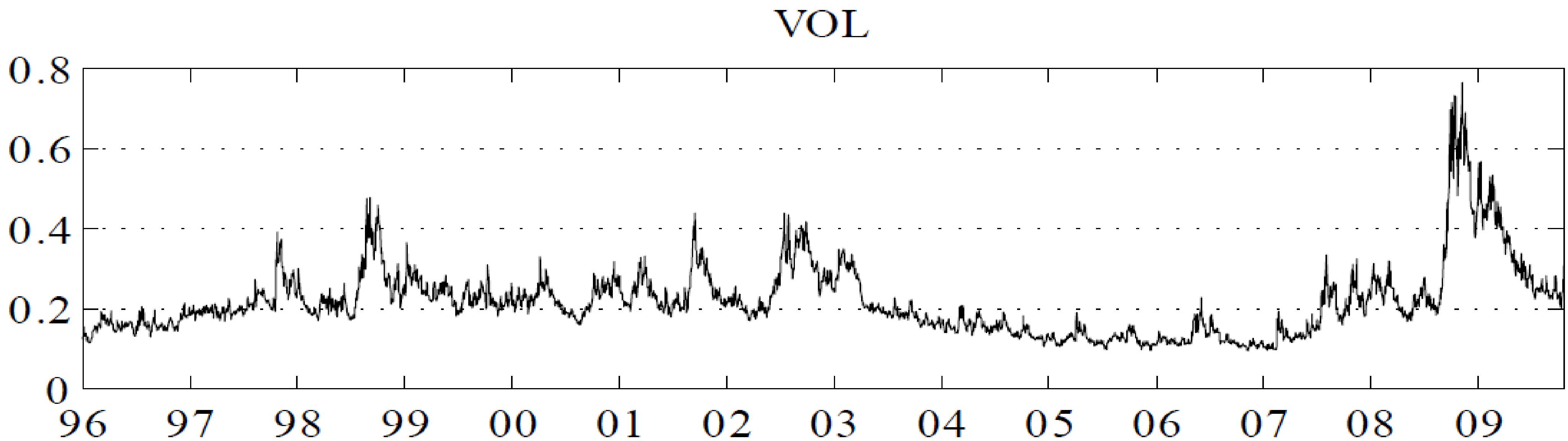
$$\boxed{M_{0,1} \equiv E_0\left[\left(\frac{S_T - S_0}{S_0}\right)\right] = e^{rT} - 1}$$

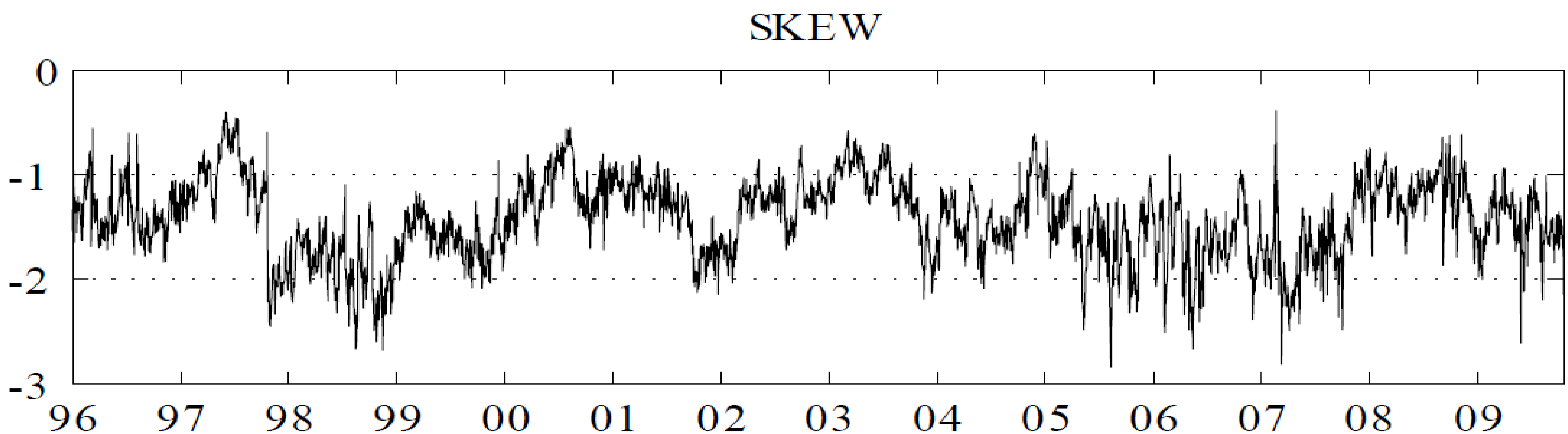
$$SKEW_0(T) = \frac{e^{rT} M_{0,3} - 3M_{0,1} e^{rT} M_{0,2} + 2M_{0,1}^3}{\left[e^{rT} M_{0,2} - M_{0,1}^2\right]^{\frac{3}{2}}}$$

$$KURT_0(T) = \frac{e^{rT} M_{0,4} - 4M_{0,1} e^{rT} M_{0,3} + 6e^{rT} M_{0,1}^2 M_{0,2} - 3M_{0,1}^4}{\left[e^{rT} M_{0,2} - M_{0,1}^2\right]^2}$$

- Using S&P 500 index options over January 1996 - September 2009 we plot higher moments of log returns for the one-month horizon
  - The volatility series is very highly correlated with the VIX index, with a correlation of 0.997
- The estimate of skewness is negative for every day in the sample
- The estimate of kurtosis is always higher than 3
- Both skewness and kurtosis do not show significant or persistent alterations during the 2008-2009 Great Financial Crisis

Model-free option-implied skewness and kurtosis

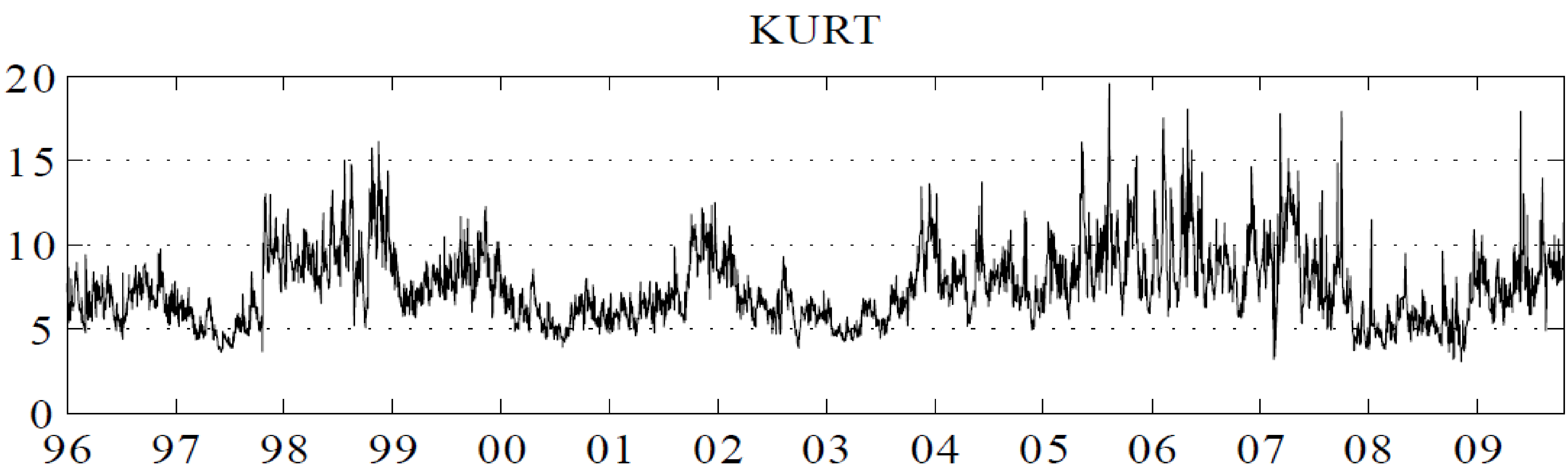




- In February 2011, the CBOE began publishing the CBOE S&P 500 Skew Index computed according to this methodology

Model-based option-implied skewness and kurtosis

- Jarrow and Rudd have proposed an option price approximation based of Edgeworth expansions of BS, $C_0^{BS}(T, X)$



- There are also parametric, model-based methods to extract skewness and kurtosis from option prices
- Some are based on functional «expansions»/approximations of BS
- Most famous approach is due to Jarrow and Rudd (1982, JFE) who propose a method where the density of the security price at option maturity, T, is approximated using an Edgeworth series expansion

Model-based option-implied skewness and kurtosis

- Jarrow and Rudd's pricing formula is:

$$C_0^{JR}(T,X) \approx C_0^{BS}(T,X) - e^{-rT}\frac{(K_3 - K_3(\Psi))}{3!}\frac{d\psi(T,X)}{dX} + \\ + e^{-rT}\frac{(K_4 - K_4(\Psi))}{4!}\frac{d^2\psi(T,X)}{dX^2}$$

- K_j is the jth cumulant of the actual density, $K_j(\Psi)$ is the cumulant of the lognormal density and other quantities are reported in Appendix A
- If one approximates around a log-normal density, then

$$VAR(X) = \exp\left(2\left(\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T\right) + \sigma^2 T\right)(\exp(\sigma^2 T) - 1)$$

$$SKEW(X) = (\exp(\sigma^2 T) + 2)\sqrt{\exp(\sigma^2 T) - 1}$$

$$KURT(X) = \exp(4\sigma^2 T) + 2\exp(3\sigma^2 T) + 3\exp(2\sigma^2 T) - 3$$

- The model now has 3 parameters to estimate, VAR, SKEW and KURT

- As an alternative to the Edgeworth expansion, Corrado and Su (1996, JFR) consider a Gram-Charlier series expansion:

$$C_0^{CS}(T,X) = C_0^{BS}(T,X) + \frac{1}{3!}S_0\sigma\sqrt{T}\left(\left(2\sigma\sqrt{T} - d\right)N'(d) + \sigma^2 T N(d)\right)SKEW$$

Model-based option-implied skewness and kurtosis

- Also parametric jump-diffusion models may be used to forecast skewness and kurtosis from option prices

$$+\frac{1}{4!}S_0\sigma\sqrt{T}\left(\left(d^2-1-3\sigma\sqrt{T}\left(d-\sigma\sqrt{T}\right)\right)N'(d)+\sigma^3T^{3/2}N(d)\right)(KURT-3)$$

- Additional parametric models have become popular in the literature to infer skewness and kurtosis from option prices
- E.g., in Bates (2000, JoE), the futures price *F* is assumed to follow a jump-diffusion:

Correlated with coefficient ρ

$$\begin{aligned} dF/F &= -\lambda\bar{k}dt+\sqrt{V}dz_1+kdq, \\ dV &= \kappa\left(\theta-V\right)dt+\sigma_V\sqrt{V}dz_2 \end{aligned}$$

- *q* is a Poisson counter with instantaneous intensity λ, and *k* is a log-normal return jump, $\ln(1+k)\sim N\left[\ln\left(1+\bar{k}\right)-\delta^2/2,\delta^2\right]$
- Higher-order moments can now be computed as a function of the unknown parameters, to be estimated
- Options can also be used to forecast the density of underlying asset returns

Model-free risk-neutral density forecasts

- The option-implied conditional density for the underlying at maturity *T* is the **forward second derivative of an ATM call**

- Breeden and Litzenberger (1978, JBus) and Banz and Miller (1978, JBus) show that the option-implied density can be extracted from a set of European option prices with a continuum of strike prices
  - This result is a special case of Carr and Madan's result reviewed above
- The value of a European call, C_0, is the discounted expected value of payoff on expiry date *T*, i.e., under the implied measure, $f_0(S_T)$:

$$C_0(T,X) = e^{-rT}\int_0^{\infty} \max\{S_T - X, 0\}\, f_0(S_T)\, dS_T = e^{-rT}\int_X^{\infty} (S_T - X)\, f_0(S_T)\, dS_T$$

- Take the partial derivative of C_0 with respect to the strike price X:

$$\frac{\partial C_0(T,X)}{\partial X} = -e^{-rT}\left[1 - \tilde{F}_0(X)\right] \Rightarrow \tilde{F}_0(S_T) = 1 + e^{rT}\left.\frac{\partial C_0(T,X)}{\partial X}\right|_{X=S_T}$$

- The conditional density function (PDF) denoted by f_0(X) is obtained as:

$$\boxed{f_0(S_T) = e^{rT}\left.\frac{\partial^2 C_0(T,X)}{\partial X^2}\right|_{X=S_T}}$$

Model-free risk-neutral density forecasts

- A discrete strike approximation of the conditional PDF in terms of calls is:

$$f_0(X_n) \approx e^{rT}\frac{C_0(T,X_{n+1}) - 2C_0(T,X_n) + C_0(T,X_{n-1})}{(\Delta X)^2}$$

- Because of put-call parity, $S_0 + P_0 = C_0 + Xe^{-rT}$ can use puts instead:

$$f_0(S_T) = e^{rT}\frac{\partial^2 P_0(T,X)}{\partial X^2}\bigg|_{X=S_T}$$

- In practice, we can obtain an approximation to the CDF using finite differences of call or put prices observed at discrete strike prices:

$$\tilde{F}_0(X_n) \approx 1 + e^{rT}\left(\frac{C_0(T,X_{n+1}) - C_0(T,X_{n-1})}{X_{n+1} - X_{n-1}}\right)$$

$$\tilde{F}_0(X_n) \approx e^{rT}\left(\frac{P_0(T,X_{n+1}) - P_0(T,X_{n-1})}{X_{n+1} - X_{n-1}}\right)$$

  - In terms of the log return, the CDF and PDF are

$$\tilde{F}_{0,R_T}(x) = F_0\left(e^{x+\ln S_0}\right) \qquad \text{and} \qquad f_{0,R_T}(x) = e^{x+\ln S_0} f_0\left(e^{x+\ln S_0}\right)$$

Parametric and approximated RN density forecasts

- The key issue in implementing this method is that typically only a limited number of options are traded, with a handful of strikes
- Tricks exist: e.g., the simple but flexible ad-hoc BS (AHBS) model constructs the density forecast off a BS implied volatility curve
- In a first step, estimate a second-order polynomial or other well-fitting function for implied BS volatility as a function of strike and maturity, to obtain fitted BSIV values:

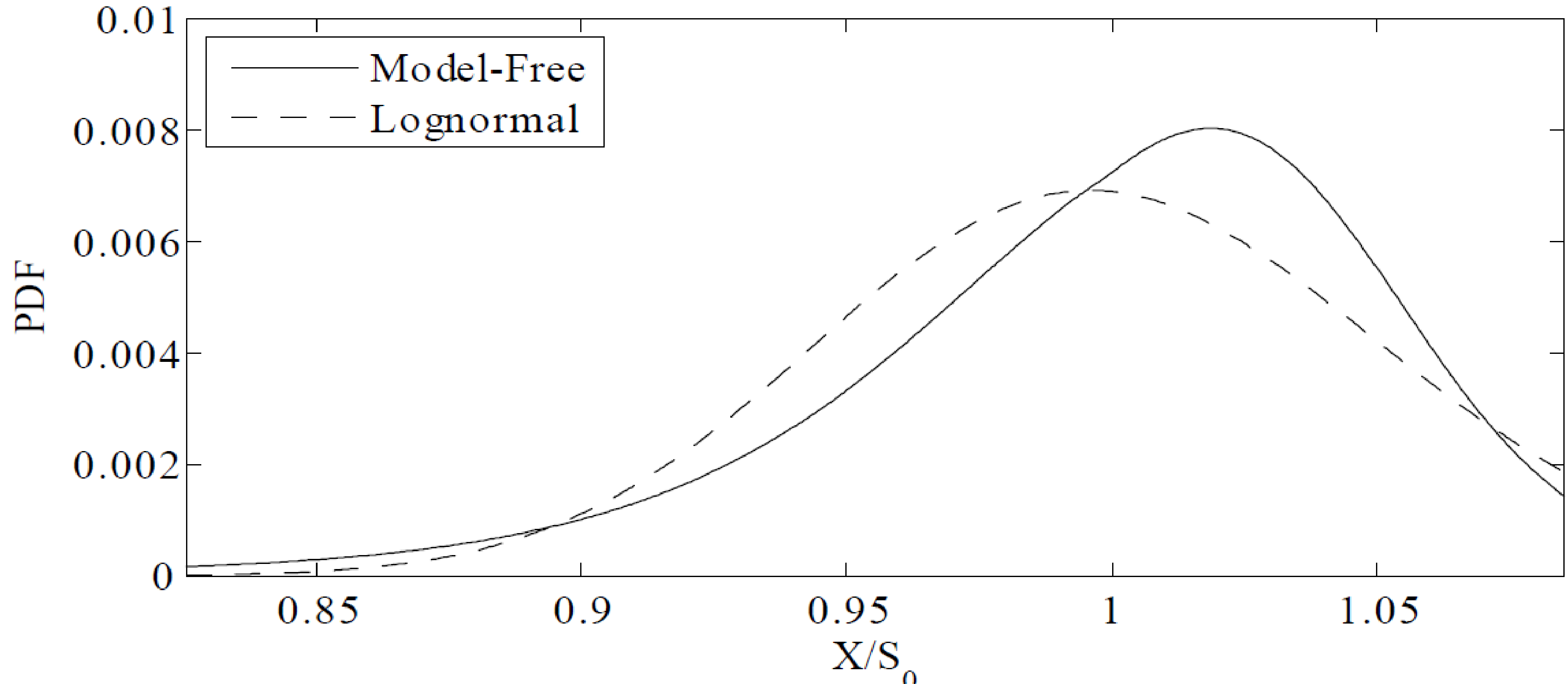
$$BSIV\,(X,T) = a_0 + a_1 X + a_2 X^2 + a_3 T + a_4 T^2 + a_5 XT$$

- Second, using this estimated polynomial, we generate a set of fixed maturity IVs across a grid of strikes
- Call prices can then be obtained using the BS formula:

$$C_0^{AHBS}\,(X,T) = C_0^{BS}\,(T, X, S_0, r; BSIV\,(X,T))$$

  - Option-implied density can be obtained using the second derivative
  - The figure shows the CDF and PDF obtained when applying a smoothing cubic spline using BSIV data on 30-day OTM calls and puts on the S&P 500 index on October 22, 2009 vs. the lognormal

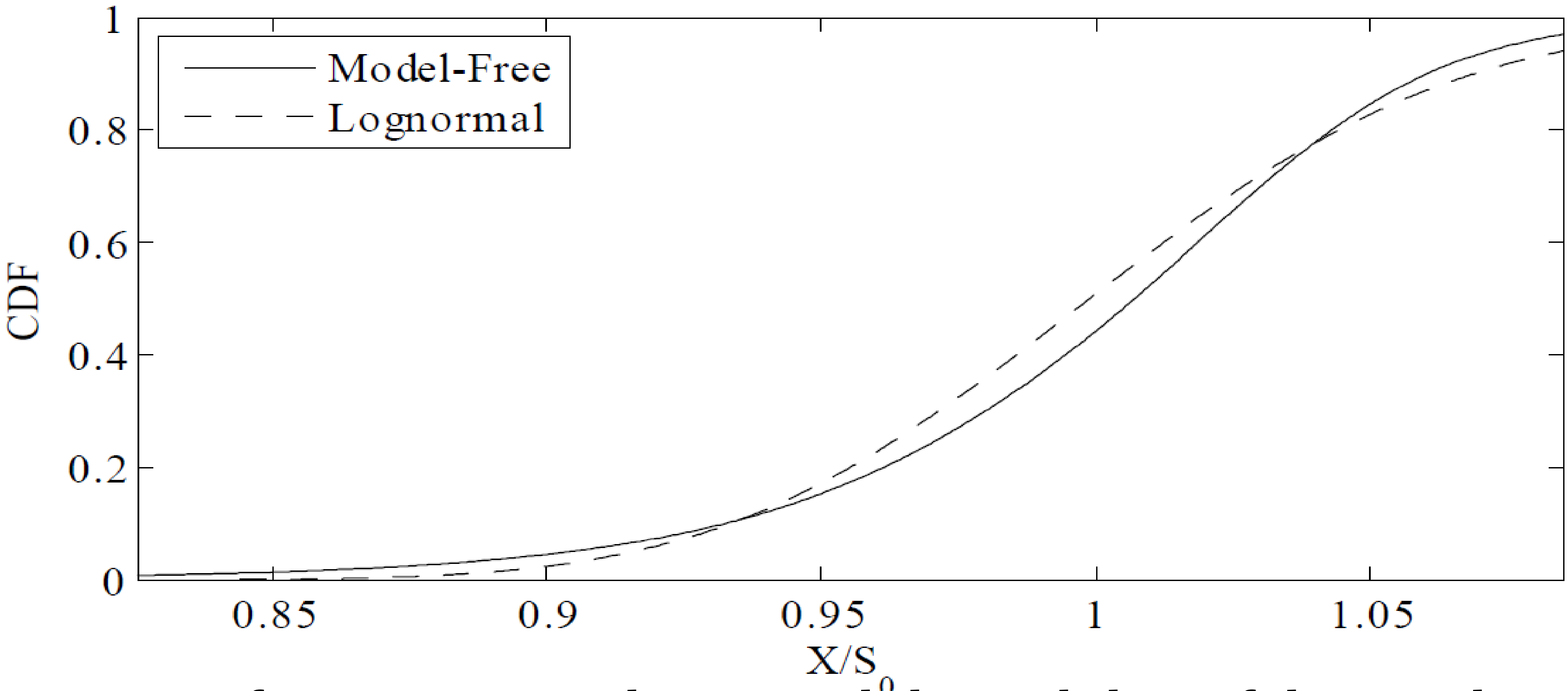
Parametric and approximated RN density forecasts



- The implied distribution is clearly more negatively skewed than the lognormal distribution

- The limited empirical evidence on alternative methods fails to reach clear conclusions as to which method is to be preferred
- Because the resulting densities are often not markedly different from each other using different estimation methods, it makes sense to use methods that are computationally easy

Parametric and approximated RN density forecasts



- Because of computational ease and the stability of the resulting parameter estimates, the smoothed implied volatility function method is a good choice for many purposes
- So far we have constructed forecasting objects using the so-called risk-neutral measure implied from options
- When forecasting properties of the underlying asset we ideally want to use the physical measure

From risk-neutral to physical forecasts

- Because forecasting occurs in the physical measure space, it is often useful to know the mapping between Q and P

- Knowing the mapping btw. the two measures is therefore required
  - Use superscript Q to describe the option-implied density used above and we use superscript P to denote the physical density
- Black and Scholes (1973) assume the physical stock price process:

$$dS = (r + \mu)\, Sdt + \sigma S dz$$

  where μ is the equity risk premium
  - In the complete markets, BS world the option is a redundant asset perfectly replicated by trading the stock and a risk-free bond
- The option price is independent of the degree of risk-aversion of investors because they can replicate the option using a dynamic trading strategy in the underlying asset
- Principle of risk-neutral valuation, all derivatives can be valued using the risk-neutral expected pay-off discounted at the risk free rate: $C_0\,(X, T) = \exp\,(-rT)\, E_0^Q\,[\max\,\{S_T - X, 0\}]$

From risk-neutral to physical forecasts

- Using Ito's lemma implies that log returns are normally distributed

$$f_0^P(\ln(S_T)) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(\frac{1}{2\sigma^2 T}\left(\ln(S_T) - \ln(S_0) + \left(r + \mu - \frac{\sigma^2}{2}\right)T\right)^2\right)$$

  which shows that the the only difference in drift is represented by the equity risk premium
    - In a BS world, the option-implied RN density forecast will therefore have the correct volatility and functional form but a mean biased downward because of the equity premium
- Because the RN mean of the asset return is the risk-free rate, the option price has no predictive content for the mean return
- In the incomplete markets case we can still assume a pricing relationship of the form

$$\begin{aligned} C_0(X,T) &= \exp(-rT)\,E_0^Q[\max\{S_T - X, 0\}] \\ &= \exp(-rT)\int_X^{\infty} \max\{\exp(\ln(S_T)) - X, 0\}\, f_0^Q(\ln(S_T))\, dS_T \end{aligned}$$

- But the link between the Q and P distributions is not unique and a pricing kernel M_T must be assumed to link the two distributions

From risk-neutral to physical forecasts

$$M_T = \exp(-rT)\frac{f_0^Q(\ln(S_T))}{f_0^P(\ln(S_T))} \Rightarrow \begin{aligned} C_0(X,T) &= \exp(-rT)E_0^Q[\max\{S_T - X, 0\}] \\ &= E_0^P[M_T \max\{S_T - X, 0\}] \end{aligned}$$

- The pricing kernel (or stochastic discount factor) describes how in equilibrium investors trade off the current (known) option price versus the future (stochastic) pay-off
- For instance, Heston's model allows for stochastic volatility implying that the option, which depends on volatility, cannot be perfectly replicated by the stock and bond
- Heston (1993) assumes that the price of an asset follows

$$dS = (r + \mu V)\,S dt + \sqrt{V} S dz_1$$

$$dV = \kappa^P\left(\theta^P - V\right)dt + \sigma_V \sqrt{V} dz_2$$

  where the two innovations are correlated with parameter ρ
- The mapping between the P and Q-parameters is given by

$$\kappa = \kappa^P + \lambda,\ \ \theta = \theta^P \frac{\kappa^P}{\kappa}$$

From risk-neutral to physical forecasts

- The P and Q processes imply a pricing kernel of the form

$$M_T = M_0 \left(\frac{S_T}{S_0}\right)^{\gamma} \exp\left(\delta T + \eta \int_0^T V(s)ds + \xi(V_T - V_0)\right)$$

where ξ is a variance preference parameter

- The risk premia μ and λ are related to the preference parameters by

$$\mu = -\gamma - \xi \sigma_V \rho \qquad \lambda = -\rho \sigma_V \gamma - \sigma_V^2 \xi$$

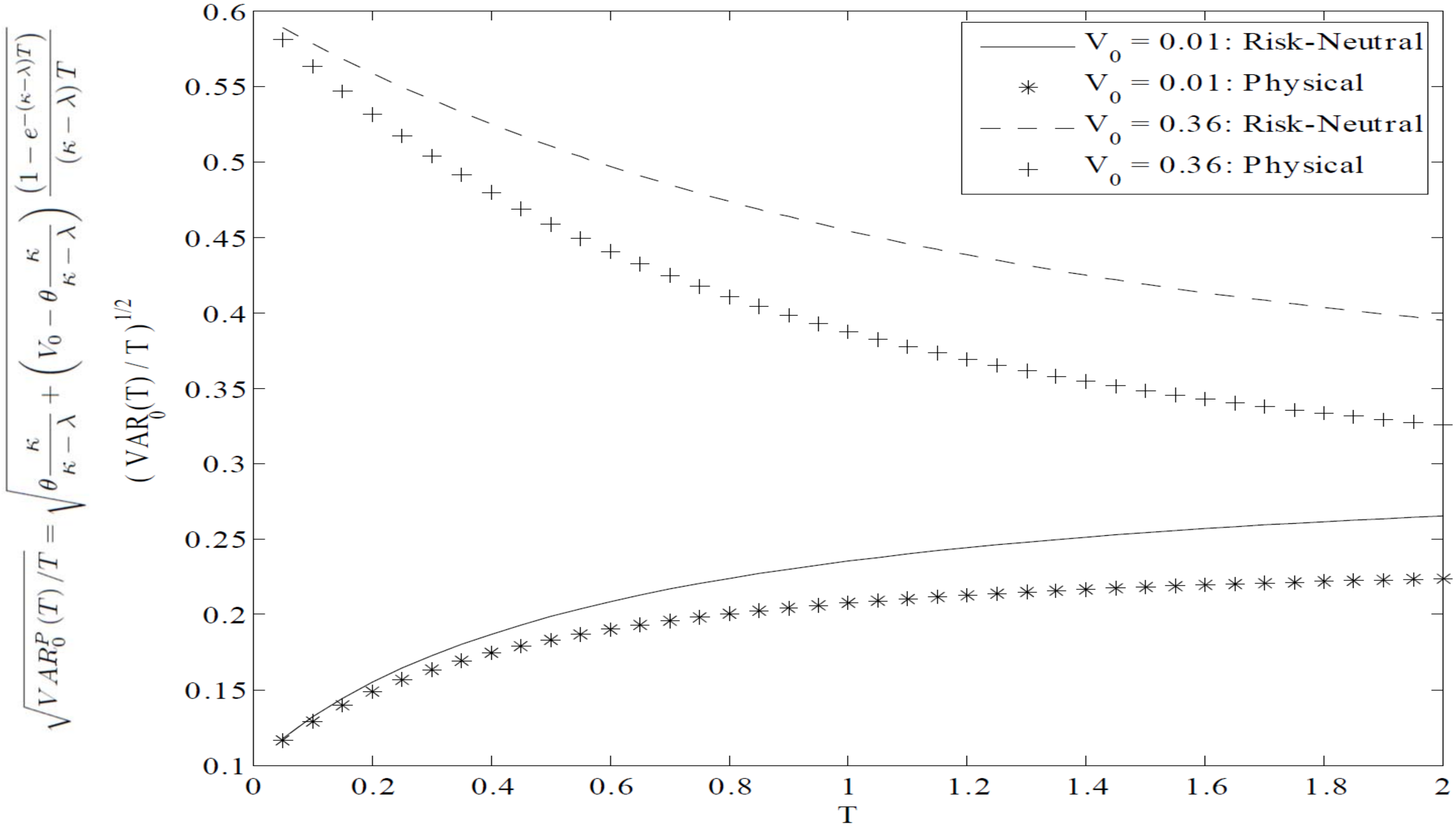
- In order to appreciate the differences btw. P- and Q-forecasts of variance, let's examine the role played by ξ:

$$\begin{aligned} VAR_0^P(T) &= \theta^P T + (V_0 - \theta^P) \frac{\left(1 - e^{-\kappa^P T}\right)}{\kappa^P} \\ &= \theta \frac{\kappa}{\kappa - \lambda} T + \left(V_0 - \theta \frac{\kappa}{\kappa - \lambda}\right) \frac{\left(1 - e^{-(\kappa - \lambda)T}\right)}{\kappa - \lambda} \end{aligned}$$

- Under P-measure the expected variance in Heston's model differs from the RN forecast

$$VAR_0(T) \equiv E_0\left[\int_0^T V_t dt\right] = \theta T + (V_0 - \theta) \frac{\left(1 - e^{-\kappa T}\right)}{\kappa}$$

From risk-neutral to physical forecasts



- For short horizons and when the current volatility is low then the effect of the volatility risk premium is relatively small
- However for long-horizons the effect is much larger.

Conclusion

- The literature contains a large body of evidence supporting the use of option-implied information to predict physical objects of interest
- It is certainly not mandatory that the option-implied information is mapped into the physical measure to generate forecasts
- However, some empirical studies have found that transforming option-implied to physical information improves forecasting performance in certain situations
- We would expect the option-implied distribution or moments to be biased predictors of their physical counterpart
- Yet this bias may be small, and attempting to remove it can create problems of its own, for instance because based on imposing restrictions on investor preferences
- More generally, the existence of a bias does not prevent the option-implied information from being a useful predictor of the future object of interest

Appendix A: Meaning of coefficients in Jarrow-Rudd's formula

$$C_0^{JR}(T,X) \approx C_0^{BS}(T,X) - e^{-rT}\frac{(K_3 - K_3(\Psi))}{3!}\frac{d\psi(T,X)}{dX} +$$

$$+ e^{-rT}\frac{(K_4 - K_4(\Psi))}{4!}\frac{d^2\psi(T,X)}{dX^2}$$

$$\psi(T,X) = \left(X\sigma\sqrt{T2\pi}\right)^{-1}\exp\left\{-\frac{1}{2}\left(d-\sigma\sqrt{T}\right)^2\right\}$$

$$\frac{d\psi(T,X)}{dX} = \frac{\psi(T,X)\left(d-2\sigma\sqrt{T}\right)}{X\sigma\sqrt{T}}$$

$$\frac{d^2\psi(T,X)}{dX^2} = \frac{\psi(T,X)}{X^2\sigma^2 T}\left[\left(d-2\sigma\sqrt{T}\right)^2 - \sigma\sqrt{T}\left(d-2\sigma\sqrt{T}\right) - 1\right]$$

$$d = \frac{\ln(S_0/X) + T\left(r+\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}$$