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Network Models of Financial Contagion and Connectedness

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20254 – Advanced Quantitative Methods for Asset Pricing and Structuring

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Plan of the Talk

- Generalities and Motivation
- Using VAR Models Variance Decompositions to Measure Connectedness and Identify Return and Volatility Spillovers
- Cholesky vs. Generalized Variance Decompositions
- Relationship to Network Connectedness Models: Variance Decompositions, Adjacency, Nodes Degree, and Distance
- Relationship to Other Systemic Risk Measures: Marginal Expected Shortfall, CoVar, and Delta-CoVar
- In-Sample vs. Out-of-Sample Variance Decompositions: Does It Matter?
- Dealing with Large Scale Networks with LASSO/Net Models
- Open Research Questions

Generalities and Motivation

- Financial crises occur with notable regularity; moreover, they display hard-to-miss similarities (e.g., Reinhart and Rogoff, 2008)
- During crises, for example, the volatility in financial markets generally increases sharply and spills over across markets
- Naturally, one would like to be able to measure and monitor such spillovers, both to provide "early warning systems" for emergent crises, and to track the progress of extant crises
- One such measure, connectedness, is becoming central to modern risk management and to systemic risk measurement
- Correlation-based measures remain widespread, yet they measure only pairwise association and are largely wed to linear, Gaussian thinking, making them of limited value
- Luckily Diebold and Yilmaz (2014, JoE) have clarified the links btw. connectedness and variance decompositions
- Variance decompositions define directed network graphs chara-cterrized by an adjacency matrix == the tableau of variance shares

Generalities and Motivation

- One of the most fundamental types of risk is market risk, the risk of changes in ptf. value due to changes in value of its constituents
- Connectedness is part of any market risk assessment, because it separates the risk of a ptf. from the risk of its components
- The likelihood of extreme mkt movements, associated with most assets moving in the same direction, depends on connectedness
  - Concepts like counterparty credit risk are directly linked to aspects of connectedness
  - Counterparty risk is fundamentally multilateral rather than bilateral—really a sort of "congestion risk," or "gridlock risk"
  - Connectedness is also related to the concept of liquidity risk
- Numerous subtleties arise, however, in the statistical measurement of connectedness in finance and macroeconomics
- In a complex-network context, "links" are not binary (existing or not existing), but are weighted according to the economic interaction under consideration, such as traded volumes, invested capital, and their weight can change over time

From Variance Decomposition to Spillover Indices

- Diebold and Yilmaz (2009, EJ) base their analysis of spillovers on a well-known tool, variance decompositions applied to standard vector autoregressions for N (de-meaned) variables,

$$\boldsymbol{x}_t = \boldsymbol{\Phi}\boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}_t$$

Understanding connectedness from $\boldsymbol{\Phi}$ is fruitless
VDs apply transformations that reveal connectedness

  - $\boldsymbol{x}_t$ can be either a vector of asset returns or of estimated volatilities and $\boldsymbol{\epsilon}_t$ contains structural shocks
- By covariance stationarity, the MA representation of the VAR exists and is given by $\boldsymbol{x}_t = [\boldsymbol{I}_N - \boldsymbol{\Phi}L]^{-1}\boldsymbol{\epsilon}_t = \boldsymbol{\Theta}(L)\boldsymbol{\epsilon}_t$, see Appendix
  - It will prove useful to rewrite the MA representation as

$$\boldsymbol{x}_t = \underbrace{\boldsymbol{\Theta}(L)\boldsymbol{Q}^{-1}}_{\boldsymbol{A}(L)}\underbrace{\boldsymbol{Q}\boldsymbol{\epsilon}_t}_{\boldsymbol{u}_t} = \boldsymbol{A}(L)\boldsymbol{u}_t$$

  where $Var[\boldsymbol{u}_t] = E[\boldsymbol{Q}\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}'_t\boldsymbol{Q}'] = \boldsymbol{Q}Var[\boldsymbol{\epsilon}_t]\boldsymbol{Q}' = \boldsymbol{I}_N$, i.e., $\mathbf{Q}$ is the unique triangular Cholesky factor for $\boldsymbol{\Sigma} \equiv Var[\boldsymbol{\epsilon}_t]$
- Because $\boldsymbol{x}_{t+1} = \boldsymbol{A}(L)\boldsymbol{u}_{t+1}$ but minimum MSFE forecast is $\hat{\boldsymbol{x}}_{t+1} = \boldsymbol{\Phi}\boldsymbol{x}_t$, we have that the 1-step ahead vector of forecast error is:

$$\boldsymbol{e}_{t+1} \equiv \boldsymbol{x}_{t+1} - \hat{\boldsymbol{x}}_{t+1} = \boldsymbol{A}_0\boldsymbol{u}_{t+1}$$

From Variance Decomposition to Spillover Indices

- This implies: $Var[\boldsymbol{e}_{t+1}] = \boldsymbol{A}_0 Var[\boldsymbol{u}_{t+1}]\boldsymbol{A'}_0 = \boldsymbol{A}_0\boldsymbol{A'}_0$
- Let us define the own variance shares to be the fractions of the 1-step-ahead error variances in forecasting variable *i* due to shocks to x_i, for i = 1, 2, …, N
- The cross variance shares, or spillovers, are the fractions of the 1-step-ahead error variances in forecasting variable *i* due to shocks to x_j, for *j* = 1, 2, …, N, *j* ≠ *i*
- We can convert total spillover to an easily-interpreted percentage index *S* by expressing it relative to total forecast error variance:

$$S = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} a_{0,ij}^2}{\operatorname{trace}\left(\mathbf{A}_0\mathbf{A}_0'\right)} \times 100$$

- For the general case of a *N*-variable VAR(*p*) and *H*-step-ahead forecasts, we have

$$S = \frac{\sum_{h=0}^{H-1} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} a_{h,ij}^2}{\sum_{h=0}^{H-1} \operatorname{trace}\left(\mathbf{A}_h\mathbf{A}_h'\right)} \times 100$$

Return (Mean) Spillovers

Spillover Table, Global Stock Market Returns, 10/1/1992–23/11/2007

To	From																			
	US	UK	FRA	GER	HKG	JPN	AUS	IDN	KOR	MYS	PHL	SGP	TAI	THA	ARG	BRA	CHL	MEX	TUR	Contribution From Others
US	93.6	1.6	1.5	0.0	0.3	0.2	0.1	0.1	0.2	0.3	0.2	0.2	0.3	0.2	0.1	0.1	0.0	0.5	0.3	6
UK	40.3	55.7	0.7	0.4	0.1	0.5	0.1	0.2	0.2	0.3	0.2	0.0	0.1	0.1	0.1	0.1	0.0	0.4	0.5	44
FRA	38.3	21.7	37.2	0.1	0.0	0.2	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.3	0.1	0.1	0.1	0.1	0.3	63
GER	40.8	15.9	13.0	27.6	0.1	0.1	0.3	0.4	0.6	0.1	0.3	0.3	0.0	0.2	0.0	0.1	0.0	0.1	0.1	72
HKG	15.3	8.7	1.7	1.4	69.9	0.3	0.0	0.1	0.0	0.3	0.1	0.0	0.2	0.9	0.3	0.0	0.1	0.3	0.4	30
JPN	12.1	3.1	1.8	0.9	2.3	77.7	0.2	0.3	0.3	0.1	0.2	0.3	0.3	0.1	0.1	0.0	0.0	0.1	0.1	22
AUS	23.2	6.0	1.3	0.2	6.4	2.3	56.8	0.1	0.4	0.2	0.2	0.2	0.4	0.5	0.1	0.3	0.1	0.6	0.7	43
IDN	6.0	1.6	1.2	0.7	6.4	1.6	0.4	77.0	0.7	0.4	0.1	0.9	0.2	1.0	0.7	0.1	0.3	0.1	0.4	23
KOR	8.3	2.6	1.3	0.7	5.6	3.7	1.0	1.2	72.8	0.0	0.0	0.1	0.1	1.3	0.2	0.2	0.1	0.1	0.7	27
MYS	4.1	2.2	0.6	1.3	10.5	1.5	0.4	6.6	0.5	69.2	0.1	0.1	0.2	1.1	0.1	0.6	0.4	0.2	0.3	31
PHL	11.1	1.6	0.3	0.2	8.1	0.4	0.9	7.2	0.1	2.9	62.9	0.3	0.4	1.5	1.6	0.1	0.0	0.1	0.2	37
SGP	16.8	4.8	0.6	0.9	18.5	1.3	0.4	3.2	1.6	3.6	1.7	43.1	0.3	1.1	0.8	0.5	0.1	0.3	0.4	57
TAI	6.4	1.3	1.2	1.8	5.3	2.8	0.4	0.4	2.0	1.0	1.0	0.9	73.6	0.4	0.8	0.3	0.1	0.3	0.0	26
THA	6.3	2.4	1.0	0.7	7.8	0.2	0.8	7.6	4.6	4.0	2.3	2.2	0.3	58.2	0.5	0.2	0.1	0.4	0.3	42
ARG	11.9	2.1	1.6	0.1	1.3	0.8	1.3	0.4	0.4	0.6	0.4	0.6	1.1	0.2	75.3	0.1	0.1	1.4	0.3	25
BRA	14.1	1.3	1.0	0.7	1.3	1.4	1.6	0.5	0.5	0.7	1.0	0.8	0.1	0.7	7.1	65.8	0.1	0.6	0.7	34
CHL	11.8	1.1	1.0	0.0	3.2	0.6	1.4	2.3	0.3	0.3	0.1	0.9	0.3	0.8	2.9	4.0	65.8	2.7	0.4	34
MEX	22.2	3.5	1.2	0.4	3.0	0.3	1.2	0.2	0.3	0.9	1.0	0.1	0.3	0.5	5.4	1.6	0.3	56.9	0.6	43
TUR	3.0	2.5	0.2	0.7	0.6	0.9	0.6	0.1	0.6	0.3	0.6	0.1	0.9	0.8	0.5	1.1	0.6	0.2	85.8	14
Contribution to others	292	84	31	11	81	19	11	31	14	16	10	8	6	12	21	9	3	8	7	675.0
Contribution including own	386	140	68	39	151	97	68	108	86	85	73	51	79	70	97	75	68	65	92	Spillover index = 35.5%

Notes: The underlying variance decomposition is based upon a weekly VAR of order 2, identified using a Cholesky factorisation with the ordering as shown in the column heading. The (i, j)-th value is the estimated contribution to the variance of the 10-week-ahead real stock return forecast error of country i coming from innovations to real stock returns of country j. The mnemonics are defined as in Table 1.

- The *ij*th entry in the table is the estimated contribution to the forecast error variance of country *i* coming from innovations to country *j*
- Almost 40% of forecast error variance comes from spillovers

Volatility Spillovers

Spillover Table, Global Stock Market Volatility, 10/1/1992–23/11/2007

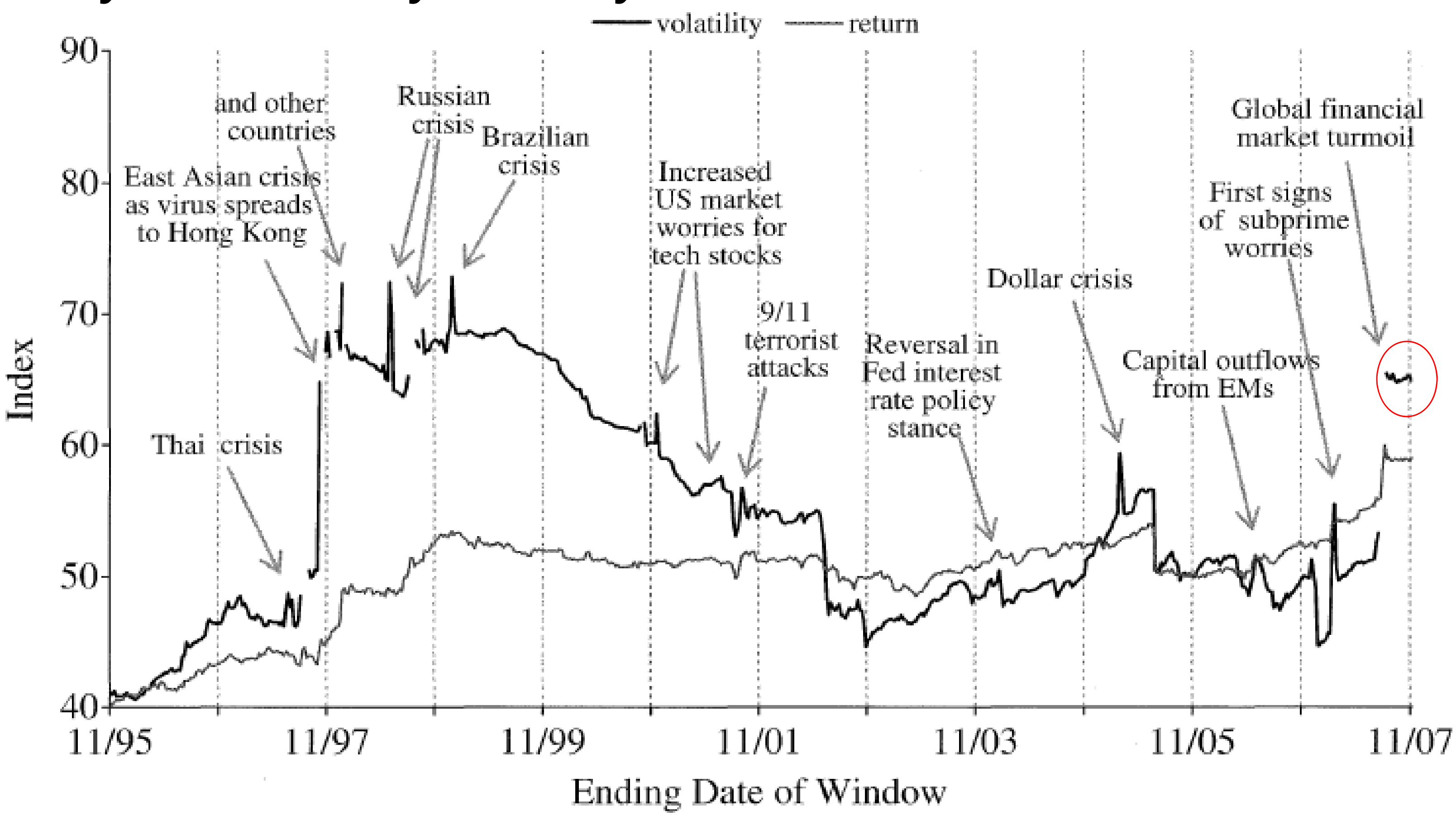
To	From US	UK	FRA	GER	HKG	JPN	AUS	IDN	KOR	MYS	PHL	SGP	TAI	THA	ARG	BRA	CHL	MEX	TUR	Contribution From Others
US	63.9	14.9	3.9	1.9	4.9	0.2	1.8	0.3	1.6	0.9	0.4	2.6	0.3	0.1	0.1	0.0	0.1	0.2	2.0	36
UK	22.9	54.5	5.0	1.3	7.4	0.5	2.1	0.3	1.0	0.8	0.1	2.4	0.2	0.2	0.4	0.2	0.1	0.1	0.7	46
FRA	24.0	32.8	27.3	0.2	5.4	0.2	2.8	0.4	0.3	1.2	0.4	2.4	0.2	0.3	0.6	0.3	0.1	0.1	0.9	73
GER	26.9	29.5	13.6	13.7	4.8	0.2	3.9	0.2	0.2	1.3	0.8	2.0	0.2	0.4	0.6	0.3	0.1	0.2	1.0	86
HKG	2.0	0.5	0.7	0.0	87.7	0.1	0.1	0.4	1.4	0.5	1.5	3.4	0.6	0.4	0.0	0.1	0.0	0.1	0.3	12
JPN	2.7	3.3	0.4	0.7	1.6	82.9	0.1	0.1	0.9	1.1	0.1	1.6	0.3	0.0	0.6	0.3	0.3	0.2	2.8	17
AUS	8.9	2.2	0.3	0.6	43.9	0.2	34.7	1.2	1.7	1.3	0.1	2.8	0.1	1.0	0.1	0.2	0.2	0.3	0.1	65
IDN	2.8	0.9	0.3	1.0	6.1	0.3	0.6	71.4	6.9	2.3	2.5	2.8	0.7	0.0	0.0	0.3	0.2	0.2	0.9	29
KOR	2.5	0.6	0.4	0.4	9.1	1.0	1.0	10.3	67.5	1.3	0.9	2.5	0.8	0.2	0.1	0.1	0.2	0.3	0.8	32
MYS	1.3	0.6	0.3	0.6	7.2	1.0	0.9	0.8	1.7	70.7	3.1	6.1	0.3	0.5	0.9	0.6	0.1	1.5	1.9	29
PHL	2.1	0.3	0.3	0.4	8.9	0.3	0.4	8.8	3.0	6.1	66.7	1.5	0.2	0.2	0.2	0.2	0.1	0.2	0.3	33
SGP	12.5	4.1	0.6	0.1	12.2	0.8	0.8	7.6	7.2	2.8	1.5	45.8	0.5	0.1	0.7	0.7	0.0	0.7	1.2	54
TAI	8.5	0.4	0.4	0.2	2.8	0.7	1.3	0.5	9.5	0.7	1.7	0.6	69.0	0.2	0.4	0.8	0.2	0.7	1.3	31
THA	0.5	0.7	0.4	0.3	9.0	0.2	0.3	3.6	2.9	0.4	0.8	5.3	0.2	73.9	0.1	0.5	0.1	0.7	0.2	26
ARG	3.5	1.5	1.6	0.4	2.7	0.5	1.2	0.3	0.1	2.1	0.2	0.8	0.4	0.3	81.0	0.9	0.8	0.6	1.0	19
BRA	4.5	2.3	1.4	0.3	12.6	0.4	3.3	1.0	0.3	10.0	0.7	3.4	0.5	0.3	11.7	45.2	0.3	0.9	0.8	55
CHL	3.5	0.7	0.7	0.3	2.7	0.1	3.6	1.1	0.2	1.8	0.3	1.8	0.3	0.4	3.6	5.0	73.7	0.2	0.1	26
MEX	6.5	1.3	0.7	0.3	25.0	0.2	4.8	0.3	0.5	2.4	0.3	2.1	0.2	0.5	6.3	3.0	0.3	44.1	1.1	56
TUR	2.8	1.7	0.8	0.7	3.9	0.3	1.2	0.3	1.1	2.7	0.5	0.9	4.0	0.1	0.7	0.3	0.2	1.1	76.8	23
Contribution to others	138	98	32	10	170	7	30	38	41	40	16	45	10	5	27	14	3	8	17	749.6
Contribution including own	202	153	59	23	258	90	65	109	108	111	83	91	79	79	108	59	77	52	94	Spillover Index = 39.5%

Notes: The underlying variance decomposition is based upon a weekly VAR of order 2, identified using a Cholesky factorisation with the ordering as shown in the column heading. The (i, j)-th value is the estimated contribution to the variance of the 10-week-ahead stock return volatility forecast error of country i coming from innovations to the stock return volatility of country j. We calculate Chile's volatility using the Santiago Stock Exchange IGPA Index for January 1992–May 2004, and using the Santiago Stock Exchange IPSA index for June 2004 onward. The mnemonics are defined as in Table 1.

- Spillovers are important in both returns and volatilities and, on average - that is, unconditionally - return and volatility spillovers are of the same magnitude

Spillover Plots

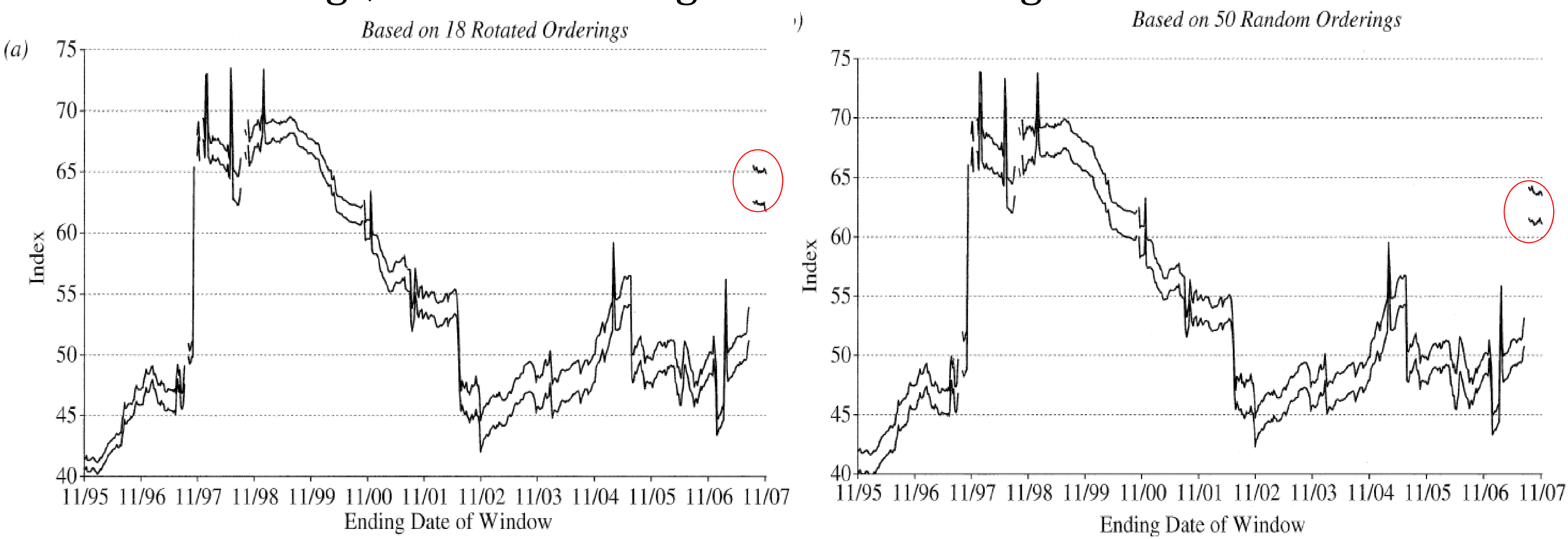
- At any given point in time - that is, conditionally - mean and volatility spillovers may be very different and, more generally, their dynamics may be very different



  - These measures are estimated using 200-week rolling samples
  - Even as the estimation window moves beyond the mid-1990s, the **return** Spillover Plots never decline to their earlier lower range

Volatility Spillovers

- This is consistent with an increase in financial market integration
- The Spillover Plot for volatility is radically different, ranging widely and responding to economic events
- It is then typical to explore robustness to VAR ordering, plotting max and minimum volatility spillovers across a variety of alternative VAR orderings, estimated using 200-week rolling windows



- Return spillovers display no bursts but a gently increasing trend; volatility spillovers, in contrast, display no trend but clear bursts

Generalized Variance Decompositions

- Diebold and Yilmaz's (DY) framework has several limitations
- First, DY relies on the Cholesky identification and the resulting variance decompositions can be dependent on variable ordering
- Solution: use a generalized VAR framework in which forecast-error variance decompositions are invariant to variable ordering
  - Calculation of variance decompositions requires orthogonal innovations, whereas VAR innovations are contemporaneously correlated
  - Identification schemes such as that based on Cholesky factorization achieve orthogonality, but the variance decompositions then depend on the ordering of the variables
- Second DY only estimate the total spillovers (from/to each market *i*, to/from all other markets, added across *i*), but one would also like to examine directional net spillovers
- Technically, the GVAR framework of Koop, Pesaran, and Potter (1996, JoE) and Pesaran and Shin (1998, EL) produces decompositions which are invariant to the ordering, as in DY (2012, IJF)

Generalized Variance Decompositions

- Instead of attempting to orthogonalize shocks, the generalized KPPS approach allows correlated shocks but accounts for them using the historically observed correlation patterns of errors, **under normality**
- As the shocks are not orthogonalized, the sum of the contributions to the variance of the forecast error (the row sum of the elements of the variance decomposition table) is **not necessarily 1$\Longrightarrow$ normalize**
- Denoting the KPPS H-step-ahead forecast error variance decompositions by $\theta_{ij}^{g}(H)$,

Std error jth equation

$$\theta_{ij}^{g}(H) = \frac{\sigma_{jj}^{-1} \sum_{h=0}^{H-1} (e_i' A_h \Sigma e_j)^2}{\sum_{h=0}^{H-1} (e_i' A_h \Sigma A_h' e_i)}$$

Note that $\sum_{j=1}^{N} \theta_{ij}^{g}(H) \neq 1$

- To compute the spillover index, normalize each entry of the variance decomposition matrix by the row sum as:

$$\tilde{\theta}_{ij}^{g}(H) = \frac{\theta_{ij}^{g}(H)}{\sum_{j=1}^{N} \theta_{ij}^{g}(H)}$$

- Using these estimates we can construct the **total volatility spillover index**:

$$S^{g}(H) = \frac{100}{N} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)$$

Estimating Directional and Net Spillovers

- The generalized VAR approach enables us to also learn about the direction of volatility spillovers because generalized IRFs and variance decompositions are invariant to the ordering of variables
- Use normalized elements of generalized decomposition matrix:

$$S_{i\cdot}^{g}(H)=\frac{\sum_{\substack{j=1\\ j\neq i}}^{N}\tilde{\theta}_{ij}^{g}(H)}{\sum_{i,j=1}^{N}\tilde{\theta}_{ij}^{g}(H)}\cdot 100=\frac{\sum_{\substack{j=1\\ j\neq i}}^{N}\tilde{\theta}_{ij}^{g}(H)}{N}\cdot 100 \qquad S_{\cdot i}^{g}(H)=\frac{\sum_{\substack{j=1\\ j\neq i}}^{N}\tilde{\theta}_{ji}^{g}(H)}{\sum_{i,j=1}^{N}\tilde{\theta}_{ji}^{g}(H)}\cdot 100=\frac{\sum_{\substack{j=1\\ j\neq i}}^{N}\tilde{\theta}_{ji}^{g}(H)}{N}\cdot 100$$

From all markets to mkt *i*

From market *i* to all others

- We obtain the net volatility spillover from *i* to all other markets *j* as $S_{i}^{g}(H)=S_{\cdot i}^{g}(H)-S_{i\cdot}^{g}(H)$, the difference btw. gross volatility shocks transmitted to and those received from all other markets
- Also interesting to examine the net pairwise volatility spillovers:

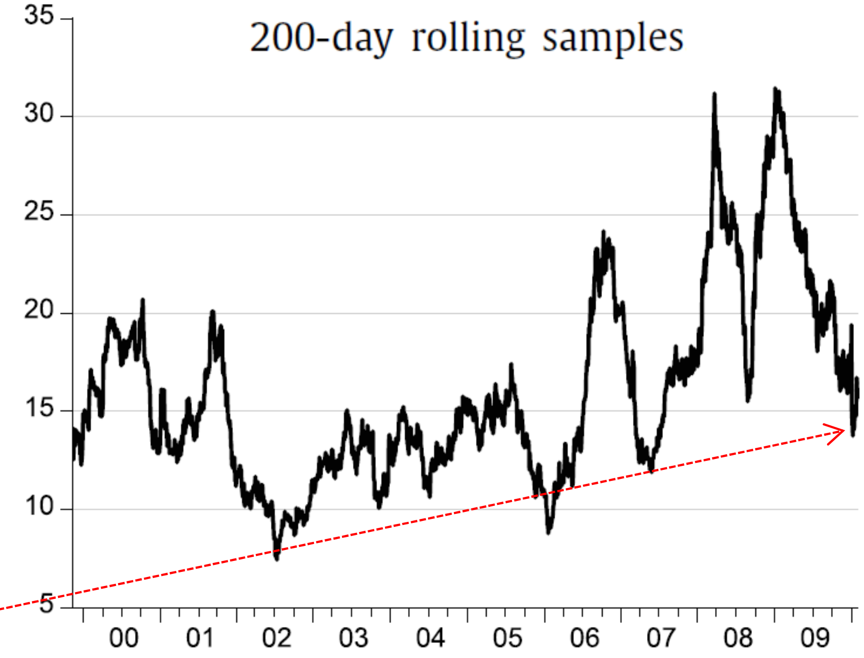
$$S_{ij}^{g}(H)=\left(\frac{\tilde{\theta}_{ji}^{g}(H)-\tilde{\theta}_{ij}^{g}(H)}{N}\right)\cdot 100$$

Estimating Directional and Net Spillovers

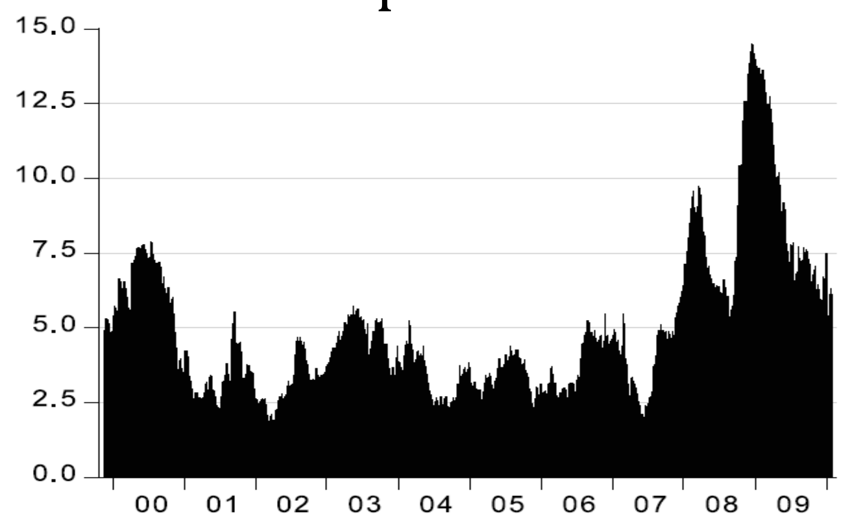
Volatility spillover table, four asset classes.

	Stocks	Bonds	Commodities	FX	Directional FROM others
Stocks	88.76	7.28	0.34	3.62	11.24
Bonds	10.17	81.49	2.69	5.65	18.51
Commodities	0.46	3.69	93.71	2.14	6.29
FX	5.66	6.99	1.59	85.76	14.24
Directional TO others	16.29	17.95	4.63	11.41	
Directional including own	105.0	99.4	98.3	97.2	Total spillover index (50.3/400): 12.6%

All of the results are based on vector autoregressions of order 4 and generalized variance decompositions of 10-day-ahead volatility forecast



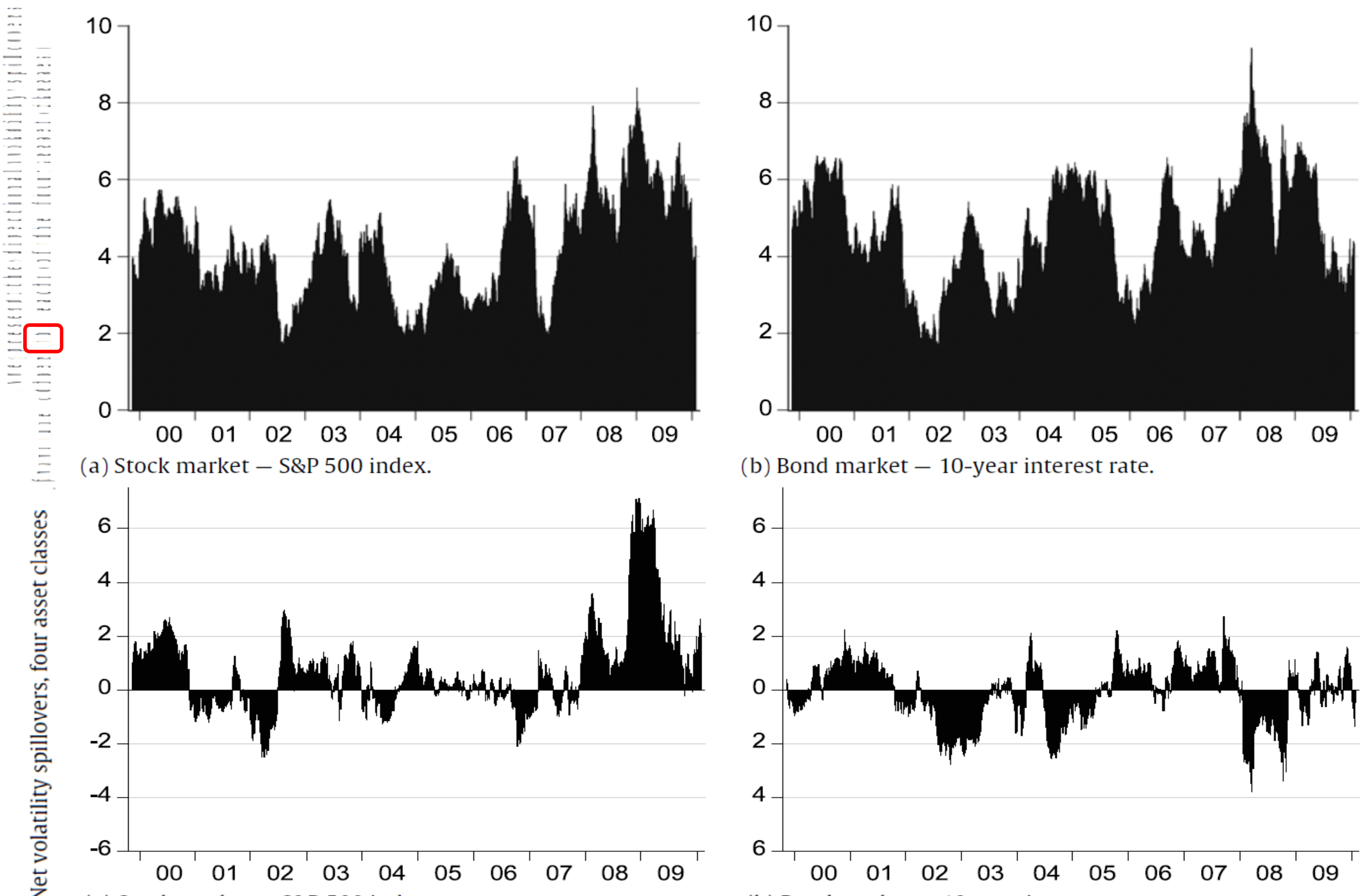
- Total volatility spillover indicates that, on average, 12.6% of the forecast error variance comes from spillovers—both the total and directional spillovers over the full sample period were quite low



(a) Stock market — S&P 500 index.

(b) Bond market — 10-year interest rate.

Estimating Directional and Net Spillovers



(a) Stock market — S&P 500 index. (b) Bond market — 10-year interest rate.

(a) Stock market — S&P 500 index. (b) Bond market — 10-year interest rate.

- During the financial crisis, the volatility from the stock market was transmitted to all three markets, but especially to the FX market (close to 5%), following the collapse of the Lehman Brothers

Unresolved Issues

- What exactly are confidence bands for these contagion indices?
- Should they reflect only parameter or also model uncertainty?

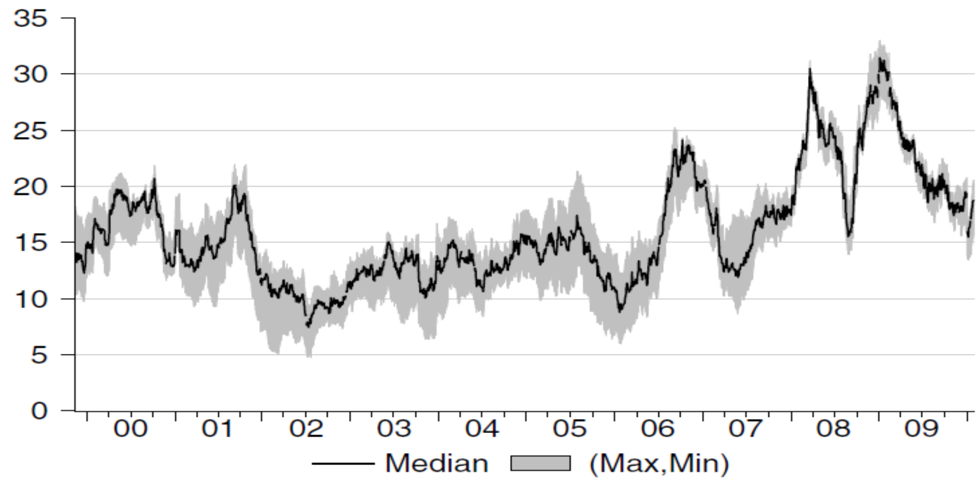


Fig. A.1. Sensitivity of the index to the VAR lag structure (max, min and median values of the index for VAR orders of 2–6).

- What it is the meaning of contagion and spillover dynamics that depend on some "horizon"?

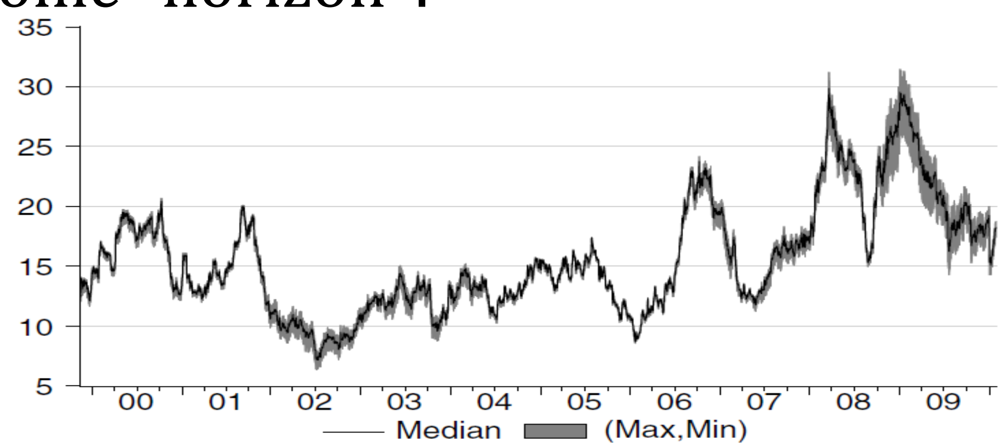


Fig. A.2. Sensitivity of the index to the forecast horizon (min, max and median values over 5- to 10-day horizons).

Relationship to Connectedness Network Measures

- Consider again the variance decomposition/connectedness table:

	x_1	x_2	$\cdots$	x_N	From others
x_1	d^H_{11}	d^H_{12}	$\cdots$	d^H_{1N}	$\Sigma^N_{j=1} d^H_{1j}, j \neq 1$
x_2	d^H_{21}	d^H_{22}	$\cdots$	d^H_{2N}	$\Sigma^N_{j=1} d^H_{2j}, j \neq 2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_N	d^H_{N1}	d^H_{N2}	$\cdots$	d^H_{NN}	$\Sigma^N_{j=1} d^H_{Nj}, j \neq N$
To others	$\Sigma^N_{i=1} d^H_{i1}$ $i \neq 1$	$\Sigma^N_{i=1} d^H_{i2}$ $i \neq 2$	$\cdots$	$\Sigma^N_{i=1} d^H_{iN}$ $i \neq N$	$\frac{1}{N} \Sigma^N_{i,j=1} d^H_{ij}$ $i \neq j$

- Define pairwise directional connectedness from *j* to *i* as $C^H_{i \leftarrow j} = d^H_{ij}$
- In general, $C^H_{i \leftarrow j} \neq C^H_{j \leftarrow i}$, so there are $N^2 - N$ separate pairwise directional connectedness measures
- Interested in "net", as opposed to "gross", pairwise directional connectedness: $C^H_{ij} = C^H_{j \leftarrow i} - C^H_{i \leftarrow j}$; there are $(N^2 - N)/2$ net pairwise directional connectedness measures

Relationship to Connectedness Network Measures

- The row sum of off-diagonal elements gives share of the H-step error variance of variable i coming from shocks arising in other variables so that total directional connectedness from others is: $$C^H_{i\leftarrow\bullet} = \sum_{\substack{j=1\\ j\neq i}}^{N} d^H_{ij}$$
- Similarly, by computing column sum of off-diagonal elements, total directional connectedness to others from j is $$C^H_{\bullet\leftarrow j} = \sum_{\substack{i=1\\ i\neq j}}^{N} d^H_{ij}$$
- There are $2N$ total directional connectedness measures, N "to others", or "transmitted", and N "from others"
- The N net total directional connectedness are: $C^H_i = C^H_{\bullet\leftarrow i} - C^H_{i\leftarrow\bullet}$
- The grand total of the off-diagonal entries measures total connectedness is: $$C^H = \frac{1}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{N} d^H_{ij}$$
- As we have seen, the connectedness table may be based on GVDs
  - They treat each variable as "first in the ordering" by allowing for correlated shocks while simultaneously accounting for the correlation among them, under a normality assumption

Relationship to Connectedness Network Measures

- In this case, we have

$$d_{ij}^{gH} = \frac{\sigma_{jj}^{-1} \sum_{h=0}^{H-1} (e_i' A_h \Sigma e_j)^2}{\sum_{h=0}^{H-1} (e_i' A_h \Sigma A_h' e_i)} \qquad \tilde{d}_{ij}^{g} = \frac{d_{ij}^{g}}{\sum_{j=1}^{N} d_{ij}^{g}}$$

Scaled to sum to 1

- C^H depends on the set of variables $\boldsymbol{x}$ whose connectedness is to be examined, the predictive horizon H, the dynamics captured by $\mathbf{A}(h)$, and the approximating VAR(p) model, C($\boldsymbol{x}$, H, $\mathbf{A}(h)$, $M(\theta)$)
- We may also see the connection table and all of its elements to vary over time, and write $C_t(x, H, A_t(h), M(\theta_t))$ to be estimated as

$$\widehat{C}_t(x, H, A_t(h), M(\hat{\theta}_t))$$

  - Varying H lets us break connectedness in "long-run", "short-run", etc.
  - In the limit as $H \to \infty$, we obtain an unconditional VD
  - Many choices are possible to allow for time-varying parameters
  - A simple scheme involves use of a rolling estimation window
- A network $\mathcal{N}$ is composed of N nodes and L links between nodes
- Distance s_{ij} between 2 nodes i and j is smallest number of links that must be traversed to go from i to j; $\mathcal{N}$ is connected if $s_{ij} \leq N - 1, \forall i, j$

Appendix: Algebra of VMA(∞) Representation

The in-sample variance decomposition method is based on the generalized impulse function of Pesaran and Shin (1998). Similarly to Diebold and Yilmaz (2014, 2016), the generalized error variance decomposition is preferred over a standard Cholesky-factor decomposition because it is independent from the ordering of the variables. Consider a vector stochastic process $\{\boldsymbol{x}_t\}$ of N random variables which follows a vector autoregressive model of order p,

$$\boldsymbol{x}_t = \sum_{i=1}^{p} \boldsymbol{A}_i \boldsymbol{x}_{t-i} + \boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\varepsilon}_t\,, \quad t = 1, 2, \dots, T \qquad (*)$$

where $\boldsymbol{w}_t$ is a $q \times 1$ vector of deterministic variables, $\boldsymbol{A}_i$ and $\boldsymbol{B}$ are $N \times N$ and $N \times q$ coefficient matrices, and $\boldsymbol{\varepsilon}_t$ is a N-dimensional innovation process with $E(\varepsilon_t) = \boldsymbol{0}$, $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Sigma}_\varepsilon^{IS}$ and $E(\boldsymbol{\varepsilon}_t | \boldsymbol{w}_t) = \boldsymbol{0}$ for $\forall t$, $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_{t'}') = \boldsymbol{O}$ for $t \neq t'$. Assuming that the process in (*) is covariance stationary, the VAR(p) model in (*) can be expressed as an infinite-order vector moving average process,

$$\boldsymbol{x}_t = \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{B} \boldsymbol{w}_{t-i}\,, \quad t = 1, 2, \dots, T,$$

Appendix: Algebra of VMA(∞) Representation

The sequence of coefficient matrices $\boldsymbol{\Phi}_i$ can be recursively calculated as:

$$\boldsymbol{\Phi}_i = 0 \;\; if \;\; i < 0, \qquad \boldsymbol{\Phi}_0 = \boldsymbol{I}_N, \qquad \boldsymbol{\Phi}_i = \sum_{l=1}^{p} \boldsymbol{A}_l \boldsymbol{\Phi}_{i-l}\,.$$

Therefore $\boldsymbol{\Phi}_1 = \boldsymbol{A}_1$, $\boldsymbol{\Phi}_2 = \boldsymbol{A}_1\boldsymbol{A}_1$, etc.