Single Name Credit Derivatives

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The opinion expressed here are solely those of the author and do not represent in any way those of her employers

Reference Book

D. Brigo and F. Mercurio
Interest Rate Models – Theory and Practice. With Smile, Inflation and Credit
Springer (2006)

Outline

Outline

Credit Risk: Definition

Credit risk is associated with an underlying asset issued by a **reference entity** (government, financial, corporate) and originates from the inability of the reference entity to meet its **contractual obligations** with its counterparties.

Obligations

Borrowed money obligations

- financial leverage
- fixed amount claims
- bonds and loans

Payment obligations

- hedging/speculative purposes
- contingent claims
- credit derivatives

Credit Derivatives

"**Credit derivatives** are bilateral financial contracts that **isolate** specific aspects of **credit risk** from an underlying instrument and **transfer** that risk between two parties."

The J.P. Morgan Guide to Credit Derivatives (1999)

They can be **single name** or **multi-name:**

- Defaultable (zero) coupon bonds
- **Credit Default Swaps (**CDS**)**
- CDS indices
- **Collateralized Debt Obligations (**CDO**)**
- ...

Credit Derivatives: ISDA

The **2014 ISDA Credit Derivative Definitions** are the **market standard** definitions for credit derivative transactions and contain the building blocks for all credit derivative transactions.

They represent a response to market developments since the financial crisis and incorporate/extend:

- the **2003 ISDA Credit Derivative Definitions**
- the **2009 CDS Big Bang**, namely:
  - the **2009 ISDA Credit Derivatives Determinations Committees**
  - the **2009 Auction Settlement and Restructuring Supplement** to the 2003 ISDA Credit Derivatives Definitions

In particular, they define the **credit event**.

2003 ISDA Credit Derivative Definitions

Credit events defined by the **2003 ISDA agreement**[1] are:

1. bankruptcy,
2. failure to pay,
3. debt restructuring
4. repudiation/moratorium
5. obligation acceleration

A credit event occurs in relation to obligations of a reference entity.
ISDA contractual Standard refers to **borrowed money obligations** and prescribes **physical settlement**.

[1] hiip://credit-deriv.com/isdadefinitions.htm

2009 CDS Big and Small Bang

In order to achieve more **consistency, standardization and timely trade matching**, the following main changes to the CDS contract are introduced:

1. **hardwire auction mechanism** for CDS following a credit event
2. **Determination Committees (DC)** to make binding determinations of whether credit and succession events have occurred as well as the terms of any auction

Recovery Auction

- The 2003 ISDA contractual standard prescribes **physical settlement** and fixes the **deliverable obligations** in terms of **bonds and loans**. Deliverable obligations are the driver of the **recovery rate**.
- The **DC** determines the list of all deliverable obligations and establishes whether a recovery auction will take place.
- The **recovery auction**[2] settles the recovery price which will be the same for all standard contracts.
- The mechanism of recovery auctions transform the CDS into a **cash settled** contract.

[2] www.creditfixings.com

March 2012: the Greek Bonds

- **1 March 2012:** ISDA declared that the restructuring of Greek bonds did not constitute a credit event
  - `Greek Restructuring: Why Isn't It (Yet) a Credit Event?, Sherman and Sterling`
  - hiips://www.theguardian.com/business/2012/mar/01/gr eece-credit-event-default-q-a
- **9 March 2012:** ISDA confirmed that a credit event has occurred ...
  - hiips://www.isda.org/a/p0EDE/greek-sovereign-cds-cr edit-event-faqs-03-09.pdf

[... 18 October 2011: Brussels gave green light to ban "naked" CDS on sovereign debt in an attempt to curb what some policymakers see as hedge fund bets on the euro zone crisis.]

2014 ISDA Credit Derivative Definitions

New Definitions constitute a major reform which address numerous issues arisen in the context of the Eurozone crisis:

- **Government Intervention:** a new Credit Event, triggered by a government initiated bail-in
- **Subordinated/Senior CDS Split:** to avoid a situation where senior CDS contracts are triggered by a credit event affecting only subordinated debt
- **Asset Package Delivery Events:** to address concerns prompted by the potential lack of any Deliverable Obligations
- **Standard Reference Obligations:** to increase liquidity, it will enable parties to trade without specifying a Reference Obligation in their contract

Credit Risk Measurement

Credit risk measurement is based on these **fundamental parameters:**

- **Probability of Default** (PD)
  The likelihood that the borrower will fail to make full and timely repayment of its financial obligations.
- **Exposure at Default** (EAD)
  The expected value of the loan at the time of default.
- **Loss Given Default** (L_{GD}):
  The amount of the loss if there is a default, expressed as a percentage of the EAD.
- **Recovery Rate** ($\text{Rec} = 1 - \text{L}_{\text{GD}}$):
  The proportion of the EAD the bank recovers.

Probability of Default

The **default event**, being unpredictable, is commonly described by a random variable τ, denoting the **default time**.

The probability of default occurring before time T, given that default has not happened until time t, is given by the so-called **cumulative default probability**:

$$\boxed{p(t, T) = \Pr(\tau \leq T \,|\, \tau > t)} \tag{1}$$

- **Probability of default between two times** T_1 and T_2:

$$p(t, T_1, T_2) = \Pr(T_1 < \tau \leq T_2 \,|\, \tau > t) = p(t, T_2) - p(t, T_1)$$

- **Marginal default probability**, i.e. the probability of default in the period $(T_1, T_2]$, conditional on having survived until the beginning of time T_1:

$$\boxed{p_M(t, T_1, T_2) = \Pr(T_1 < \tau \leq T_2 \,|\, \tau > T_1) = \frac{p(t, T_2) - p(t, T_1)}{1 - p(t, T_1)}} \tag{2}$$

Probability of Default: Example I

We follow Ballotta *et al* in order to illustrate the **relationship between marginal and cumulative default probabilities**. Consider $t = 0$ and $T_i = 1, 2, 3, \ldots$ year.

- $\mathbf{T_1 = 1}$:

$$p(0,1) = p_M(0,0,1)$$

- $\mathbf{T_2 = 2}$:

$$p(0,2) = \underbrace{p(0,1)}_{\text{default in 1st year}} \underbrace{+}_{\text{OR}} \underbrace{(1 - p_M(0,0,1))}_{\text{survive 1st year}} \times \underbrace{p_M(0,1,2)}_{\text{default in 2nd year}}$$

Solving for $p_M(0,1,2)$ we obtain a result in agreement with Eq. (2):

$$p_M(0,1,2) = \frac{p(0,2) - p(0,1)}{1 - p(0,1)}$$

Probability of Default: Example II

- $\mathbf{T_3 = 3}$:

$$p(0,3) = \underbrace{p(0,1)}_{\text{default in 1st year}} \underbrace{+}_{\text{OR}} \underbrace{(1 - p_M(0,0,1))}_{\text{survive 1st year}} \times \underbrace{p_M(0,1,2)}_{\text{default in 2nd year}} \underbrace{+}_{\text{OR}}$$

$$\underbrace{(1 - p_M(0,0,1))}_{\text{survive 1st year}} \times \underbrace{(1 - p_M(0,1,2))}_{\text{survive 2nd year}} \times \underbrace{p_M(0,2,3)}_{\text{default in 3rd year}}$$

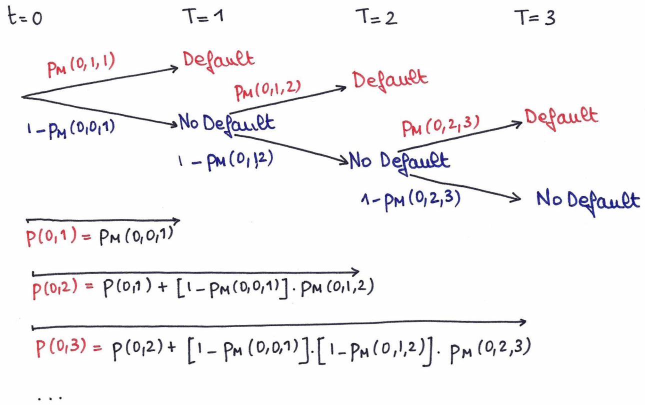
$$= p(0,2) + (1 - p_M(0,0,1))(1 - p_M(0,1,2)) \times p_M(0,2,3)$$

Solving for $p_M(0,2,3)$ we get, in accordance with Eq. (2):

$$p_M(0,2,3) = \frac{p(0,3) - p(0,2)}{1 - p(0,2)}$$

- and so on...

Probability of Default: Graphical Illustration



Recovery Rate

Recovery rates can vary widely, as they are affected by a number of factors:

- **Type of instrument and its seniority**
  The recovery rate is directly proportional to the instrument's seniority, i.e. an instrument that is more senior in the capital structure will usually have a higher recovery rate than one that is lower down in the capital structure.
- **Corporate issues: capital structure and leverage**
  Debt instruments issued by a company with a relatively lower level of debt in relation to its assets may have higher recovery rates than a company with substantially more debt.
- **Macroeconomic conditions: economic cycle, liquidity conditions...**
  If a large number of companies are defaulting on their debt (e.g. in a deep recession) the recovery rates may be lower than during normal economic times.

For a review of recovery rate modeling see Altman *et al* (2005). In this course, we will assume **recovery rates** (and LGD) to be **constant** and given by **historical estimates of rating agencies**.

Outline

Credit Default Swaps (CDS)

Credit Default Swaps (CDS**)** are **basic protection contracts** that provide the protection buyer an insurance against the occurrence of **credit events** of a reference entity, in exchange for periodic payments to the protection seller.

- The 2003 ISDA protocol specifies the credit events, which occur in relation to **borrowed money obligations** of a reference entity.
- Upon a credit event, the 2003 ISDA contractual standard prescribes **physical settlement** and fixes the **deliverable obligations** in terms of **bonds and loans**.
- Deliverable obligations are the driver of the **recovery rate**: the recovery price is settled by the market by means of a **recovery auction** (www.creditfixings.com) and it is the same for all standard contracts. Basically, the mechanism of recovery auctions transform the CDS into a **cash settled** contract.

Rationale for CDS Modeling

The **market** for a large number of reference entities is quite liquid and CDS spreads are determined by **supply and demand**. There is no need to use pricing models to value CDS contracts at inception. However, models are necessary in order to:

- extract market default probability from CDS quotations through **bootstrapping**
- **price more complex** credit derivatives, by calibrating models to CDS market quotes
- determine the **mark-to-market** MTM value of existing CDS positions.

Pricing models for default are mainly of two kinds:

1. **reduced form (intensity)** models
2. **structural** models

CDS Features

A CDS contract ensures **protection** against default of a reference entity.

The protection buyer A pays to the protection seller B a given rate R, at times $T_{a+1}, \ldots, T_b$ or until default (the **premium leg**), in exchange for a single protection payment LGD (the **protection leg**) at default time τ of a reference entity C, provided that $T_a < \tau \leq T_b$. Schematically:

Protection	$\rightarrow$	protection LGD at default τ if $T_a < \tau \leq T_b$	$\rightarrow$	Protection
Seller B	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ	$\leftarrow$	Buyer A

At evaluation time t, the amount R is set at a value $R_{ab}(t)$ that makes the **contract fair**, i.e. such that the present value of the two exchanged flows is zero. This is how the market quotes CDS.

Modeling Assumptions and Notations

Let us denote:

- the year fraction between two consecutive dates as $\alpha_i \equiv T_i - T_{i-1}$
- the first date among the $\{T_i\}_{i=a+1,\dots,b}$ following t as $T_{\beta(t)}$, i.e. $t \in [T_{\beta(t)-1}, T_{\beta(t)})$
- the stochastic discount factor as $D(t,T) = B(t)/B(T)$, where

$$B(t) = e^{\int_0^t r_u d_u}$$

  represents the bank account numeraire, r being the instantaneous short interest rate.

Let us assume

- the Loss Given Default $\text{LGD} \equiv 1 - \text{Rec}$ to be deterministic
- the default time τ and the interest rate process r to be independent
- unit notional amount

CDS Cash Flows

Premium Leg payments

The premium leg pays

- at each time T_i, if default has not occurred yet, the periodic fee: $R\,\alpha_i\,\mathbb{1}_{\{\tau \geq T_i\}}$
- at default time τ, if $\tau < T_b$, the accrued interest: $R\,(\tau - T_{\beta(\tau)-1})\,\mathbb{1}_{\{T_a<\tau<T_b\}}$

where $\mathbb{1}_{\{X\}}$ is the indicator function of set X.

Protection Leg payments

The protection leg pays

- at default time τ, if $\tau \leq T_b$, the Loss Given Default: $\text{LGD}\,\mathbb{1}_{\{\tau<T_b\}}$

Discounted payoff seen from B

$$\begin{aligned}\Pi_{\text{CDS}_{ab}}(t) &= \Pi_{\text{PremL}_{ab}}(t) - \Pi_{\text{ProtL}_{ab}}(t) \quad (3)\\ &= D(t,\tau)(\tau - T_{\beta(\tau)-1})\,R\,\mathbb{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\,\alpha_i\,R\,\mathbb{1}_{\{\tau\geq T_i\}}\\ &\quad -D(t,\tau)\,\text{LGD}\,\mathbb{1}_{\{T_a<\tau\leq T_b\}}\end{aligned}$$

Outline

General Pricing Framework

Let use denote by $\text{CDS}_{a,b}(t, R, \text{L}_{\text{GD}})$ the **price** at time t of the CDS.

At $t = 0$, the price is given by the expectation of the discounted payoff $\Pi_{\text{CDS}_{ab}}(0)$ under the **risk neutral measure** $\mathbb{Q}$:

$$\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = \mathbb{E}[\Pi_{\text{CDS}_{ab}}(0)]\,. \tag{4}$$

This is equivalent to pricing (analytically) both legs.

Goal

In general, the resulting pricing formulas depend on the choice of the underlying model (**intensity based** or **structural**) used to describe the dynamics of:

- the default event
- interest rates
- the loss given default (when stochastic)

However, such formulas will not be used to price CDS that are already quoted in the market.

Rather, they will be **inverted** in correspondence of CDS market quotes to **calibrate** the models to the CDS quotes themselves.

Model Independent Framework

Before tackling the general problem of deriving pricing formulas for CDS under specific model assumptions, which will be dealt with in the next lectures, we show how to derive **model independent formulas**, given, at time $t = 0$:

- the initial **zero coupon curve** (bonds) observed in the market: $P^{\text{mkt}}(0, T_i)$
- the **survival probabilities**

Survival Function (Li 2000)...

- Let $F(t)$ denote the **distribution function of default time τ**:

$$F(t) := \Pr(\tau \leq t) \quad t \geq 0 \text{ and } F(0) = 0.$$

  If F is differentiable, its derivative f represents the density function and:

$$F(t) = \int_{-\infty}^{t} f(u)du\,.$$

- The probability that default does not occur before time t represents the **survival probability** and is described the survival function $S(t)$:

$$S(t) := \Pr(\tau \geq t) = \int_{t}^{\infty} f(u)du = 1 - F(t)\,.$$

Premium Leg I

The premium leg price is given by the expected value of the premium leg payoff under the risk neutral measure $\mathbb{Q}$:

$$\text{PremL}_{ab}(R) = \mathbb{E}\left[D(0,\tau)(\tau - T_{\beta(\tau)-1}) R \, \mathbb{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(0,T_i)\alpha_i R \, \mathbb{1}_{\{\tau \geq T_i\}} \right]$$

Under independence between default τ and interest rates, the **premium leg price** is:

$$\text{PremL}_{ab}(R) = R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) \, d\mathbb{Q}(\tau \geq t) + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \, \mathbb{Q}(\tau \geq T_i) \right]$$

(See Appendix 1 for the proof.)

Premium Leg II

Where, the infinitesimal survival probability $d\mathbb{Q}(\tau \geq t)$ is given by:

$$\begin{aligned} d\mathbb{Q}(\tau \geq t) &= d[1 - \mathbb{Q}(\tau < t)] \\ &= -d\mathbb{Q}(\tau < t) \\ &= -\left[\mathbb{Q}(\tau < t + dt) - \mathbb{Q}(\tau < t)\right] \\ &= -\mathbb{Q}(\tau \in [t, t + dt)) \\ &= -\mathbb{E}[\mathbb{1}_{\{\tau \in [t, t+dt)\}}] \end{aligned}$$

and the zero coupon bond $P(0, t)$ is given by:

$$P(0, t) = \mathbb{E}[D(0, t)]$$

Protection Leg

The protection leg price is given by the expected value of the protection leg payoff under the risk neutral measure $\mathbb{Q}$:

$$\text{ProtL}_{ab} = \mathbb{E}\left[\text{LGD}\, D(0,\tau)\, \mathbb{1}_{\{T_a<\tau\leq T_b\}}\right]$$

Under independence between default τ and interest rates, the **protection leg price** is:

$$\text{ProtL}_{ab} = -\text{LGD} \int_{T_a}^{T_b} P(0,t)\, d\mathbb{Q}(\tau \geq t)$$

(See Appendix 2 for the proof.)

CDS Model Independent Pricing Formula

The CDS price at time $t = 0$ is given by:

$$\mathrm{CDS}_{ab}(0, R, \mathrm{LGD}; \mathbb{Q}(\tau > \cdot)) = \mathrm{PremL}_{ab}(\mathbb{Q}(\tau > \cdot)) - \mathrm{ProtL}_{ab}(\mathbb{Q}(\tau > \cdot))$$
$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(u)-1})\, d\mathbb{Q}(\tau \geq t) + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i\, \mathbb{Q}(\tau \geq T_i)\right]$$
$$+ \mathrm{LGD}\int_{T_a}^{T_b} P(0,t)\, d\mathbb{Q}(\tau \geq t)$$

The integrals in the survival probabilities can be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.

CDS Market Quotes

The market quotes, at time 0, the **fair** CDS rate

$$R = R_{0,b}^{\text{mkt MID}}(0)$$

with initial protection time

$$T_a = 0$$

and final protection time

$$T_b \in \{1y, 3y, 5y, 7y, 10y\}\,.$$

CDS Rates

$R_{0,b}^{\text{mkt MID}}(0)$ is obtained by solving:

$$\text{CDS}_{ab}(0, R_{0,b}^{\text{mkt MID}}(0), \text{LGD}; \mathbb{Q}(\tau > \cdot)) = 0$$

which gives:

$$R_{0,b}^{\text{mkt MID}}(0) = \frac{\text{LGD} \int_0^{T_b} P(0,t)\, d\mathbb{Q}(\tau \geq t)}{\int_0^{T_b} P(0,t)(t - T_{\beta(t)-1})\, d\mathbb{Q}(\tau \geq t) - \sum_{i=1}^{b} P(0,T_i)\alpha_i\, \mathbb{Q}(\tau \geq T_i)}$$

Using the initial zero coupon curve (bonds) observed in the market, $P^{\text{mkt}}(0, T_i)$, and the market quotes of $R_{0,b}^{\text{mkt MID}}(0)$, starting from $T_b = 1y$, we can find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$ and iteratively bootstrap all the survival probabilities $\{\mathbb{Q}(\tau \geq t), t \in (1y, 3y]\}$ and so on up to $T_b = 10y$.

This is a way to strip survival probabilities from CDS quotes in a **model independent way**, with no need to assume an intensity or structural model of default!

However, the market in doing the above stripping typically resorts to **hazard functions** associated with the default time.

...Hazard Functions (Li 2000)

Recalling the survival function introduced above, survival probabilities can be equivalently expressed in terms of the so-called hazard functions.

- The default rate at time t, conditional on survival until time t or later, defines the **hazard rate function**:

$$h(t) := \lim_{dt \to 0} \frac{\Pr(t \leq \tau < t + dt)}{dt \Pr(\tau \geq t)} = \frac{F(t + dt) - F(t)}{dt\, S(t)} = \frac{f(t)}{S(t)} = -\frac{S'(t)}{S(t)} \tag{5}$$

- The **cumulative hazard function** $H(t)$ is defined as:

$$H(t) := \int_0^t h(u)du = -\int_0^t d(\ln S(u)) = -\ln S(t) \tag{6}$$

Then, the survival function $S(t)$, describing the **survival probability** up to time t, can be expressed in terms of hazard functions as:

$$S(t) = e^{-\int_0^t h(u)du} = e^{-H(t)}$$

Outline

General Framework

GOAL: Quoted spread $\Rightarrow$ survival probabilities $\Rightarrow$ intensity

Assume the existence of a **deterministic default intensity** $\gamma(t)$, such that $\gamma(t)\,dt$ represents the probability rate of defaulting in $[t, t + dt)$ having not defaulted before t:

$$\mathbb{Q}(\tau \in [t, t + dt) | \tau > t) = \gamma(t)dt$$

Recalling definitions (5) and (6) of hazard functions, the **cumulated intensity** is nothing but a *cumulated hazard rate*:

$$\Gamma(t) := \int_0^t \gamma(u)\,du$$

The survival probability (under the risk neutral measure) can be written as

$$\mathbb{Q}(\tau \geq t) = e^{-\Gamma(t)} \tag{7}$$

Also,

$$d\mathbb{Q}(\tau \geq t) = -\gamma(t)\,e^{-\Gamma(t)}\,dt$$

Interest Rate Analogy

Usually, the actual model assumes for the default time τ a more complex structure (e.g. stochastic intensity). However, γ^{mkt} are retained as a mere quoting mechanism for CDS rate market quotes.

If the **intensity is stochastic** ($\lambda(t)$) the survival probability formula (7) must be generalized as follows:

$$\mathbb{Q}(\tau \geq t) = \mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right]$$

This is just the price of a zero coupon bond in an interest rate model with short rate r replaced by λ. In this way, survival probabilities can be interpreted as **zero coupon bonds** and intensity λ (or γ in the deterministic case) as **instantaneous credit spreads**.

In the following, we will show a very simple quoting mechanism, based on **constant intensity**. Further examples, based on deterministic and stochastic intensities, will be dealt with extensively, when considering Reduced Form (Intensity) Models.

Constant Intensity Formula

The market makes intensive use of a simple formula for calibrating a constant intensity (hazard rate) $\gamma(t) = \gamma$ to a **single** CDS rate, $R_{0,b}$:

$$\boxed{\gamma = \frac{R_{0,b}}{\text{L}_{\text{GD}}}}$$

Since γ can be interpreted as an instantaneous credit spread, this interpretation can be extended to R as well. Therefore, the CDS premium rate R admits a double interpretation, either as a **credit spread** or as a **default probability**.

This formula is handy, since it does not require the interest rate curve. However, it does not take into account the term structure of CDS, being based on a single quote for R. It can be used in any situation where one needs a **quick calibration** of the default intensity to a single CDS quote (e.g. the liquid 5y one).

Constant Intensity: Formula Derivation I

Assumptions

- independence between default time and interest rates
- the premium leg pays *continuously* until default the premium rate R of the CDS: i.e. in the interval $[t, t + dt)$ the premium leg pays $R\,dt$.

Premium Leg

$$\text{PremL} = \mathbb{E}\left[\int_0^T D(0,t)\,\mathbb{1}_{\{\tau>t\}}\,R\,dt\right] = R\int_0^T \mathbb{E}[D(0,t)\,\mathbb{1}_{\{\tau>t\}}]\,dt$$

$$= R\int_0^T \mathbb{E}[D(0,t)]\,\mathbb{E}[\mathbb{1}_{\{\tau>t\}}]\,dt = R\int_0^T P(0,t)\,\mathbb{Q}(\tau>t)\,dt$$

Constant Intensity: Formula Derivation II

Protection Leg

$$\text{ProtL} = \mathbb{E}\left[\text{LGD}\, D(0,\tau)\, \mathbb{1}_{\{\tau \le T\}}\right] = \text{LGD} \int_0^T \mathbb{E}\left[D(0,t)\, \mathbb{1}_{\{\tau \in [t, t+dt)\}}\right]$$

$$= \text{LGD} \int_0^T P(0,t)\, \mathbb{Q}(\tau \in [t, t+dt)) = -\text{LGD} \int_0^T P(0,t)\, d\mathbb{Q}(\tau > t)$$

Given the constant intensity assumption, i.e.

$$\mathbb{Q}(\tau > t) = e^{-\gamma\, t} \quad \text{and} \quad d\mathbb{Q}(\tau > t) = -\gamma\, e^{-\gamma\, t} dt = -\gamma\, \mathbb{Q}(\tau > t) dt$$

it follows:

$$\text{ProtL} = \gamma\, \text{LGD} \int_0^T P(0,t)\, \mathbb{Q}(\tau > t) dt\,.$$

Premium Leg = Protection Leg

$$R \int_0^T P(0,t)\, \mathbb{Q}(\tau > t)\, dt = \gamma\, \text{LGD} \int_0^T P(0,t)\, \mathbb{Q}(\tau > t) dt \;\Rightarrow\; \boxed{\gamma = \frac{R}{\text{LGD}}}$$

Outline

Appendix 1

$$\mathrm{PremL}_{ab}(R) = \mathbb{E}\left[D(0,\tau)(\tau - T_{\beta(\tau)-1}) R\, \mathbb{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(0,T_i)\alpha_i R\, \mathbb{1}_{\{\tau \geq T_i\}} \right]$$

$$= \mathbb{E}\left[\int_0^{\infty} D(0,t)(t - T_{\beta(t)-1}) R\, \mathbb{1}_{\{T_a<t<T_b\}}\, \mathbb{1}_{\{\tau\in[t,t+dt)\}} \right] + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)]\alpha_i R\, \mathbb{E}[\mathbb{1}_{\{\tau \geq T_i\}}]$$

$$= \mathbb{E}\left[\int_{T_a}^{T_b} D(0,t)(t - T_{\beta(t)-1}) R\, \mathbb{1}_{\{\tau\in[t,t+dt)\}} \right] + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\, \mathbb{Q}(\tau \geq T_i)$$

$$= \int_{T_a}^{T_b} \mathbb{E}[D(0,t)](t - T_{\beta(t)-1}) R\, \mathbb{E}[\mathbb{1}_{\{\tau\in[t,t+dt)\}}] + R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i\, \mathbb{Q}(\tau \geq T_i)$$

$$= \int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) R\, \mathbb{Q}(\tau \in [t, t+dt)) + R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i\, \mathbb{Q}(\tau \geq T_i)$$

$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\, d\mathbb{Q}(\tau \geq t) + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i\, \mathbb{Q}(\tau \geq T_i) \right] \qquad \square$$

Appendix 2

$$
\begin{aligned}
\text{ProtL}_{ab} &= \mathbb{E}\left[\text{LGD}\, D(0,\tau)\, \mathbb{1}_{\{T_a<\tau\leq T_b\}}\right] \\
&= \text{LGD}\, \mathbb{E}\left[\int_0^{\infty} D(0,t)\, \mathbb{1}_{\{T_a<t\leq T_b\}}\, \mathbb{1}_{\{\tau\in[t,t+dt)\}}\right] \\
&= \text{LGD} \int_{T_a}^{T_b} \mathbb{E}[D(0,t)\, \mathbb{1}_{\{\tau\in[t,t+dt)\}}] \\
&= \text{LGD} \int_{T_a}^{T_b} \mathbb{E}[D(0,t)]\, \mathbb{E}[\mathbb{1}_{\{\tau\in[t,t+dt)\}}] \\
&= \text{LGD} \int_{T_a}^{T_b} P(0,t)\, \mathbb{Q}(\tau\in[t,t+dt)) \\
&= -\text{LGD} \int_{T_a}^{T_b} P(0,t)\, d\mathbb{Q}(\tau\geq t) \qquad \square
\end{aligned}
$$

Outline

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