(Advanced) Multi-Name Credit Derivatives

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Disclaimer

The opinion expressed here are solely those of the author and do not represent in any way those of her employers

Main References

- **Overview**

  Brigo, D., Pallavicini, A., and Torresetti, R. (2010). Credit Models and the Crisis, or: How I learned to stop worrying and love the CDOs. *Credit Models and the Crisis: A journey into CDOs, Copulas, Correlations and Dynamic Models, Wiley, Chichester*

- **Bespoke** CDOs

  Baheti, P., and Morgan, S. (2007). Base Correlation Mapping. *Lehman Brothers Quantitative Research Quarterly (Q1)*

- **Implied Copula Model**

  Hull, J., and White, A. (2005). The Perfect Copula: The Valuation of Correlation-Dependent Derivatives Using the Hazard Rate Path Approach, *Working Paper*.

Outline

Outline

Introduction

CDOs look like contracts selling (or buying) **insurance on portions of the loss** of a portfolio.

The valuation problem is trying to determine the **fair price** of this insurance.

Pricing (marking to market) a tranche: taking expectations of the future **tranche losses** under the risk neutral measure.

Introduction: Tranching I

Tranching is a **non-linear operation**, which requires the knowledge of the **whole loss distribution** of the pool of names. The expectation will depend on all moments of the loss and not just the expected loss.

Figure: Source: Brigo (2010)

Introduction: Tranching II

The complete description of the portfolio loss is obtained in two alternative ways, through:

- the knowledge of the **whole distribution** (e.g. Monte Carlo simulation)
- **Single name marginal distributions + dependence structure = copula**

Dependency is commonly called
"correlation"
(abuse of language).

The dependence of the tranche on correlation is crucial. The market assumes a **Gaussian Copula** connecting the defaults of the n names belonging to the portfolio.

Introduction: Correlation

Consider a standard (liquid) index composed of $n = 125$ names (e.g. DJ-iTraxx Index). The copula is parameterized by a matrix of $125 \times 124/2 = 7750$ pairwise correlation values.

Implied Correlation

However, when looking at a given tranche:

7750 parameters $\longrightarrow$ **1** parameter

The **unique correlation parameter** is reverse-engineered to reproduce the price of the liquid tranche under consideration. This is the **implied correlation** and once obtained it is used to value related products.

Two types of correlation can be implied from the market: **compound correlation** and **base correlation** (the market has chosen this one as a quotation standard).

Introduction: Compound Correlation

Two tranches on the same pool (same maturity) yield different values of compound correlation. This implies that the two tranches are priced with **two models** having different and inconsistent loss distributions.

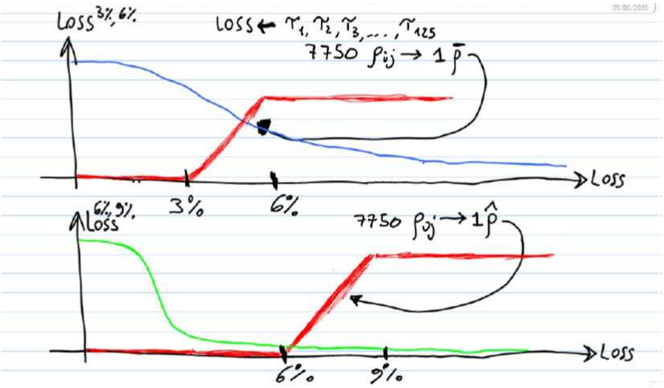


Figure: Source: Brigo (2010)

Introduction: Non-Invertibility of Compound Correlation

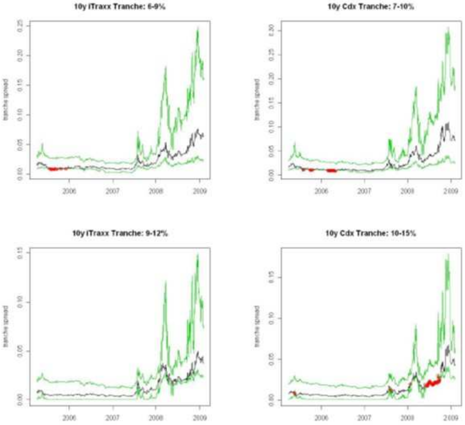


Figure: DJi-Traxx (left charts) and CDX (right charts) 10 year Compound Correlation Invertibility. Red dots highlight the dates in which market spreads were not invertible. Source: Brigo *et al* (2010)

Introduction: Multiple Solutions to Compound Correlation

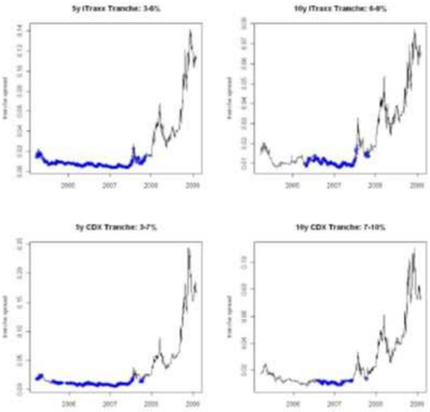


Figure: Upper charts: DJi-Traxx 5 (left charts) and 10 (right charts) year Compound Correlation uniqueness. Lower charts: CDX. Blue dots highlit the dates where more than one compound correlation could reprice the tranche market spread. Source: Brigo *et al* (2010)

Introduction: Base Correlation

The market prefers an alternative definition of implied correlation, the base correlation, which decomposes e.g. the 3% − 6% tranche in terms of the 0% − 3% and the 0% − 6% equity tranches, using two different correlations (and hence distributions) for those. Therefore, base correlation, though allowing for an easier interpolation, is inconsistent even at the single tranche level.

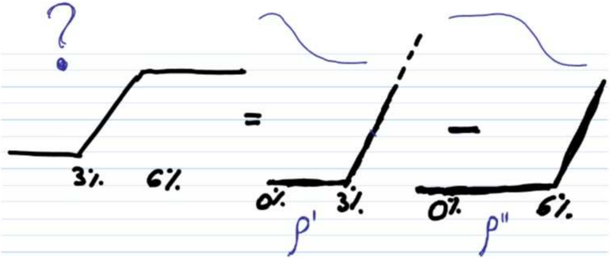


Figure: Source: Brigo (2010)

Introduction: Open Problems

The **One Factor Gaussian Copula** model and **implied base correlation** have become the **market standard** for valuing CDOs and similar instruments.

However, such model presents the following issues:

- inconsistency across the **capital structure**
- inconsistency across **maturities**
- difficult pricing of **bespoke**[1] portfolios and tranches.

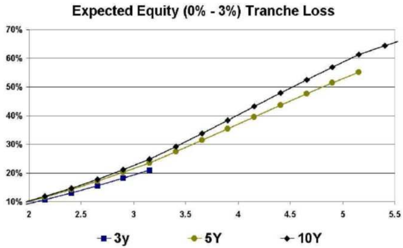
[1] Portfolios constructed specifically for one structured credit derivative, for which there is no liquid information on implied correlation.

Inconsistency Across the Capital Structure

The phenomenon of **correlation skew** means that, in order to match the observed market prices, the correlation must depend on the **position in the capital structure** of the particular tranche being priced.

Inconsistency across the capital structure means that there exist **different models** associated to **different tranches** (compound correlation) or even to the **same tranche** (base correlation).

Inconsistency Across Maturities



The expected $[0\% - 3\%]$ tranche loss calibrated to the 3y, 5y and 10y $[0\% - 3\%]$ tranches on April, 26th 2006 (in the One-Factor Homogeneous Finite Pool Gaussian Copula model) do not overlap. Source: Brigo *et al* (2010)

When valuing the same expected tranche loss $\mathbb{E}_0[\text{Loss}^{tr}_{0\%,3\%}(T_i)]$ for $T_i <$3y, we are using **three different numbers** depending on the tranche maturity even though the pool of underlying credit references is the same!

Outline

Introduction

To value a tranche in the one-factor Gaussian copula framework, we need to know the correlations to apply to the underlying credits so that we can build the portfolio loss distribution for a **range of times** up to the maturity of the trade.

These correlations may depend on time, and in the simplest implementation of the BC framework they are **constant** across all the credits in the portfolio.

The phenomenon of **correlation skew** means that the correlation must depend on the position in the **capital structure** of the particular tranche being priced if the model is to match the observed market prices.

How to Build the Correlation Surface

The correlation surface for standard tranches is built by means of a **calibration to market prices of standard tranches**.

The calibration can be performed in different ways. In the following, we present two of them:

1. **flat correlation** for a given maturity and detachment point
2. **boostrapping algorithm**

Flat Correlation I

This definition of correlation surface follows closely the copula approach, which only models the **terminal distribution** at a given **time horizon** and therefore it cannot be used consistently to introduce a dynamics for the underlying risks.

Correlation Surface

We assume that for a given time horizon T and a given base tranche detachment K (**strike**), the terminal loss distribution can be constructed by imposing a **flat pairwise correlation**

$$\rho := \rho(K, T)$$

among default indicators in the underlying pool.

We call:

- $\rho(K, T)$ the **correlation surface**
- the curve $\rho(K, \bar{T})$, for a given time horizon $\bar{T}$, the **correlation skew** for maturity $\bar{T}$

Flat Correlation II

The points on the correlation surface are obtained by reproducing the market prices of **standard tranches**, so it is possible for instance to build a correlation surface for the DJ-iTraxx Europe index and for the DJ-CDX NA IG.

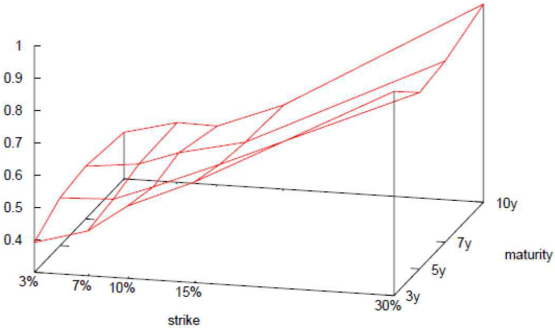


Figure: Base correlation surface for CDX IG Series 11 on January, 8th 2009. Source: Prampolini and Dinnis (2009).

Bootstrapping Algorithm

Alternatively, the base correlation (BC) surface can be obtained by calibration to the liquid tranche market using a bootstrapping algorithm.

Example: CDX IG index

Liquid tranches trade at strikes of 3%, 7%, 10%, 15%, and 30%, and maturities of $5y$, $7y$, and $10y$. The bootstrapping algorithm goes through the following steps:

1. we start from the shortest maturity $(T = 5y)$ and calibrate base correlations across the capital structure, i.e. recursively from the first detachment point till the last
2. for the following maturity $(T = 7y)$, the correlation at $(K, T) = (3\%, 7y)$ is calculated by matching the market price for the 7y equity tranche using the previously calibrated BCs for all times before 5y
3. in this way, the BC surface can be obtained out to the 10y maturity

Time and Strike Dimensions in Pricing I

How do the **time** and **strike dimensions** of the correlation surface impact the **pricing** of base tranches?

Time and Strike Dimensions in Pricing II

The discounted payoff of the legs of a base tranche:

$$\Pi_{\mathrm{PremL}_{0,K}}(0) = \sum_{i=1}^{b} D(0,T_i)\, R^{0,K}_{0,T}(0) \int_{T_{i-1}}^{T_i} \mathrm{OutSt}^{tr}_{0,K}(t)dt$$
$$\approx R^{0,K}_{0,T}(0) \sum_{i=1}^{b} D(0,T_i)\, \alpha_i \left[1 - \mathrm{Loss}^{tr}_{0,K}(T_i)\right]$$

$$\Pi_{\mathrm{ProtL}_{0,K}}(0) = \int_0^T D(0,t) d\mathrm{Loss}^{tr}_{0,K}(t) \approx \sum_{i=1}^{b} D(0,T_i)[\mathrm{Loss}^{tr}_{0,K}(T_i) - \mathrm{Loss}^{tr}_{0,K}(T_{i-1})]$$

depends on the **loss distribution at all times between time 0 and the maturity** $T \equiv T_b$ of the tranche.

A common approximation consists in discretizing the leg payoffs at quarterly intervals in coincidence with the premium payment dates.

Time and Strike Dimensions in Pricing III

In order to preserve consistency, for the pricing of a base tranche the **full term structure** of (time dependent) correlations for a given strike, from the time origin to the maturity of the deal, is used.

For example, the 5 year correlation at 6% *detachment* $\rho(6\%; 5)$ *of DJ-iTraxx is used to build the loss distribution at the 5 year* **point in time** *for all three DJ-iTraxx Series standard base tranches detaching at* 6%*: with 5 year, 7 year and 10 year maturity.*

In contrast, for each time horizon, **only one** point of the correlation skew is involved in the calculation of a base tranche cash flows: the correlation associated with the base tranche detachment point (**strike**).

Standard Tranches

- **Strike dimension**
  One does **not need** any interpolation or extrapolation assumptions in the strike dimension when **calibrating** the correlation surface to the market prices of standard tranches.

- **Time dimension**
  In contrast, **interpolation and extrapolation** in the time dimension are necessary to produce **quarterly loss distributions** for the **pricing** of the standard base tranche legs.

Extrapolation in the Time Dimension

Extrapolation assumptions that extend the correlation surface from the first available tranche maturity backwards to the time origin have an impact on the pricing of the tranches.

For instance, given that the shortest available standard tranche maturity for the DJ-iTraxx Europe is 5 years, a common practice in the market is to build the relevant correlation surface on the assumption that

$$\rho^{iTraxx}(K, t) := \rho^{iTraxx}(K, 5y) \quad t < 5y$$

for any strike K.

This practice may lead to inconsistent expected loss surfaces. In general **time-extrapolation** below the shortest available standard tranche maturity is better performed in the base expected loss space.

Non-Standard Tranches

One of the issues when **evaluating non-standard tranches** is how to extend the pillar information set to obtain the surface via interpolation or extrapolation of the **skew** as appropriate.

The problem of building a **consistent correlation surface** from the pillars associated with a set of standard tranches is still elusive and possibly may not have a solution in the gaussian factor modeling framework.

Outline

Bespoke CDOs: Introduction

Bespoke portfolio

A portfolio constructed specifically for one structured credit derivative, for which there is **no liquid information** on implied correlation.

The pricing of CDO tranches on **bespoke portfolios** depends crucially on the assumptions about the default **correlations** between the names in the underlying pool.

A **liquid index tranche market** allows to obtain implied correlations for a range of standardized (index) portfolios from the observed market prices. It is market practice to achieve this by calibrating a one–factor Gaussian copula model with base correlation (BC) to the liquid indices.

Mapping procedures are then used to obtain **base correlations for bespoke** CDO tranches, allowing pricing and risk-management of these instruments.

Bespoke CDOs: Mapping I

Goal

The goal consists in building a **base correlation surface for the bespoke portfolio** at the strikes of interest, by starting from the standard (index) BC surface through the definition of a **mapping rule.**

The general method is used to generate the BCs for the bespoke portfolio at the **standard maturities**, values at other times being obtained by interpolation as for the standard index.

Bespoke CDOs: Mapping II

The procedure goes through the following steps:

1. we build the base correlation surface for the standard index
2. we select a base tranche with detachment point $K_{Bespoke}$ on the bespoke portfolio
3. through a **mapping rule** we associate to the selected bespoken tranche an **equivalent** base tranche on a standard (index) portfolio with strike K_{Index}^{Eq}
4. the correlation used to price the bespoke tranche is then taken to be the correlation at the equivalent standard strike, i.e.

$$\boxed{\rho_B(K_{Bespoke}, T) = \rho_I(K_{Index}^{Eq}, T)}$$

Bespoke CDOs: Mapping Methods I

Different mapping methods are distinguished by the way they **define equivalence** between a bespoke and a standard tranche.

These methods work by:

- defining a quantity that **measures the risk** in a tranche and treating it as a **market invariant**.
- Calibration to liquid indices indicates the **correlation parameter** that should be used to price a particular **level of risk**.
- This value is then used to price **bespoke** tranches with the **same risk**.

If a particular mapping rule is consistent with the market, then plots of the associated risk measure against correlation should coincide, independent of the particular portfolios we consider.

Bespoke CDOs: Mapping Methods II

Mapping rules used by market practitioners include:

1. No Mapping (NM)
2. At The Money (ATM) mapping
3. Probability Matching (PM)
4. Tranche Loss Proportion (TLP)

For a review of Mapping Methods see Baheti and Morgan (2007), Turc *et al* (2006).

Bespoke CDOs: Mapping Desiderata

The mapping method

- should be **intuitive**, in the sense that changes in the correlation should be easily attributable to changes in the market environment
- should have a plausible **theoretical justification**
- should be **stable** with respect to small changes in the market environment
- should **not introduce arbitrage** in the bespoke where there is none in the index
- should be easy to implement and **always give a solution**
- should be able to map to a **wide range** of portfolios in terms of risk
- should reflect effects of sector or spread concentration in the bespoke portfolio

Bespoke CDOs: No Mapping (NM)

- **Mapping rule**

  The bespoke strike $K_{Bespoke}$ and the equivalent standard strike K_{Index}^{Eq} are trivially related by:

$$K_{Index}^{Eq} = K_{Bespoke}$$

- **Invariant measure of risk**

  The invariant measure of risk is the **tranche strike** and the calibrated BC surface for the standard index is used directly to price bespokes.

- **Theoretical justification**

  In practice, this approach is used as a **benchmark** against which the other mapping methods can be measured. The difference in bespoke pricing between NM and other methods can be attributed purely to the different correlation assumptions made, as differences in the spread and dispersion of the credits between the bespoke and the index are included in the NM calculation.

Bespoke CDOs: At The Money (ATM) I

- **Mapping rule**

  The bespoke and equivalent standard strikes are related by

  $$\frac{K^{Eq}_{Index}}{\text{EPL}_{Index}} = \frac{K_{Bespoke}}{\text{EPL}_{Bespoke}}$$

  where $\text{EPL} = \mathbb{E}[\int_0^T D(0,t)d\text{Loss}(t)]$ is the expected portfolio loss.

- **Invariant measure of risk**

  The EPL sets the **scale** for the level of risk in the portfolio and the invariant measure of risk in a tranche is therefore the **strike as a fraction of this expected loss**. Two tranches are considered equivalent if their strikes are in the same region of the capital structure of their respective portfolios, as measured by the EPL.

- **Theoretical justification**

  This rule has a theoretical justification if we consider mapping between two portfolios with similar compositions, in terms of spread levels and dispersion.

Bespoke CDOs: At The Money (ATM) II

- **Example**

  Consider a bespoke portfolio that contains exactly the same credits as the reference portfolio but the contract specifies a fixed recovery rate that is a constant multiple of the value for the index. In this case, the loss on the bespoke will be a constant multiple of the loss on the index in all possible states of the world.

  Suppose that the recovery rate for the index is 40% while for the bespoke it is 0%. In this case, the losses on the bespoke will always be a factor of 10/6 of those on the index and a 10% tranche on the bespoke will experience the same relative losses as a 6% tranche on the index. The 10% strike on the bespoke should be priced with the same correlation as the 6% strike on the index.

Bespoke CDOs: At The Money (ATM) III

- **Pros**

  Easy to implement.

- **Cons**

  1. It is based on the **first moment of the portfolio loss distribution** and does not consider **spread dispersion**. Two portfolios with the same EPL but very different spread distributions will be priced with the same correlation.

     For example, it does not distinguish between a 40bp homogeneous portfolio (100 CDS trading at 40bp) and a portfolio with all names trading tighter (say at 30bp) except one CDS trading close to default (say at 1000bp). This is a problem for equity tranches because in the first case (homogeneous portfolio), an equity tranche is not very risky while in the second case it is extremely risky.

  2. If the bespoke portfolio is much safer or much riskier than the index, then the **standard equivalent strike** K^{Eq}_{Index} can be above the maximum standard strike or below the minimum standard strike, requiring **extrapolation**.

Bespoke CDOs: Probability Matching (PM) I

- **Mapping rule**

  The bespoke and equivalent standard strikes are related by

$$\mathbb{P}(\text{Loss}_{Index}(T) < K^{Eq}_{Index}, \rho_I) = \mathbb{P}(\text{Loss}_{Bespoke}(T) < K_{Bespoke}, \rho_I)$$

  where $\rho_I := \rho_I(K^{Eq}_{Index}, T)$ is the base correlation calculated on the index surface and $\mathbb{P}(\text{Loss}_{Index}(T))$ and $\mathbb{P}(\text{Loss}_{Bespoke}(T))$ are, respectively, the cumulative loss on the standard and bespoke portfolios at maturity T.

  Two base tranches are priced with the same correlation if they have the **same probability** of being wiped out, which follows from the fact that $\mathbb{P}(\text{Loss} > K) = 1 - \mathbb{P}(\text{Loss} < K)$.

- **Invariant measure of risk**

  The invariant measure of risk is the **probability that an investor loses his entire investment**.

Bespoke CDOs: Probability Matching (PM) II

- **Theoretical justification**

  Changing the correlation in a portfolio does not change the expected loss but rather redistributes losses around the capital structure. The effect of a change in correlation is therefore a **change in the shape** of the underlying loss distribution. The PM mapping method tries to capture this effect by directly comparing the loss distributions of two portfolios.

- **Pros**

  This method works well when taking into account the portfolio dispersion.

Bespoke CDOs: Probability Matching (PM) III

- **Cons**
  1. This method is not completely straightforward as computing the probability of elimination of a bespoke tranche requires a **correlation assumption** which itself depends on the equivalent strike.
  2. Computing equivalent strikes is numerically difficult when using **deterministic recovery rates**. Under this assumption, the loss distribution function is not continuous and subtle numerical schemes are required to create a continuous loss distribution.
  3. The method may not work well if the bespoke portfolio is much riskier than the index, as the equivalent strike may be below the lowest standard strike and **extrapolation assumptions** may be needed.

Bespoke CDOs: Tranche Loss Proportion (TLP) I

- **Mapping rule**

  The bespoke and equivalent standard strikes are related by

$$\frac{\text{ETL}_{Index}(K^{Eq}_{Index}, \rho_I(K^{Eq}_{Index}, T))}{\text{EPL}_{Index}} = \frac{\text{ETL}_{Bespoke}(K_{Bespoke}, \rho_I(K^{Eq}_{Index}, T))}{\text{EPL}_{Bespoke}}.$$

  Here the expected tranche loss function (ETL) is defined as

$$\text{ETL}(K, \rho) = \mathbb{E}[\min(\text{Loss}(T), K)]$$

  and depends on the correlation ρ.

- **Invariant measure of risk**

  The market invariant risk measure is the **fraction of the total expected portfolio loss** which resides **in a given base tranche**.

Bespoke CDOs: Tranche Loss Proportion (TLP) II

- **Theoretical justification**

  The rationale behind the TLP mapping is similar to that behind the PM approach. The correlation skew can be seen as a means of adjusting the loss distribution implied by the one-factor Gaussian copula model to get the correct market prices.The TLP is a good proxy for the **relative risk in a tranche**, so matching this quantity can be seen as a way of tracking the market-implied changes to the Gaussian copula prices.

- **Pros**

  The TLP methodology works well in practice for most bespoke portfolios, either tight or wide:

  1. it always finds a solution as the expected loss ratio of the bespoke tranche is between 0% and 100% and any ratio between 0% and 100% corresponds to one index tranche
  2. it gives equivalent strikes that are most of the time inside the quoted tranches on indices.

Bespoke CDOs: Tranche Loss Proportion (TLP) III

- **Cons**
  1. TLP takes dispersion into account but sometimes in a counterintuitive way.

     Turc et al (2006) present cases in which if one name widens significantly inside a portfolio, the equivalent strikes do not change much compared to other methods. Also, they present cases in which equivalent tranches instead of becoming more junior (which is logical as the bespoke portfolio is riskier if one name comes close to default), become more senior.
  2. Another problem occurs when a name comes close to default. There is a jump in the equivalent strike as soon as one name defaults like in the ATM approach.

Bespoke CDOs: Summary of Mapping Methods

	Invariant Measure of Risk	Pros	Cons
NM	Tranche strike	• Benchmark • Easy to implement	• Too simplistic
ATM	Strike as a fraction of expected ptf. loss	• Easy to implement • Widely used	• No spread dispersion • Exprapolation needed • Problems with names close to default
PM	Probability of loosing the entire investment	• Account for spread dispersion	• Dependent on correlation assumption • Problems with deterministic Rec • Extrapolation needed
TLP	Fraction of expected ptf. loss in a given tranche	• Alwasy find a solution for equivalent strike • No need for extrapolation	• Counter-intuitive spread dispersion • Problems with names close to default

Bespoke CDOs: Test on Mapping I

In general, quantitative tests of mapping methods are hard to find, as there is less transparency in the prices of bespoke tranches than there is for the liquid indices.

Tests on Standard Indices

A useful quantitative test is to investigate how a mapping performs for **two standard indices** by treating one as a bespoke and mapping it to the other. We can then compare the prices obtained from the mapping with the correct values observed in the market.

See Baheti and Morgan (2007), Turc *et al* (2006), for examples of tests on mapping methods and comparison of different methods.

Bespoke CDOs: Test on Mapping II

Following Baheti and Morgan (2007), we present their results of the analysis conducted on January, 31 2007 using as **standard index** the **DJ-CDX IG7** (Investment Grade) and as **bespoke portfolios**:

1. the **DJ-iTraxx S6** index

   This index is chosen because of its **similarity** in terms of expected portfolio loss and average spread levels to the standard index on the reference date (although the DJ-CDX index has significantly greater spread dispersion than DJ-iTraxx)

2. the **DJ-CDX HY7** index

   This mapping represents a more extreme test of the methods because the spread levels are **very different** (the 5y expected loss on HY7 is about 11.4% compared with about 1.6% for IG7).

Standard Index (DJ-CDX IG7)

Term	Tranche	Upfront Or Spread	Swap
5Y	0-3%	20.09	31.37
	3-7%	64.3	
	7-10%	12.3	
	10-15%	4.7	
	15-30%	2.4	
7Y	0-3%	38.95	43.53
	3-7%	176.9	
	7-10%	34.9	
	10-15%	15.0	
	15-30%	6.1	
10Y	0-3%	50.51	55.81
	3-7%	425.9	
	7-10%	92.4	
	10-15%	42.2	
	15-30%	13.7	

Figure: DJ-CDX IG7 tranche and swap quotes on January, 31 2007 Source: Baheti and Morgan (2007)

Bespoke Index (DJ-iTraxx S6) I

Term	Tranche	Market	NM	ATM	PM	TLP
5Y	0-3%	10.53	10.83	9.58	9.74	10.35
	3-6%	42.2	38.5	36.0	43.2	42.9
	6-9%	12.3	7.2	6.4	11.1	10.3
	9-12%	5.6	2.5	1.0	4.4	4.7
	12-22%	2.2	0.8	0.0	2.3	1.9
7Y	0-3%	25.51	27.11	24.21	23.95	25.49
	3-6%	111.5	103.2	90.6	106.8	109.8
	6-9%	33.4	19.7	17.9	27.4	27.6
	9-12%	15.5	7.2	6.4	13.3	13.1
	12-22%	5.2	2.9	0.4	6.3	5.6
10Y	0-3%	40.38	43.02	40.74	39.92	41.48
	3-6%	325.4	317.9	269.9	280.1	302.7
	6-9%	85.5	68.1	54.0	68.7	73.3
	9-12%	39.0	23.3	30.4	40.5	35.9
	12-22%	13.9	11.3	7.3	16.6	16.2

Figure: DJ-iTraxx S6 tranche prices. Market quotes on January, 31 2007 *vs* prices obtained through mapping methods. Source: Baheti and Morgan (2007)

Bespoke Index (DJ-iTraxx S6) II

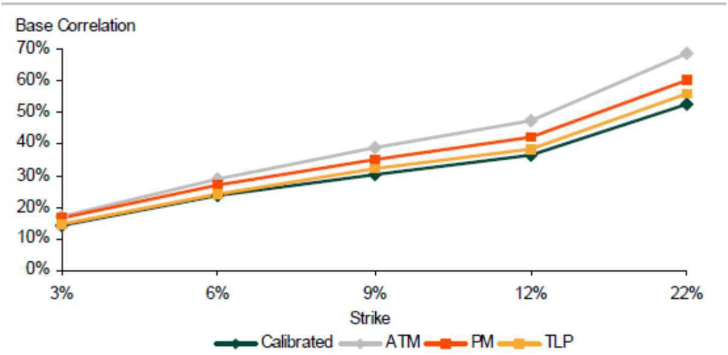


Figure: DJ-iTraxx S6 5y BC skew on January, 31 2007 *vs* BC skew obtained through mapping methods. Source: Baheti and Morgan (2007)

Bespoke Index (DJ-iTraxx S6): Results

- From the price Table:
  - For the 5y and 7y case the TLP approach generally works better, followed by PM.
  - The same is true for the 10y term, although here TLP overestimates the **equity tranche** price. Both PM and ATM work better for this tranche.
  - The NM method significantly overestimates the price of the equity tranche at all maturities, but otherwise it seems to work better than ATM which generally gives very poor results, especially for **senior tranches**.
- The skew obtained from the TLP mapping is the closest to the calibrated curve, followed by PM and then ATM.
- The results are qualitatively the same across different dates and in general the TLP mapping method seems to perform better in a comparison of the DJ-CDX IG7 and DJ-iTraxx S6 indices, followed by PM and then ATM.

Bespoke Index (DJ-CDX HY7)

Term	Tranche	Market	NM	ATM	PM	TLP
5Y	0-10%	68.75	61.73	74.92	76.26	74.79
	10-15%	26.07	19.31	28.85	26.26	22.78
	15-25%	225.7	230.2	155.2	129.9	136.7
	25-35%	56.1	134.2	21.3	27.6	28.1

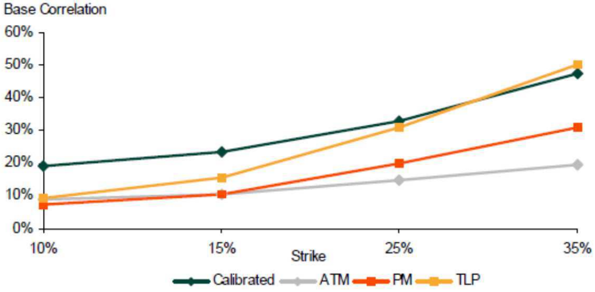


Figure: DJ-CDX HY7 prices (Table) and BC skew (plot) on January, 31 2007 *vs* prices and BC skew obtained through mapping methods. Source: Baheti and Morgan (2007)

Bespoke Index (DJ-CDX HY7): Results

- All the mapping methods **perform badly** in this comparison, consistently putting **too much risk in the equity** tranche and **too little risk in the senior** part of the capital structure (NM is an exception).
- A possible explanation is that there is a limit to the amount a market participant would be willing to pay upfront for 5y protection. The market therefore trades at lower levels for the high-yield equity tranches than that implied from the investment grade universe. The corresponding **correlations** are therefore **higher** than those predicted by mapping to DJ-CDX IG7.
- Since the expected portfolio loss is not a correlation-dependent quantity, the corollary of this is that the mapping methods put **less risk in the senior tranches** than is observed in the market.

Outline

Idea

The **Implied Copula (or Perfect Copula)** approach has been introduced by Hull and White (2005, 2006) to address the following issues associated to the Gaussian Copula/Base Correlation model:

- inconsistency across the capital structure
- pricing of bespoke CDO tranches.

Main Idea

The Implied Copula approach retains the concept of **copula**, but shifts the focus from implied correlations to the **implied probability distribution of hazard rate** paths. Hull and White (2005-2006) show how the Implied Copula model is able to fit exactly market prices.

Goal

Assumptions

- Homogeneous portfolio: all names have the same default probabilities and the same recovery rate
- A number of alternatives for the term structure of hazard rates is chosen. These alternatives are called **hazard rate scenarios** and there exists one for each value of the systemic factor Y

$$\text{One value of factor } Y \Leftrightarrow \text{ one hazard rate scenario}$$

Goal

To search for **probabilities** to apply to any hazard rate scenario such that CDO tranches market quotes are matched.

Hazard Rate Scenarios I

- Given a homogeneous, constant in time, hazard rate λ, the survival probability for name i is given by:

$$\mathbb{Q}(\tau_i > t) = \mathbb{E}[e^{-\lambda t}]$$

- The scenario distribution of hazard rates is defined as:

$$\lambda | Y = \begin{cases} \text{conditional hazard rate} & \text{systemic scenario} & \text{scenario probability} \\ \lambda_1 & Y = y_1 & p_1 \\ \lambda_2 & Y = y_2 & p_2 \\ \vdots & \vdots & \vdots \\ \lambda_s & Y = y_s & p_s \end{cases}$$

Hazard Rate Scenarios II

- The **conditional default probability** of a single name in scenario j is:

$$\mathbb{Q}(\tau_i < t | Y = y_j) = 1 - e^{-\lambda_j t}$$

- The **unconditional default probability** of a single name is obtained by summing over all possible scenarios:

$$\mathbb{Q}(\tau_i < t) = \mathbb{E}[\mathbb{Q}(\tau_i < t | Y)] = \sum_{j=1}^{s} p_j \, \mathbb{Q}(\tau_i < t | Y = y_j) = \sum_{j=1}^{s} p_j \left(1 - e^{-\lambda_j t}\right)$$

- Conditional on Y all default times τ_i are independent and have the same hazard rate, given by the hazard rate scenarios. Each **tranche price** can be computed by summing over all tranches across different scenarios, each weighted by the corresponding scenario probability

$$\text{Tranche}^{A,B}(0, R) = \sum_{j=1}^{s} p_j \, \text{Tranche}^{A,B}(0, R; \{\lambda_j\})$$

Tranches in each scenarios can be computed with different methods (e.g. Monte Carlo, large pool model approximation, etc...)

Implied Copula Calibration

Goal

Given the values of hazard rate in each scenario (e.g. Hull and White specify them **exogenously**), calibration to market quotes allows to find the probability weights p_j of each scenario.

Usually the minimization involves:

- for a given maturity (e.g. $T = 5y$), 5 (market tranche premia) + 1 (market index quote)
- $s \sim 30$ scenarios
- smoothing tricks (e.g. penalty functions which penalize changes in convexity)
- a non flat recovery rate (otherwise the minimization may not yield a solution). According to Hamilton *et al* (2005)

$$\text{Rec}_j = 0.52 - 6.9\left(1 - e^{-\lambda_j 5y}\right)$$

Implied Copula Calibration in Formula

$$\begin{aligned}
\mathbf{p}^* = \text{argmin}_{p_1,\dots,p_s} \sum_{i=1}^{5} & \left[\sum_{j=1}^{s} p_j \text{Tranche}^{A_i,B_i}(0, R_{0,5y}^{A_i,B_i,\text{mkt}}; \{\lambda_j, \text{Rec}_j\}) \right]^2 \\
& + \left[125 \left(\sum_{j=1}^{s} p_j \, R_{0,5y}^{\text{mkt}} \sum_{k=1}^{b} \alpha_k \, P^{\text{mkt}}(0, T_k) \, e^{-\lambda_j T_k} \right.\right. \\
& \left.\left. - \sum_{j=1}^{s} p_j \, \text{LGD}_j \sum_{k=1}^{b} P^{\text{mkt}}(0, T_k) \left(e^{-\lambda_j T_{k-1}} - e^{-\lambda_j T_k} \right) \right) \right]^2 \\
& + c \sum_{j=2}^{s-1} \frac{2(p_{j+1} + p_{j-1} - 2p_j)^2}{e^{-\lambda_{j-1} 5y} - e^{-\lambda_{j+1} 5y}}
\end{aligned}$$

Implied Copula Calibration: Results I

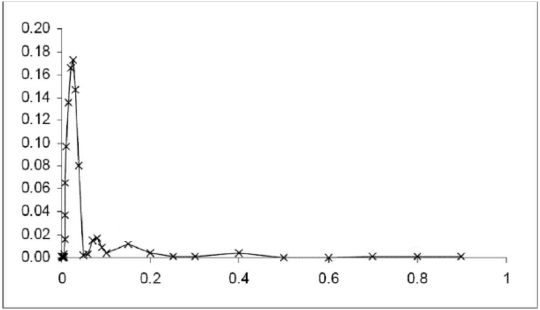


Figure: Implied distribution for the 5y default rates for DJ-iTraxx on August, 24 2004. Source: Hull and White (2005)

The 5y default rate peaks at 2.5%. The chance that the 5y cumulative default rate will be more than 10% is about 2.6%.

Implied Copula Calibration: Results II

A loss distribution consistent across the capital structure, for a single maturity, features **modes in the right tail**. These probability masses on the far right tail imply the possibility of default for **large clusters** (possibly sectors) of names of the economy.

According to Longstaff and Rajan (2007)

"*if the market expects defaults to cluster in some way, this clearly has implications for the behavior of the corresponding stocks – clustered default risk in bond markets necessarily implies related non diversifiable event risk in the equity market. As another example, the pricing of senior CDO tranches opens a new window on the upper tail of the distribution of potential credit losses in the economy. This information is essential in understanding the systemic risk faced by financial institutions the possibility of contagion across business and credit cycles and the risk of credit crunches and liquidity crises in the capital markets.*"

Implied Copula Approach through the Crisis

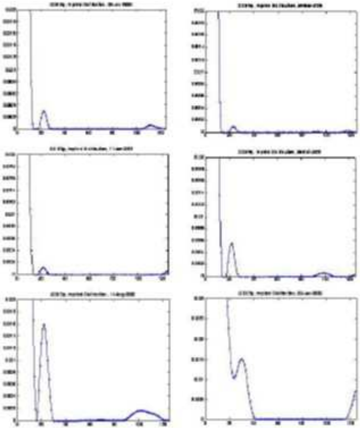


Figure: Implied distribution calibrated to the DJ-CDX 5y tranches from March 2005 to January 2009. Source: Torresetti *et al* (2006b)

Implied Copula: Bespoke Tranches

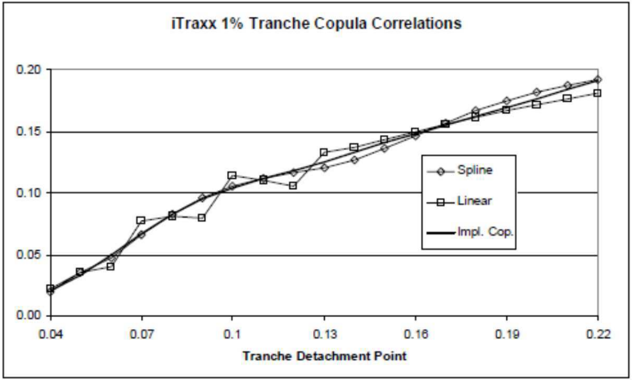


Figure: Tranche correlations when the tranche width is 1% calculated using the base correlation and Implied Copula approach. Linear and spline interpolation schemes are used in the implementation of the base correlation case. Source: Hull and White (2006).

Conclusions

The Implied Copula approach:

- captures the phenomenon of **clustered (sector) defaults** associated to masses in the far right tail of the loss distribution
- calibrates **consistently across the capital structure**

but **cannot calibrate across maturities**, since it is inherently a **static model**.

Outline

Beyond Copula

We are left with the issue of addressing **inconsistency across maturities**.

Copula-based approaches are inherently static and does not allow to solve the problem.

Different approaches have been introduced to tackle the problem:

1. **Expected Tranche Loss (ETL)** method
2. **dynamical loss models**

Expected Tranche Loss (ETL) Method I

The Expected Tranche Loss (ETL) method has been introduced for the first time by Torresetti *et al* (2006a).

The idea is based on the observation that expected tranche losses for different detachment points and maturities can be viewed as the **building blocks** on which synthetic CDO formulas are built with linear operations (but under some non-linear constraints):

$$R_{0,T}^{A,B}(0) = \frac{\sum_{i=1}^{b} P(0, T_i)[\mathbb{E}[\text{Loss}_{A,B}^{tr}(T_i)] - \mathbb{E}[\text{Loss}_{A,B}^{tr}(T_{i-1})]]}{\sum_{i=1}^{b} P(0, T_i)\alpha_i[1 - \mathbb{E}(\text{Loss}_{A,B}^{tr}(T_i))]}$$

If a term structure of tranche spreads $R_{0,T}^{A,B}(0)$ for different maturities T is given, then it is possible to **strip back** the expectations in a **model independent way**, under some **minimal interpolation assumptions**.

Expected Tranche Loss (ETL) Method II

- The ETL implied surface can be used to value **tranches with nonstandard attachments and maturities** as an alternative to implied correlation.
- Deriving **hedge ratios** as well as **extrapolation** may prove **difficult**.
- ETL is not really a model but rather a **model-independent stripping algorithm**, although the particular choice of interpolation (e.g. linear or spline) may be viewed as a modeling choice.
- ETL is not helpful in pricing more advanced derivatives such as tranche options or cancelable tranches.

ETL does **not specify an explicit dynamics** for the loss of the pool but it represents an interpolation method.

Dynamical Loss Models I

Idea

In the framework of Dynamical Loss approaches the modeling focus is directed towards **aggregated objects**, such as the **pool loss** and the **number of defaults**, rather then single name defaults, building up the portfolio loss.

Different models have been proposed in **literature** among which:

- the **Generalized-Poisson Loss (GPL)** model by Brigo *et al* (2006a, 2006b) (and extensions)
- other models by Bennani (2005), Schönbucher (2005), Di Graziano and Rogers (2005), Elouerkhaoui (2006), Sidenius *et al* (2005) and Errais, Giesecke and Goldberg (2006).

Dynamical Loss Models II

Pros

- **Consistent calibration** across capital structure and maturities.
- Able to price tranche options, forward starting CDOs and in general loss dynamics dependent payoffs.
- Able to capture **clustered defaults** in some sectors (systemic risk), represented by probability masses in the far right tail of the density function.

Cons

- **Difficulty** of all loss models to account for **single name** data and to allow for single name sensitivities.
- **Partial hedges** with respect to single name are not possible.
- Even the few models achieving single name consistency have **not** been developed and **tested enough** to become operational on the trading floor.

Generalized Poisson (Cluster) Loss Model (GPCL) I

The Generalized Poisson (Cluster) Loss Model (GPCL) introduced by Brigo *et al* (2007) models the loss as a **sum of independent Poisson processes**, each associated to the defaults of a different number of entities, and capped at the pool size to avoid infinite defaults.

The intuition of these driving Poisson processes is that of **defaults of sectors**.

GPL model is able to reproduce the **tail multi-modal feature** that the Implied Copula approach proved to be indispensable to reprice accurately the market spreads of CDO tranches on a single maturity.

Generalized Poisson (Cluster) Loss Model (GPCL) II

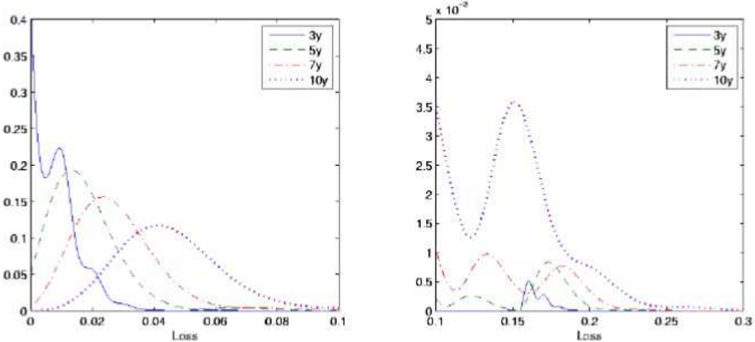


Figure: Loss distribution evolution of the GPL model with minimum jump size of 50bp at all the quoted maturities up to ten years, drawn as a continuous line. Source: Brigo *et al* (2010)

Outline

The Market's Choice

Despite all the issues and inconsistencies related to the **Gaussian Copula/implied correlation** approach, such model is still used in its base correlation formulation, although under some extensions such as **random recovery** (see Prampolini and Dinnis (2009))

The reasons for this are complex:

- the difficulty of all the loss models in accounting for single name data and to allow for single name sensitivities and partial hedges with respect to single names. As these issues are crucial in many situations, the market practice remains with base correlation
- loss models have not been developed and tested enough to become operational on a trading floor or in a large risk management platform
- when one model has been coded in the libraries of a financial institution, changing the model implies a long path involving a number of issues that have little to do with modeling and more to do with IT problems, integration with other systems etc...

CDOs on Other Asset Classes

Here we have described in detail Synthetic Corporate CDOs.

However, CDOs, especially Cash, are available on other asset classes, such as loans (CLO), residential mortgage portfolios (RMBS), commercial mortgages portfolios (CMBS), etc... For many of these CDOs, and especially **RMBS**, quite related to the asset class that triggered the crisis, the problem is in the **data** rather than in the models.

Notice that **synthetic CDOs** on corporates, where the Implied correlation/copula model has been used massively, are **not the ones that lead to the major losses**!

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