

(Informal) Introduction to Stochastic Calculus

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Disclaimer

The opinion expressed here are solely those of the author and do not represent in any way those of her employers

Main References

- D. Brigo and F. Mercurio

Interest Rate Models – Theory and Practice. With Smile, Inflation and Credit
Springer (2006)

Appendix C

- S. Shreve

Stochastic Calculus for Finance II
Springer (2004)

Chapters 1-6

Outline

- 1 From Deterministic to Stochastic Differential Equations
- 2 Ito's formula
- 3 Examples
- 4 Change of Measure
- 5 No-Arbitrage Pricing
- 6 Exercises

Preamble

[...] In **continuous-time finance**, we work within the framework of a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$. We normally have a fixed final time T , and then have a filtration, which is a collection of σ -algebras $\{\mathcal{F}(t); 0 \leq t \leq T\}$, indexed by the time variable t . We interpret $\mathcal{F}(t)$ as the **information** available at time t . For $0 \leq s \leq t \leq T$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. In other words, information increases over time. Within this context, an **adapted stochastic process** is a collection of random variables $\{X(t); 0 \leq t \leq T\}$, also indexed by time, such that for every t , $X(t)$ is $\mathcal{F}(t)$ -measurable; the information at time t is sufficient to evaluate the random variable $X(t)$. We think of $X(t)$ as the **price of some asset** at time t and $\mathcal{F}(t)$ as the information obtained by watching all the prices in the **market** up to time t .

Two important classes of adapted stochastic processes are **martingales** and **Markov processes**.

Shreve, Chapter 2

Probability Space: Definition

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be interpreted as an **experiment**, where:

- $\omega \in \Omega$ represents a **generic result** of the experiment
- Ω is the set of **all possible outcomes** of the random experiment
- a subset $A \subset \Omega$ represents an **event**
- \mathcal{F} is a **collection of subsets** of Ω which forms a **σ -algebra** (σ -field)
- \mathbb{P} is a **probability measure**

(See Shreve, Chapter 1)

Information

In order to price a derivative security in the no-arbitrage framework, we need to model mathematically the **information** on which our future decisions (**contingency plans**) are based.

- Given a non empty set Ω and a positive number T , assume that for each $t \in [0, T]$ there is a σ -algebra \mathcal{F}_t .
 \mathcal{F}_t represents the **information up to time t** .
- If $t \leq u$, every set in \mathcal{F}_t is also in \mathcal{F}_u , i.e. $\mathcal{F}_t \subseteq \mathcal{F}_u \subseteq \mathcal{F}$.
“The information increases in time”, never exceeding the whole set of events
- The family of σ -fields $(\mathcal{F}_t)_{t \geq 0}$ is called **filtration**.
A filtration tells us the information that we will have at future times, i.e. when we get to time t we will know for each set in \mathcal{F}_t whether the true ω lies in that set.

(See Shreve, Chapter 2)

Random Variables and Stochastic Processes: Definitions

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t)_t$, $0 \leq t \leq T$:

- A **random variable** X is defined as a **measurable function** from the set of possible outcomes Ω to \mathbb{R} , i.e. $X : \Omega \rightarrow \mathbb{R}$
(+ some technical conditions – See Shreve, Chapter 1)
- A **stochastic process** $(X_t)_t$ is defined as a **collection of random variables, indexed** by $t \in [0, T]$.
For each experiment result ω , **the map** $t \mapsto X_t(\omega)$ is called the **path** of X associated to ω .
- A stochastic process is said to be **adapted** if, for each t , the random variable X_t is \mathcal{F}_t -measurable.

Expectations: Definitions

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- The **expectation** (or expected value) of X is defined as

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$$

provided that X is integrable i.e. $\int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty$

- Let $\mathcal{G} \subset \mathcal{F}$ be a sub-algebra of \mathcal{F} .

The **conditional expectation** of X given \mathcal{G} is any **random variable** which satisfies:

- Measurability:** $\mathbb{E}[X|\mathcal{G}]$ is \mathcal{G} -measurable
- Partial averaging:**

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega) \quad \forall A \in \mathcal{G}$$

(See Shreve, Chapter 1,2)

Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ and X, Y be (integrable) random variables.

- **Linearity of conditional expectations**

$$\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$$

- **Taking out what is known**

If Y and XY are integral r.v and X is \mathcal{G} -measurable then:

$$\mathbb{E}[XY|\mathcal{G}] = X \mathbb{E}[Y|\mathcal{G}]$$

- **Independence**

If X is integrable and independent of \mathcal{G}

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$$

- **Iterated conditioning (tower rule)**

If $\mathcal{H} \subset \mathcal{G}$ and X is an integrable r.v., then:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$$

(See Shreve, Chapter 2)

Martingales I

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_t$, where $0 \leq t \leq T$. Consider a process $(X_t)_t$ satisfying the following conditions:

- **Measurability:**

\mathcal{F}_t includes all the information on X_t up to time t , i.e. $(X_t)_t$ is **adapted** to $(\mathcal{F}_t)_t$.

- **Integrability:**

The relevant expected values exist.

If:

$$\mathbb{E}[X_T | \mathcal{F}_t] = X_t \quad \text{for each } 0 \leq t \leq T \quad (1)$$

we say the process is a **martingale**. It has no tendency to rise or fall.

Martingales II

In other words...

- if t is the present time, the expected value at a future time T , given the current information, is equal to the **current value**
- a martingale represents a picture of a **fair game**, where it is not possible to lose or gain on average
- the martingale property is suited to model the **absence of arbitrage**, i.e. there is no safe way to make money from nothing (**no free lunch**)

► [Go to No-Arbitrage Pricing](#)

Submartingales, Supermartingales and Semimartingales

- A **submartingale** is a similar process $(X_t)_t$ satisfying:

$$\mathbb{E}[X_T | \mathcal{F}_t] \geq X_t \quad \text{for each } t \leq T$$

i.e. the expected value of the process grows in time.

- A **supermartingale** satisfies:

$$\mathbb{E}[X_T | \mathcal{F}_t] \leq X_t \quad \text{for each } t \leq T$$

i.e. the expected value of the process decreases in time.

- A process $(X_t)_t$ that is either a submartingale or a supermartingale is termed a **semimartingale**.

► Go to Martingales: Exercises

Quadratic Variation: Definition

Given a stochastic process Y_t with continuous paths, its **quadratic variation** is defined as:

$$\langle Y \rangle_T = \lim_{\|\Pi\| \rightarrow 0} \sum_{i=1}^n [Y_{t_i}(\omega) - Y_{t_{i-1}}(\omega)]^2$$

where $0 = t_0 < t_1 < \dots < t_n = T$ and $\Pi = \{t_0, t_1, \dots, t_n\}$ is a **partition** of the interval $[0, T]$. $\|\Pi\|$ represents the maximum step size of the partition.

In form of a **second order integral**:

$$\langle Y \rangle_T = " \int_0^T [dY_s(\omega)]^2 "$$

or in **differential form**:

$$"d\langle Y \rangle_t = dY_t(\omega) dY_t(\omega)"$$

Quadratic Variation: Deterministic Process

A process whose paths are **differentiable** for almost all ω satisfies $\langle Y \rangle_t = 0$.

If Y is the **deterministic process** $Y : t \mapsto t$, then $dY_t = 0$ and

$$dt \, dt = 0$$

Quadratic Covariation: Definition

The **quadratic covariation** of two stochastic processes Y_t and Z_t , with continuous paths, is defined as follows:

$$\langle Y, Z \rangle_T = \lim_{\|\Pi\| \rightarrow 0} \sum_{i=1}^n [Y_{t_i}(\omega) - Y_{t_{i-1}}(\omega)] [Z_{t_i}(\omega) - Z_{t_{i-1}}(\omega)]$$

or in form of a **second order integral**:

$$\langle Y, Z \rangle_T = \int_0^T dY_s(\omega) dZ_s(\omega)$$

or in **differential form**:

$$d\langle Y, Z \rangle_t = dY_t(\omega) dZ_t(\omega)$$

Deterministic Differential Equations (DDE)

EXAMPLE: Population Growth Model

- Let $x(t) = x_t \in \mathbb{R}, x_t \geq 0$, denote the population at time t , and assume for simplicity a constant (proportional) population growth rate, so that the change in the population at t is given by the **deterministic differential equation**:

$$dx_t = K x_t dt, \quad x_0$$

where K is a real constant.

- Assume now that **x_0 is a random variable $X_0(\omega)$** and that the population growth is modeled by the following differential equation:

$$dX_t(\omega) = K X_t(\omega) dt, \quad X_0(\omega)$$

From Deterministic to Stochastic Differential Equations

- The solution to this equation is:

$$X_t(\omega) = X_0(\omega) e^{Kt}$$

where all the randomness comes from the initial condition $X_0(\omega)$.

- As a further step, suppose that even K is not known for certain, but that also our knowledge of K is perturbed by some randomness, which we model as the increment of a stochastic process $\{W_t(\omega)\}$, $t \geq 0$, so that

$$dX_t(\omega) = (K dt + dW_t(\omega)) X_t(\omega), \quad X_0(\omega), \quad K \geq 0 \quad (2)$$

where $dW_t(\omega)$ represents a **noise process** that adds randomness to K .

Eq. (2) represents an example of **stochastic differential equation (SDE)**.

Stochastic Differential Equations (SDE)

- More generally, a **SDE** is written as

$$dX_t(\omega) = f_t(X_t(\omega)) dt + \sigma_t(X_t(\omega)) dW_t(\omega), \quad X_0(\omega) \quad (3)$$

The function **f** corresponds to the deterministic part of the SDE and is called the **drift**. The function σ_t is called the **diffusion coefficient**.

The randomness enters the SDE from two sources:

- the **noise term** $dW_t(\omega)$
 - the **initial condition** $X_0(\omega)$
- The solution **X** of the SDE is also called a **diffusion process**.
In general the corresponding paths $t \mapsto X_t(\omega)$ are continuous.

Noise Term

Which kind of **process** is suitable to describe the
noise term $dW_t(\omega)$?

Brownian Motion

- The process whose increments $dW_t(\omega)$ are candidates to represent the noise process in the SDE given by Eq. (3) is the **Brownian motion**.
- The **most important properties** of Brownian motion are that:
 - it is a **martingale**
 - it accumulates **quadratic variation at rate one per unit time**.
This makes **stochastic calculus** different from **ordinary calculus**.

Brownian Motion: Definition

Definition

Given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, for each $\omega \in \Omega$ there is a **continuous function** W_t , $t \geq 0$ such that it depends on ω and $\mathbf{W}_0 = \mathbf{0}$. Then, \mathbf{W}_t is a **Brownian motion** if and only if for any $0 < s < t < u$ and any $h > 0$ it has:

- **Independent increments:** $W_u(\omega) - W_t(\omega)$ independent of $W_t(\omega) - W_s(\omega)$
- **Stationary increments:** $W_{t+h}(\omega) - W_{s+h}(\omega) \sim W_t(\omega) - W_s(\omega)$
- **Gaussian increments:** $W_t(\omega) - W_s(\omega) \sim \mathcal{N}(0, t - s)$

Although the paths are continuous, they are (almost surely) **nowhere differentiable**, i.e.

$$\dot{W}_t(\omega) = \frac{d}{dt} W_t(\omega)$$

does not exist.

Property 1: Martingality

Brownian motion is a **martingale**.

PROOF

Let $0 \leq s \leq t$. Then:

$$\begin{aligned}\mathbb{E}[W_t | \mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s | \mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s | \mathcal{F}_s] + \mathbb{E}[W_s | \mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s] + W_s \\ &= W_s\end{aligned}$$

□

Property 2: Quadratic Variation

The **quadratic variation** of a Brownian motion W is given *almost surely* by:

$$\langle W \rangle_T = T \quad \text{for each } T$$

or, equivalently:

$$dW_t(\omega) dW_t(\omega) = dt$$

Brownian motion **accumulates quadratic variation at rate one per unit time**

*This comes from the fact that the Brownian motion moves so quickly that **second order effects cannot be neglected**. Instead, a process with differentiable trajectories cannot move so quickly and therefore second order effects do not contribute (derivatives are continuous).*

See Shreve, Chapter 3 for the proof.

Property 2.bis: Quadratic Covariation

If W is a **Brownian motion** and Z a **deterministic** process $t \mapsto z_t$ it follows:

$$\langle W, z \rangle_T = 0 \quad \text{for each } T$$

or, equivalently:

$$dW_t(\omega) dz_t = 0$$

Integral Form of an SDE

The integral form of the general SDE, given by Eq. (3), i.e.

$$X_t(\omega) = X_0(\omega) + \int_0^t f_s(X_s(\omega)) ds + \int_0^t \sigma_s(X_s(\omega)) dW_s(\omega) \quad (4)$$

contains two types of integrals:

- $\int_0^t f_s(X_s(\omega)) ds$ is a **Riemann-Stieltjes integral**.
- $\int_0^t \sigma_s(X_s(\omega)) dW_s(\omega)$ is a **stochastic generalization of the Riemann-Stieltjes integral**, such that the result depends on the chosen points of the sub-partitions used in the limit that defines the integral.

Stochastic Integrals

A **stochastic integral** is an integral of the type:

$$\int_0^T \phi_t(\omega) dW_t(\omega)$$

where ϕ_t is an adapted process, and W_t a Brownian motion.

*The problem we face when trying to **assign a meaning** to the above integral, is that the Brownian motion paths cannot be differentiated w.r.t. time. If $g(t)$ is a differentiable function, then we can define:*

$$\int_0^T \phi_t(\omega) dg(t) = \int_0^T \phi_t(\omega) g'(t) dt$$

where the right end side is an ordinary integral w.r.t time. This will not work with Brownian motion.

Stochastic Integrals: Partitions

Take the interval $[0, T]$ and consider the following partitions of this interval:

$$T_i^n = \min \left(T, \frac{i}{2^n} \right) \quad i = 0, 1, \dots, \infty$$

where n is an integer.

- For all $i > 2^n T$ all terms collapse to T , i.e. $T_i^n = T$.
- For each n we have such a partition, and when n increases the partition contains more elements, giving a better discrete approximation of the continuous interval $[0, T]$.

Ito and Stratonovich Integrals

Then define the integral as:

$$\int_0^T \phi_s(\omega) dW_s(\omega) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \phi_{t_i^n}(\omega) [W_{T_{i+1}^n}(\omega) - W_{T_i^n}(\omega)]$$

where t_i^n is any point in the interval $[T_i^n, T_{i+1}^n)$. By choosing:

- $t_i^n := T_i^n$ (initial point), we have the **Ito integral**;
- $t_i^n := \frac{T_i^n + T_{i+1}^n}{2}$ (middle point), we have the **Stratonovich integral**.

Properties of Ito Integral

Let $I(t) = \int_0^t \phi(u) dW(u)$ be an **Ito integral**. $I(t)$ has the following properties:

- 1 **Continuity:** as a function of the upper limit t , the paths of $I(t)$ are continuous
- 2 **Adaptivity:** for each t , $I(t)$ is \mathcal{F}_t -measurable
- 3 **Linearity:** for $J(t) = \int_0^t \gamma(u) dW(u)$ then

$$I(t) \pm J(t) = \int_0^t [\phi(u) \pm \gamma(u)] dW(u)$$

$$\text{and } cI(t) = \int_0^t c \phi(u) dW(u)$$

- 4 **Martingality:** $I(t)$ is a martingale
- 5 **Ito isometry:** $\mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t \phi^2(u) du]$
- 6 **Quadratic variation** $\langle I \rangle_t = \int_0^t \phi^2(u) du$

Ito Integral vs Stratonovich Integral

$$\text{Ito} \Rightarrow \int_0^t W_s(\omega) dW_s(\omega) = \frac{W_t(\omega)^2}{2} - \frac{1}{2} t$$

$$\text{Stratonovich} \Rightarrow \int_0^t W_s(\omega) dW_s(\omega) = \frac{W_t(\omega)^2}{2}$$

Ito

- Martingale property
- No standard chain rule

Stratonovich

- No martingale property
- Standard chain rule

Solution to a General SDE

Consider the general SDE, given by Eq. (3):

$$dX_t(\omega) = f_t(t, X_t(\omega)) dt + \sigma_t(t, X_t(\omega)) dW_t(\omega), \quad X_0(\omega)$$

Existence and uniqueness of the solution are guaranteed by:

- **Lipschitz continuity:**

$$|f(t, x) - f(t, y)| \leq C|x - y| \quad \text{and} \quad |\sigma(t, x) - \sigma(t, y)| \leq C|x - y| \quad x, y \in \mathbb{R}^d$$

- **Linear growth bound:**

$$|f(t, x)| \leq D(1 + |x|) \quad \text{and} \quad |\sigma(t, x)| \leq D(1 + |x|) \quad x \in \mathbb{R}^d$$

(See Øksendal (1992) for the details.)

Interpretation of the Coefficients: DDE Case

For a **deterministic differential equation**

$$dx_t = f(x_t)dt$$

with f a smooth function, we have:

$$\lim_{h \rightarrow 0} \left. \frac{x_{t+h} - x_t}{h} \right|_{x_t=y} = f(y)$$

$$\lim_{h \rightarrow 0} \left. \frac{(x_{t+h} - x_t)^2}{h} \right|_{x_t=y} = 0$$

Interpretation of the Coefficients: SDE Case

For a **stochastic differential equation**

$$dX_t(\omega) = f(X_t(\omega))dt + \sigma(X_t(\omega))dW_t(\omega)$$

functions f and σ can be interpreted as:

$$\lim_{h \rightarrow 0} \mathbb{E} \left\{ \frac{X_{t+h}(\omega) - X_t(\omega)}{h} \middle| X_t = y \right\} = f(y)$$

$$\lim_{h \rightarrow 0} \mathbb{E} \left\{ \frac{[X_{t+h}(\omega) - X_t(\omega)]^2}{h} \middle| X_t = y \right\} = \sigma^2(y)$$

The second limit is non-zero because of the *infinite velocity* of the Brownian motion. Moreover, if the **drift f is zero**, the solution is a **martingale**.

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Deterministic Case

For a deterministic differential equation such as

$$dx_t = f(x_t)dt$$

given a smooth transformation $\phi(t, x)$, we can write the evolution of $\phi(t, x_t)$ via the **standard chain rule**:

$$d\phi(t, x_t) = \frac{\partial \phi}{\partial t}(t, x_t)dt + \frac{\partial \phi}{\partial x}(t, x_t)dx_t \quad (5)$$

Stochastic Case: Ito's Formula I

Let $\phi(t, x)$ be a smooth function and $X_t(\omega)$ the unique solution to the SDE (3).

The **chain rule – Ito's formula** reads as:

$$\begin{aligned} d\phi(t, X_t(\omega)) = & \frac{\partial \phi}{\partial t}(t, X_t(\omega))dt + \frac{\partial \phi}{\partial x}(t, X_t(\omega))dX_t(\omega) \\ & + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t, X_t(\omega))dX_t(\omega)dX_t(\omega) \end{aligned} \quad (6)$$

or, in compact notation:

$$d\phi(t, X_t) = \frac{\partial \phi}{\partial t}(t, X_t)dt + \frac{\partial \phi}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t, X_t)d\langle X \rangle_t$$

Stochastic Case: Ito's Formula II

The term $dX_t(\omega)dX_t(\omega)$ can be developed by recalling the rules on quadratic variation and covariation:

$$dW_t(\omega)dW_t(\omega) = dt, \quad dW_t(\omega)dt = 0, \quad dt dt = 0$$

thus giving:

$$\begin{aligned} d\phi(t, X_t(\omega)) = & \left[\frac{\partial \phi}{\partial t}(t, X_t(\omega)) + \frac{\partial \phi}{\partial x}(t, X_t(\omega))f(X_t(\omega)) \right. \\ & \left. + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t, X_t(\omega))\sigma^2(X_t(\omega)) \right] dt \\ & + \frac{\partial \phi}{\partial x}(t, X_t(\omega))\sigma(X_t(\omega))dW_t(\omega) \end{aligned}$$

► Go to Ito's Formula: Exercises

Leibniz Rule

It applies to **differentiation of a product of functions**.

- **Deterministic Leibniz rule**

For deterministic and differentiable functions x and y :

$$d(x_t y_t) = x_t dy_t + y_t dx_t$$

- **Stochastic Leibniz rule**

For two diffusion processes $X_t(\omega)$ and $Y_t(\omega)$:

$$d(X_t(\omega) Y_t(\omega)) = X_t(\omega) dY_t(\omega) + Y_t(\omega) dX_t(\omega) + dX_t(\omega) dY_t(\omega)$$

or, in compact notation:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t$$

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Linear SDE with Deterministic Diffusion Coefficient I

A SDE is **linear** if both its drift and diffusion coefficients are first order polynomials in the state variable.

Consider the **particular case**:

$$dX_t(\omega) = (\alpha_t + \beta_t X_t(\omega)) dt + v_t dW_t(\omega), \quad X_0(\omega) = x_0 \quad (7)$$

where α, β, v are deterministic functions of time, regular enough to ensure existence and uniqueness of the solution.

The **solution** is:

$$\begin{aligned} X_t(\omega) &= e^{\int_0^t \beta_s ds} \left[x_0 + \int_0^t e^{-\int_0^s \beta_u du} \alpha_s ds + \int_0^t e^{-\int_0^s \beta_u du} v_s dW_s(\omega) \right] \\ &= x_0 e^{\int_0^t \beta_s ds} + \int_0^t e^{\int_s^t \beta_u du} \alpha_s ds + \int_0^t e^{\int_s^t \beta_u du} v_s dW_s(\omega) \end{aligned} \quad (8)$$

Linear SDE with Deterministic Diffusion Coefficient II

The **distribution** of the solution $X_t(\omega)$ is **normal** at each time t :

$$X_t \sim \mathcal{N} \left(x_0 e^{\int_0^t \beta_s ds} + \int_0^t e^{\int_s^t \beta_u du} \alpha_s ds, \int_0^t e^{2 \int_s^t \beta_u du} v_s^2 ds \right)$$

Major examples: Vasicek SDE (1977) and Hull and White SDE (1990).

Vasicek Model (1977)

The Vasicek model has been introduced in 1977 to describe the evolution of interest rates. The dynamics of the short rate process r_t is given by the following SDE:

$$dr_t = k(\theta - r_t) dt + \sigma dW_t$$

with k , θ and σ strictly positive constants, and initial condition r_s .

1 Solution:

$$r_t = \theta + (r_s - \theta) e^{-k(t-s)} + \sigma \int_s^t e^{k(u-t)} dW_u$$

2 Distributional properties of the solution:

$$\mathbb{E}[r_t | \mathcal{F}_s] = r_s e^{-k(t-s)} + \theta [1 - e^{-k(t-s)}]$$

$$\text{var}[r_t | \mathcal{F}_s] = \frac{\sigma^2}{2k} [1 - e^{-2k(t-s)}]$$

Vasicek Model (1977): Zero Coupon Bonds

The price at time t of a zero coupon bond with maturity T is:

$$P(t, T) = A(t, T) e^{-B(t, T) r_t}$$

where:

$$A(t, T) = \left[\frac{2h \exp\{(k + h)(T - t)/2\}}{2h + (k + h)(\exp\{(T - t)h\} - 1)} \right]^{2k\theta/\sigma^2}$$
$$B(t, T) = \frac{2(\exp\{(T - t)h\} - 1)}{2h + (k + h)(\exp\{(T - t)h\} - 1)}$$
$$h = \sqrt{k^2 + 2\sigma^2}$$

Hull-White Model (1990-1994)

- **SDE:**

$$dr_t = (\vartheta(t) - k r_t) dt + \sigma dW_t$$

where k , σ are positive constants, and ϑ is chosen so as to **exactly fit the term structure of interest rates** currently observed in the market.

- The **solution**, r_t , can be expressed in terms of the **Vasicek solution**, x_t , and a **deterministic shift** extension $\varphi(t; \alpha)$, which captures the initial term structure of interest rates:

$$r_t = x_t + \varphi(t; \alpha)$$

with $\alpha = (k, \theta, \sigma)$ the parameters of the Vasicek model.

Hull-White Model (1990-1994): Market Instruments

- ZCB:**

$$\begin{aligned} P(t, T) &= \mathbb{E} \left[e^{-\int_t^T r_s ds} \right] = e^{-\int_t^T \varphi(s; \alpha)} \mathbb{E} \left[e^{-\int_t^T x_s ds} \right] \\ &= \Phi(t, T; \alpha) P^{\text{Vasicek}}(t, T) \end{aligned}$$

- Swaptions** and **caps/floors** admit a closed-form expression, as a function of the parameters α and of the initial term structure.

The **parameters** of the model, evolving under the **risk neutral measure**, are derived by **calibrating** the theoretical prices of market instruments to their corresponding market quotes:

$$\alpha^* = \underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^N \left(\Pi_i^{\text{Th}}(t, T; \alpha) - \Pi_i^{\text{Mkt}}(t, T) \right)^2$$

Lognormal Linear SDE

The **lognormal** SDE can be obtained as an exponential of a linear equation with deterministic diffusion coefficient.

Let us take $Y_t = \exp(X_t)$, where X_t evolves according to (7), i.e.:

$$d \ln Y_t(\omega) = (\alpha_t + \beta_t \ln Y_t(\omega)) dt + v_t dW_t(\omega), \quad Y_0(\omega) = \exp(x_0)$$

Equivalently, by Ito's formula we can write:

$$\begin{aligned} dY_t(\omega) &= de^{X_t(\omega)} = e^{X_t(\omega)} dX_t(\omega) + \frac{1}{2} e^{X_t(\omega)} dX_t(\omega) dX_t(\omega) \\ &= \left[\alpha_t + \beta_t \ln Y_t(\omega) + \frac{1}{2} v_t^2 \right] Y_t dt + v_t Y_t(\omega) dW_t(\omega) \end{aligned}$$

The process Y has a **lognormal marginal density**. Major examples: Black Karasinski model (1991) and **Geometric Brownian Motion**.

Geometric Brownian Motion I

The GBM is a particular case of **lognormal linear** process.

Its **evolution** is defined by:

$$dX_t(\omega) = \mu X_t(\omega) dt + \sigma X_t(\omega) dW_t(\omega), \quad X_0(\omega) = X_0$$

where μ and σ are positive constants.

By Ito's formula, one can solve the SDE, by computing $d \ln X_t$:

$$X_t(\omega) = X_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t(\omega) \right\}$$

From the work of **Black and Scholes** (1973) on, processes of this type are frequently used in **option pricing theory** to model the asset price dynamics.

Geometric Brownian Motion II

The GBM process is a **submartingale**:

$$\mathbb{E}[X_T | \mathcal{F}_t] = e^{\mu(T-t)} X_t \geq X_t$$

The process $Y_t(\omega) = e^{-\mu t} X_t(\omega)$ is a **martingale**, since we obtain:

$$dY_t(\omega) = \sigma Y_t(\omega) dW_t(\omega)$$

i.e. the drift of the process is zero.

► Go to Geometric Brownian Motion: Exercise

Square Root Process

It is characterized by a **non-linear** SDE:

$$dX_t(\omega) = (\alpha_t + \beta_t X_t(\omega)) dt + v_t \sqrt{X_t(\omega)} dW_t(\omega), \quad X_0(\omega) = X_0$$

Square root processes are naturally linked to **non-central χ -square distributions**.

Major examples: the **Cox Ingersoll and Ross (CIR)** model (1985) and a particular case of the **constant-elasticity variance (CEV)** model for stock prices:

$$dX_t(\omega) = \mu X_t(\omega) dt + \sigma \sqrt{X_t(\omega)} dW_t(\omega), \quad X_0(\omega) = X_0$$

► Go to Cox Ingersoll Ross Model

► Go to SDE: Exercise

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Change of Measure

The way a **change** in the underlying **probability measure** affects a SDE is defined by the **Girsanov theorem**

The theorem is based on the following facts:

- the SDE drift depends on the particular probability measure \mathbb{P}
- if we change the probability measure in a “regular” way, the **drift of the equation changes while the diffusion coefficient remains the same**.

The Girsanov theorem can be useful when we want to **modify the drift** coefficient of a SDE.

Radon-Nikodym Derivative

Two measures \mathbb{P}^* and \mathbb{P} on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$ are said to be **equivalent**, i.e. $\mathbb{P}^* \sim \mathbb{P}$, if they share the same sets of null probability.

When two measures are equivalent, it is possible to express the first in terms of the second through the **Radon-Nikodym derivative**.

There exists a martingale ρ_t on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that

$$\mathbb{P}^* = \int_A \rho_t(\omega) d\mathbb{P}(\omega), \quad A \in \mathcal{F}_t$$

which can be written as:

$$\left. \frac{d\mathbb{P}^*}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \rho_t$$

The process ρ_t is called the **Radon-Nikodym derivative** of \mathbb{P}^* with respect to \mathbb{P} restricted to \mathcal{F}_t .

Expected Values

When in need of computing the expected value of an integrable random variable X , it may be useful to switch from one measure to another equivalent one.

- Expectations**

$$\mathbb{E}^*[X] = \int_{\Omega} X(\omega) d\mathbb{P}^*(\omega) = \int_{\Omega} X(\omega) \frac{d\mathbb{P}^*}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) = \mathbb{E}\left[X \frac{d\mathbb{P}^*}{d\mathbb{P}}\right]$$

- Conditional expectations**

$$\mathbb{E}^*[X | \mathcal{F}_t] = \frac{\mathbb{E}\left[X \frac{d\mathbb{P}^*}{d\mathbb{P}} \middle| \mathcal{F}_t\right]}{\rho_t}$$

Girsanov Theorem

Consider SDE, with Lipschitz coefficients, under $d\mathbb{P}$:

$$dX_t(\omega) = f(X_t(\omega))dt + \sigma(X_t(\omega))dW_t(\omega), \quad x_0$$

Let be given a new drift $f^*(x)$ and assume $(f^*(x) - f(x))/\sigma(x)$ to be bounded. Define the measure \mathbb{P}^* through the Radon-Nikodym derivative:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}(\omega) \Big|_{\mathcal{F}_t} = \exp \left\{ -\frac{1}{2} \int_0^t \left(\frac{f^*(X_s(\omega)) - f(X_s(\omega))}{\sigma(X_s(\omega))} \right)^2 ds + \int_0^t \frac{f^*(X_s(\omega)) - f(X_s(\omega))}{\sigma(X_s(\omega))} dW_s(\omega) \right\}$$

Then \mathbb{P}^* is equivalent to \mathbb{P} and the process W^* defined by:

$$dW_t^*(\omega) = - \left[\frac{f^*(X_t(\omega)) - f(X_t(\omega))}{\sigma(X_t(\omega))} \right] dt + dW_t(\omega)$$

is a Brownian motion under \mathbb{P}^* and

$$dX_t(\omega) = f^*(X_t(\omega))dt + \sigma(X_t(\omega))dW_t^*(\omega), \quad x_0$$

Example: from \mathbb{P} to \mathbb{Q} I

A classical example involves moving from the **real world asset price dynamics** \mathbb{P} to the **risk neutral** one, \mathbb{Q} , i.e. from

$$dS_t = \boxed{\mu} S_t dt + \sigma S_t \boxed{dW_t^{\mathbb{P}}} \quad \text{under } \mathbb{P} \quad (9)$$

to

$$dS_t = \boxed{r} S_t dt + \sigma S_t \boxed{dW_t^{\mathbb{Q}}} \quad \text{under } \mathbb{Q} \quad (10)$$

The **risk neutral measure** \mathbb{Q} is used in **pricing problems** while the **real-world (or historical) measure** \mathbb{P} is used in **risk management**.

Example: from \mathbb{P} to \mathbb{Q} II

- Start from the asset dynamics under the **real-world measure** \mathbb{P} , eq. (9):

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}$$

- Consider the **discounted asset price process** $\tilde{S}_t = S_t e^{-rt}$.
This process satisfies the following SDE:

$$d\tilde{S}_t = (\mu - r) \tilde{S}_t dt + \sigma \tilde{S}_t dW_t^{\mathbb{P}} \quad (11)$$

- The goal is to find a measure \mathbb{Q} , equivalent to \mathbb{P} , such that the discounted asset price process is a **martingale** under the new measure, i.e.

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^{\mathbb{Q}} \quad (12)$$

Example: from \mathbb{P} to \mathbb{Q} III

- To this purpose, rewrite eq. (11) as follows:

$$d\tilde{S}_t = [(\mu - r) dt + \sigma dW_t^{\mathbb{P}}] \tilde{S}_t = \left[\frac{\mu - r}{\sigma} dt + dW_t^{\mathbb{P}} \right] \sigma \tilde{S}_t$$

and define, according to **Girsanov theorem**, a new Brownian process:

$$dW_t^* \equiv \frac{\mu - r}{\sigma} dt + dW_t^{\mathbb{P}}.$$

Therefore,

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^* \quad (13)$$

i.e. \tilde{S}_t is a martingale under the equivalent measure \mathbb{P}^* .

- Comparing (12) with (13), we obtain $\mathbb{P}^* \equiv \mathbb{Q}$.
- Going back to the **asset price process** $S_t = \tilde{S}_t e^{rt}$, we finally get eq. (10):

$$dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

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No-Arbitrage Pricing

We refer to Brigo and Mercurio, Chapter 2.

As already mentioned, **absence of arbitrage** is equivalent to the impossibility to invest zero today and receive tomorrow a non-negative amount that is positive with positive probability. In other words, two portfolios having the same payoff at a given future date must have the same price today.

Historically, **Black and Scholes (1973)** showed that, by constructing a suitable portfolio having the same instantaneous return as that of a risk-less investment, the portfolio instantaneous return was indeed equal to the instantaneous risk-free rate, which led to their celebrated **option-pricing formula**.

Harrison and Pliska Result (1983)

A financial market is **arbitrage free and complete** if and only if there exists a **unique equivalent (risk-neutral or risk-adjusted) martingale measure**.

Stylized characterization of **no-arbitrage theory via martingales**:

- The market is **free of arbitrage** if (and only if) there exists a martingale measure
- The market is **complete** if and only if the martingale measure is unique
- In an arbitrage-free market, not necessarily complete, the **price** π_t of any attainable claim is uniquely given, either by the value of the associated replicating strategy, or by the **risk neutral expectation of the discounted claim payoff under any of the equivalent (risk-neutral) martingale measures**:

$$\pi_t = \mathbb{E}[D(t, T)\Pi_T | \mathcal{F}_t]$$

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Brownian Motion: Exercises

1 Time reversal

Prove that the continuous time stochastic process defined by:

$$B_t = W_T - W_{T-t}, \quad t \in [0, T]$$

is a standard Brownian motion.

2 Brownian scaling

Let W_t be a standard Brownian motion. Given a constant $c > 0$, show that the stochastic process X_t defined by:

$$X_t = \frac{1}{\sqrt{c}} W_{ct}, \quad t > 0$$

is a standard Brownian motion.

Martingales: Exercises

- 1 Let X_1, X_2, \dots be a sequence of independent random variables in \mathcal{L}^1 such that $\mathbb{E}[X_n] = 0$ for all n . If we set:

$$S_0 = 0, \quad S_n = X_1 + X_2 + \dots + X_n, \quad \text{for } n \geq 1$$

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n), \text{ for } n \geq 1$$

prove that the process $(S_n)_n$ is an $(\mathcal{F}_n)_n$ -martingale.

- 2 Consider a filtration $(\mathcal{F}_n)_n$ and an \mathcal{F}_n -adapted stochastic process $(X_n)_n$ such that $X_0 = 0$ and $\mathbb{E}[|X_n|] \leq \infty$ for all $n \geq 0$. Also, let $(c_n)_n$ be a sequence of constants. Define $M_0 = 0$ and

$$M_n = c_n X_n - \sum_{j=1}^n c_j \mathbb{E}[X_j - X_{j-1} | \mathcal{F}_{j-1}] - \sum_{j=1}^n (c_j - c_{j-1}) X_{j-1}, \quad \text{for } n \geq 1$$

Prove that $(M_n)_n$ is an \mathcal{F}_n -martingale.

Ito's formula: Exercises

- 1 Consider a standard one-dimensional Brownian motion W_t . Use Ito's formula to calculate:

$$W_t^2 = t + 2 \int_0^t W_s dW_s$$

and

$$W_t^{27} = 351 \int_0^t W_s^{25} ds + 27 \int_0^t W_s^{26} dW_s$$

- 2 Consider a standard one-dimensional Brownian motion W_t . Given $k \geq 2$ and $t \geq 0$, use Ito's formula to prove that:

$$\mathbb{E}[W_t^k] = \frac{1}{2} k(k-1) \int_0^t \mathbb{E}[W_u^{k-2}] du$$

Use this expression to calculate $\mathbb{E}[W_t^4]$ and $\mathbb{E}[W_t^6]$.

Hint: stochastic integrals are martingales, so their expectation is zero.

Geometric Brownian Motion: Exercise

Consider the Geometric Brownian motion SDE:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

with deterministic time-dependent coefficients, μ_t and σ_t , and initial condition S_0 .

Prove that its solution is given by:

$$S_t = S_0 \exp \left\{ \int_0^t \left(\mu_u - \frac{\sigma_u^2}{2} \right) du + \int_0^t \sigma_u dW_u \right\}$$

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Cox Ingersoll Ross Model

In the Cox Ingersoll Ross model (1985) for interest rates, the dynamics of the short rate process r_t is given by the following SDE:

$$dr_t = k(\theta - r_t) dt + \sigma\sqrt{r_t}dW_t$$

with k , θ and σ strictly positive constants, and initial condition r_0 .

- 1 Show that the solution to the above SDE is given by:

$$r_t = \theta + (r_0 - \theta) e^{-kt} + \sigma \int_0^t e^{k(s-t)} \sqrt{r_s} dW_s$$

Hint: Consider the Ito processes X_t and Y_t defined by $X_t = e^{kt}$ and $Y_t = r_t$ and integrate by parts.

- 2 Calculate the mean $\mathbb{E}[r_t]$ and the variance $\text{var}(r_t)$ of the random variable r_t .

Hint: Use Ito's isometry and assume that all stochastic integrals are martingales, so they have zero expectation.

SDE: Exercise

Consider the following integral SDE:

$$Z_t = - \int_0^t Z_u du + \int_0^t e^{-u} dW_u$$

Prove that:

$$Z_t = e^{-t} W_t$$

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