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Subcalculus for set functions and cores of TU games

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Abstract

This paper introduces a subcalculus for general set functions and uses this framework to study the core of TU games. After stating a linearity theorem, we establish several theorems that characterize measure games having finite-dimensional cores. This is a very tractable class of games relevant in many economic applications.

Keywords: TU games; Non-additive set functions; Subcalculus; Core

1. Introduction

General set functions, not necessarily additive, are widely used in mathematical economics. In cooperative game theory, the key notion of transferable utility (TU) game is modelled as a general set function ν defined on a collection Σ of admissible coalitions, with the only requirement on ν that it takes on value zero at the empty set. In decision theory, non-additive set functions have been recently used to model "vague" beliefs, which in general are not representable by standard additive probabilities (see Schmeidler, 1989). Though the motivation is very different, the mathematical object is essentially the same in both cases.

This has motivated a large literature on non-additive set functions in both game and decision theory, which includes the classic book of Aumann and Shapley (1974). In mathematics as well, non-additive set functions have been the subject of many investigations, mostly in the wake of the seminal work of Choquet (1953), which anticipated most of the themes of the subsequent literature.

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Rather surprisingly, in these different strands of literature there has been little attempt to develop a systematic calculus and subcalculus for general set functions, despite of the potential insights that such basic mathematical tools could provide. Recently, Epstein and Marinacci (2001) have developed a calculus for TU games, in which the derivative is an additive set function that suitably approximates the TU game on "small" sets. This derivative is then used to study the core of TU games. In their analysis, a key role is played by linear sets (coalitions), namely sets E in Σ such that $\nu(E) + \nu(E^c) = \nu(\Omega)$, where Ω is the grand coalition. Naturally, the empty set $\emptyset$ and the grand coalition Ω are linear sets. They show that, under mild assumptions, the core shrinks to a singleton as long as the game is differentiable at some linear set. Moreover, the core consists of the derivative itself.

A limitation of their analysis is that the core may not be a singleton. This naturally leads to the question of whether it is possible to extend their approach by using superdifferentials rather than differentials. This is our purpose in the present work, where a subcalculus for TU games is introduced and exploited to characterize cores of TU games.

Our starting point was the discovery of a simple characterization of the cores by means of superdifferentials. As a matter of fact, let $\partial\nu(E)$ be the natural adaptation for TU games of the standard superdifferential of functions on Euclidean spaces. For the core of a TU game ν it holds

$$\mathrm{core}(\nu) = \partial\nu(E) \cap \partial\nu(E^c),$$

where E is any linear set (Theorem 11). Based on this simple characterization we are able to prove several novel results, as well as to provide simple proofs and a unifying framework for some important known results. In particular, our "subcalculus" framework is the natural setting in which some of the powerful methods of Convex Analysis can be used to study TU games.

More specifically, our paper is organized as follows. In Section 3 we discuss the main properties of the superdifferentials. They turn out to be similar to those of the standard superdifferentials, though the notions are less close than one might think at a first sight. Among them, it is especially important the sum rule for convex games, which ensures that $\partial(\nu_1 + \nu_2)(E) = \partial\nu_1(E) + \partial\nu_2(E)$ for all sets E in Σ. An immediate consequence of this rule is that the cores of convex games are stable under summation, that is, $\mathrm{core}(\nu_1 + \nu_2) = \mathrm{core}(\nu_1) + \mathrm{core}(\nu_2)$.

After having established a "subcalculus," Section 4 studies the relations existing among our superdifferentials, the derivatives studied by Epstein and Marinacci (2001), and the cores. The main result, Theorem 13, provides conditions ensuring that the core shrinks to a singleton as long as the differential of the game belongs to its superdifferential. This result can be viewed as an enrichment of the theory developed by Epstein and Marinacci (2001).

In Section 5 we specialize our analysis to measure games. As a matter of fact, TU games that are relevant for economic applications have often the form $\nu = g(P)$, where $P = (P_1, \dots, P_N) : \Sigma \to \mathbb{R}^N$ is a non-atomic vector measure and $g : \mathbb{R}^N \to \mathbb{R}$ is a function such that $\nu(E) = g(P(E))$ for all sets E belonging to Σ. Games of this form are called *measure games*, and standard examples include exchange economies with transferable utilities and models of production technology. While we do not expatiate here on these known issues, we refer the reader to Aumann and Shapley (1974) and Hart and

Neyman (1988) for detailed discussions of these examples and of the relevance of measure games in economic applications.

Based on our subcalculus, we provide simple conditions under which the core of a measure game consists of linear combinations $\sum_{i=1}^{N} \alpha_i P_i$ of the components $\{P_i\}_{i=1}^{N}$ of the underlying vector measure P. For example, we show that the cores have this form whenever there exists a linear and radial set $E \in \Sigma$, that is, a linear set E such that $P(E)$ belongs to the relative interior of $R(P)$, the range $\{P(E) : E \in \Sigma\} \subseteq \mathbb{R}^N$ of the vector measure P. The existence of linear and radial sets is a condition often satisfied by economic games of the form $g(P)$. In fact, these games typically feature some homogeneity condition of the function g, and it will be seen that even very mild homogeneity conditions deliver linear and radial sets.

Our results of this section generalize well-known results of Billera and Raanan (1981), as well as recent results of Einy et al. (1999). They are based on a novel linearity theorem for non-atomic vector measures (Theorem 20) that should be of independent interest. This theorem relies on results from both Measure Theory and Convex Analysis, an interplay made possible by the Lyapunov Theorem, which guarantees the range $R(P)$ to be a convex set.

Finally, in Section 6 we discuss the related works of Billera and Raanan (1981) and Einy et al. (1999), as well as the relationships between linear cores and semi-infinite linear programming. Appendix A gathers some technical lemmas and all proofs.

2. Preliminaries

Throughout the paper, Ω is the set of players and Σ the σ-algebra of admissible coalitions. Subsets of Ω are understood to be in Σ even where not stated explicitly.

A set function $\nu : \Sigma \to \mathbb{R}$ is a *game* if $\nu(\varnothing) = 0$. A game ν is

- *positive* if $\nu(E) \geq 0$ for all E;
- *bounded* if $\sup_{E\in\Sigma}|\nu(E)| < \infty$;
- *monotone* if $\nu(E) \geq \nu(E')$ whenever $E' \subseteq E$;
- *continuous at E* if $\lim_{n\to\infty}\nu(E_n) = \nu(E)$ whenever $E_n \uparrow E$ or $E_n \downarrow E$;
- *superadditive* if $\nu(E \cup E') \geq \nu(E) + \nu(E')$ for all pairwise disjoint sets E and E';
- *supermodular* (or convex) if $\nu(E \cup E') + \nu(E \cap E') \geq \nu(E) + \nu(E')$ for all sets E and E';
- *additive* (or a charge) if $\nu(E \cup E') = \nu(E) + \nu(E')$ for all pairwise disjoint sets E and E';
- *countably additive* (or a measure) if $\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$ for all countable collections of pairwise disjoint sets $\{E_i\}_{i=1}^{\infty}$.

Unless otherwise stated, charges and measures are understood to be signed. The set of all charges (measures) that are bounded with respect to the variation norm is denoted by $\mathrm{ba}(\Omega)$ ($\mathrm{ca}(\Omega)$). Generic elements of $\mathrm{ba}(\Omega)$ are denoted by m, while its non-negative elements are denoted by P.

A charge m is *non-atomic* if for all $m(E) \neq 0$ there exists $B \subseteq E$ such that $m(B) \neq 0$ and $m(E - B) \neq 0$. It is *strongly continuous* if, for every $\varepsilon > 0$, there exists a partition

$\{E_1, \dots, E_n\}$ of Ω in Σ such that $|m|(E_i) \le \varepsilon$ for all $i = 1, \dots, n$. A strongly continuous charge is non-atomic, while the two concepts are equivalent for measures (see Bhaskara Rao and Bhaskara Rao, 1983). Let $m = (m_1, \dots, m_N) : \Sigma \to \mathbb{R}^N$ be a vector charge. If each m_i is strongly continuous, then by the Lyapunov Theorem the range $R(m) = \{m(E) : E \in \Sigma\}$ is a convex subset of $\mathbb{R}^N$ (see Bhaskara Rao and Bhaskara Rao, 1983).

The game $\nu : \Sigma \to \mathbb{R}$ is a *measure game* if there exists a positive vector charge $P = (P_1, \dots, P_N) : \Sigma \to \mathbb{R}^N_+$, with each $P_i : \Sigma \to \mathbb{R}_+$ bounded and strongly continuous, and a function $g : R(P) \to \mathbb{R}$ such that

$$\nu(E) = g(P(E)) \quad \text{for all } E \in \Sigma.$$

When $N = 1$, $\nu = g(P)$ is called a *scalar measure game*.

The core of a game ν is

$$\text{core}(\nu) = \{m \in \text{ba}(\Omega) : m(\Omega) = \nu(\Omega) \text{ and } m(E) \ge \nu(E) \text{ for all } E \in \Sigma\}.$$

It is easy to see that the core is a weak*-compact subset of $\text{ba}(\Omega)$. A game ν is *exact* if $\text{core}(\nu) \ne \varnothing$ and $\nu(E) = \min_{m \in \text{core}(\nu)} m(E)$ for all $E \in \Sigma$. All positive convex games are exact (see Schmeidler, 1972). For cores of convex games this is proved in the next result, which generalizes to bounded convex games a well-known property of positive convex games.

Lemma 1. *Let $\nu : \Sigma \to \mathbb{R}$ be a bounded and convex game. Then ν is exact and given any chain $\{E_i\}_{i \in I}$, there is $m \in \text{core}(\nu)$ such that $m(E_i) = \nu(E_i)$ for all $i \in I$.*

Given a game $\nu : \Sigma \to \mathbb{R}$, a set E is *linear* if $\nu(E) + \nu(E^c) = \nu(\Omega)$. Notice that both Ω and $\varnothing$ are linear sets. Moreover, when $\text{core}(\nu) \ne \varnothing$, E is linear if and only if $\nu(E) + \nu(E^c) \ge \nu(\Omega)$. The set of linear sets is denoted by $\mathcal{A}$.

Linear sets are delivered by *efficient coalition structures*, that is, at most countable partitions $\{E_i\}_{i \in I}$ of Ω such that $\sum_{i \in I} \nu(E_i) = \nu(\Omega)$. In fact, if ν is superadditive and either ν is continuous or the partition is finite, then E_i is linear for each i in I (see Epstein and Marinacci, 2001).[1]

We close by reporting the notion of derivative for games studied by Epstein and Marinacci (2001). For any $E \in \Sigma$, let $\{E^{j,\lambda}\}_{j=1}^{n_\lambda}$ be a finite partition of E. Denote by $\{E^{j,\lambda}\}_\lambda$ the net of all finite partitions of E, where $\lambda' > \lambda$ implies that the partition corresponding to λ' refines that corresponding to λ.

Definition 2. A game $\nu : \Sigma \to \mathbb{R}$ is differentiable at $E \in \Sigma$ if there exists a charge $\delta\nu(\cdot; E) \in \text{ba}(\Omega)$ such that for all $F \subseteq E^c$ and $G \subseteq E$,

$$\sum_{j=1}^{n_\lambda} |\nu(E \cup F^{j,\lambda} - G^{j,\lambda}) - \nu(E) - \delta\nu(F^{j,\lambda}; E) + \delta\nu(G^{j,\lambda}; E)| \underset{\lambda}{\to} 0.$$

This definition is slightly different from that of Epstein and Marinacci (2001), which originates in Epstein (1999), as we do not require the charge $\delta\nu(\cdot; E)$ to be convex-ranged.

[1] In a finite setting, efficient coalition structures have been introduced by Aumann and Dreze (1974).

3. Superdifferentials

Definition 3. A game $\nu : \Sigma \to \mathbb{R}$ is superdifferentiable at $E \in \Sigma$ if there exists a charge $m \in \mathrm{ba}(\Omega)$ such that

$$\nu(A) \leq \nu(E) + m(A) - m(E), \tag{1}$$

for each $A \in \Sigma$.

The charges m that satisfy Eq. (1) are called *supergradients* and $\partial\nu(E)$ is the *superdifferential* of ν, that is, the (possibly empty) set of all supergradients.

Definition 3 is the natural adaptation to our setting of the standard notion of superdifferential of real-valued functions (see Rockafellar, 1970),[2] as it becomes evident by considering measure games $g(P) : \Sigma \to \mathbb{R}$. Recall that, given a subset $A \subseteq \mathbb{R}^N$ (e.g. $A = R(P)$), a function $g : A \to \mathbb{R}$ is *superdifferentiable* at $x_0 \in A$ if there is a vector $\chi \in \mathbb{R}^N$, called *supergradient*, such that $g(x_0) \leq g(x) + \chi \cdot (x - x_0)$ for all $x \in A$. The *superdifferential* $\partial g(x_0)$ is the set of all supergradients.

Given a set E, the two superdifferentials $\partial\nu(E)$ and $\partial g(P(E))$ are related by the following lemma, which we report for later reference (the simple proof is omitted).

Lemma 4. *Given a measure game $\nu = g(P) : \Sigma \to \mathbb{R}$, for each set $E \in \Sigma$ a charge of the form $\chi \cdot P$ belongs to $\partial\nu(E)$ if and only if the vector $\chi \in \mathbb{R}^N$ belongs to $\partial g(P(E))$.*

We now present few elementary properties of the superdifferential $\partial\nu(E)$. It is easy to check that the set $\partial\nu(E)$ is convex and weak*-closed, and that the following properties hold:

(i) $\partial\lambda\nu(E) = \lambda\partial\nu(E)$ for all $\lambda > 0$ and all $E \in \Sigma$;
(ii) $\partial\nu_1(E) + \partial\nu_2(E) \subseteq \partial(\nu_1 + \nu_2)(E)$ for all $E \in \Sigma$ and all games ν_1 and ν_2, with equality if at least one of the two games is in $\mathrm{ba}(\Omega)$.

Given $E \in \Sigma$, consider the cone K_E defined by

$$K_E = \{m \in \mathrm{ba}(\Omega) : m(G) \geq 0 \text{ and } m(F) \leq 0 \text{ for each } F \subseteq E^{\mathrm{c}} \text{ and } G \subseteq E\}.$$

Clearly, $K_\Omega = \mathrm{ba}(\Omega)^+$ and $-K_E = K_{E^{\mathrm{c}}}$. The following result shows the importance of these cones for our analysis.

Proposition 5. *Let $\nu : \Sigma \to \mathbb{R}$ be a game superdifferentiable at E. Then, $\partial\nu(E) = \partial\nu(E) + K_{E^{\mathrm{c}}}$ for each $E \in \Sigma$.*

We now consider two key properties of superdifferentials, non-emptiness and the sum rule. Our first result shows that for the important class of exact games the set $\partial\nu(E)$ is non-empty for all $E \in \Sigma$.

Proposition 6. *If the game $\nu : \Sigma \to \mathbb{R}$ is exact, then $\partial\nu(E) \neq \varnothing$ for all $E \in \Sigma$. In particular, ν is exact if and only if $\partial\nu(E) \cap \mathrm{core}(\nu) \neq \varnothing$ for all $E \in \Sigma$.*

[2] Fujishige (1991) gives a similar definition for supermodular functions defined on finite distributive lattices.

Since bounded convex games are exact, they are superdifferentiable at all sets $E \in \Sigma$ by Proposition 6. The following result shows that it is actually possible to characterize convexity through superdifferentials.

Proposition 7. *A bounded game* $\nu : \Sigma \to \mathbb{R}$ *is convex if and only if* $\partial\nu(E_1) \cap \partial\nu(E_2) \neq \varnothing$ *for every pair* $E_1 \subseteq E_2$.

The next result shows that superdifferentials preserve sums.

Theorem 8. *Given any two convex and bounded games* $\nu_1 : \Sigma \to \mathbb{R}$ *and* $\nu_2 : \Sigma \to \mathbb{R}$, *we have*

$$\partial(\nu_1 + \nu_2)(E) = \partial\nu_1(E) + \partial\nu_2(E), \tag{2}$$

for all $E \in \Sigma$.

Since $\partial\lambda\nu(E) = \lambda\partial\nu(E)$ for all $\lambda > 0$ and all sets E, we conclude that, by Theorem 8, superdifferentials of bounded convex games preserve positive linear combinations. This fundamental property immediately implies the following result, which shows that cores of bounded convex games are stable under summation.

Corollary 9. *Let* $\nu_1 : \Sigma \to \mathbb{R}$ *and* $\nu_2 : \Sigma \to \mathbb{R}$ *be any two convex and bounded games. Then,*

$$\text{core}(\nu_1 + \nu_2) = \text{core}(\nu_1) + \text{core}(\nu_2). \tag{3}$$

Notice that in general it only holds the superadditive property $\text{core}(\nu_1) + \text{core}(\nu_2) \subseteq \text{core}(\nu_1 + \nu_2)$. Equality is no longer true when ν_1 and ν_2 are exact. In this case, we have $\text{core}(\nu_1 + \nu_2) = \overline{\text{core}(\nu_1) + \text{core}(\nu_2)}$, where the upper bar denotes the m-closure of a set (see Marinacci and Montrucchio (2002b) for details).

We close by considering measure games. In this case, it is enough to study the existence of the standard superdifferential $\partial g(P(E))$ since, by Lemma 4, $\partial\nu(E)$ is non-empty whenever $\partial g(P(E))$ is non-empty.

Proposition 10. *Let* $\nu = g(P) : \Sigma \to \mathbb{R}$ *be a measure game. Then* $\partial\nu(E) \neq \varnothing$ *for all* $E \in \Sigma$ *provided one of the following conditions holds*:

(i) $g : R(P) \to \mathbb{R}$ *is Lipschitz and concave*;
(ii) ν *is superadditive and* $g : R(P) \to \mathbb{R}$ *is such that* $g(\alpha P(E)) = \alpha g(P(E))$ *for each* $\alpha \in (0, 1)$ *and each* $E \in \Sigma$.

Condition (ii) is especially important in cooperative game theory, where the TU games that satisfy condition (ii) are called market games. They play an important role in the study of exchange economies (see Hart and Neyman, 1988).

4. Cores and derivatives

The derivative for games introduced in Definition 2 was used by Epstein and Marinacci (2001) to study the cores of some TU games. In particular, they study a class of important economic games that have singleton cores and loosely speaking, they show that the singleton actually consists of the derivative of the game. Since for real-valued functions the derivative can be viewed as a singleton superdifferential, it is natural to wonder whether the superdifferentials for games that we introduced are related to the derivatives of Definition 2 and, more importantly, whether they can be used to characterize cores that are not necessarily singleton. In this section we address these natural queries.

Interestingly, as in Epstein and Marinacci (2001), also in this work linear sets play a key role. Our first result provides a subcalculus characterization of the core based on linear sets.

Theorem 11. *Consider the following conditions*:

(i) $E \in \mathcal{A}$;
(ii) $\text{core}(\nu) = \partial\nu(E) \cap \partial\nu(E^{\text{c}})$;
(iii) $\partial\nu(E) \cap \partial\nu(E^{\text{c}}) \neq \varnothing$.

We have that (i) *implies* (ii), *while the three conditions are equivalent whenever* $\text{core}(\nu)$ *is non-empty*.

In other words, $\text{core}(\nu) = \partial\nu(E) \cap \partial\nu(E^{\text{c}})$ when E is linear, regardless of whether or not $\text{core}(\nu)$ is non-empty. However, if $\text{core}(\nu)$ is non-empty, the three conditions are equivalent.

Having established a subcalculus characterization of the core, we now move to study the relations of supergradients with the derivatives of games introduced in Definition 2.

Proposition 12. *Let $\nu : \Sigma \to \mathbb{R}$ be a game superdifferentiable and differentiable at E. Then*

$$\delta\nu(\cdot; E) \in \partial\nu(E) + K_E.$$

If, in addition, E is linear and $\text{core}(\nu) \neq \varnothing$, *then,*

$$\delta\nu(\cdot; E) \in \partial\nu(E^{\text{c}}).$$

In the last result, we saw that $\delta\nu(\cdot; E) \in \partial\nu(E^{\text{c}})$ when E is linear and $\text{core}(\nu)$ is non-empty. This raises the question of when $\delta\nu(\cdot; E) \in \partial\nu(E)$, something that in standard subcalculus happens in many important cases.

Proposition 13. *Let $\nu : \Sigma \to \mathbb{R}$ be a game differentiable at a linear set A. If* $\text{core}(\nu) \neq \varnothing$, *then $\partial\nu(A) \neq \varnothing$ and the following conditions are equivalent*:

(i) $\delta\nu(\cdot; A) \in \partial\nu(A)$;
(ii) $\text{core}(\nu) = \{\delta\nu(\cdot; A)\}$;
(iii) $\delta\nu(\cdot; A) \in \text{core}(\nu)$;
(iv) $\delta\nu(A; A) = \nu(A)$ *and* $\delta\nu(A^{\text{c}}; A) = \nu(A^{\text{c}})$.

Moreover, if (i) *and* (iv) *hold for some linear set* A*, then* $\mathrm{core}(\nu)$ *is non-empty and coincides with the singleton* $\{\delta\nu(\cdot; A)\}$.

Proposition 13 can be viewed as a calculus characterization of the core and it sharpens some results of this kind proved by Epstein and Marinacci (2001). A different route is, in contrast, followed in Marinacci and Montrucchio (2002a), which provides a characterization of the cores of convex games based on standard Gateaux derivatives of the Choquet integrals associated with the games.

5. Measure games

Games relevant for economic applications have often the form of a measure game $g(P) : \Sigma \to \mathbb{R}$. In this section, we study in more detail the structure of the superdifferentials and cores of this class of games.

The natural question for cores of measure games is how to relate the underlying vector charge P with the charges in the cores. We start by establishing a simple general result of this type for the important countably additive case.

Proposition 14. *Let* $g(P) : \Sigma \to \mathbb{R}$ *be a measure game and suppose that the vector measure* P *is countably additive and that there is a linear set* A *(e.g.* $A = \emptyset$*) such that* g *is lower semicontinuous at* $P(A)$ *and* $P(A^c)$*. Then, for each* $m \in \mathrm{core}(\nu)$ *there exists a* Σ*-measurable vector function* $f = (f_1, \dots, f_N) : \Omega \to \mathbb{R}^N$ *such that, for all* $E \in \Sigma$

$$m(E) = \sum_{i=1}^{N} \int_E f_i \, \mathrm{d}P_i. \tag{4}$$

Notice that Eq. (4) provides two important pieces of information on the charges belonging to $\mathrm{core}(\nu)$: (i) they are all countably additive; (ii) they are all absolutely continuous w.r.t. the "average" measure $P^* = (1/N)\sum_{i=1}^{N} P_i$.

An especially interesting case in Proposition 14 is when to a given m in $\mathrm{core}(\nu)$ corresponds a constant vector function $f : \Omega \to \mathbb{R}^N$, that is, when there exists a vector $(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ such that $f(\omega) = (\alpha_1, \dots, \alpha_N)$ for all $\omega \in \Omega$. In this case, m is a linear combination of the underlying vector charge P, a most convenient situation. Because of their interest, we first give a name to the subset of $\mathrm{core}(\nu)$ consisting of such linear combinations.

Definition 15. The linear core of a measure game $\nu = g(P) : \Sigma \to \mathbb{R}$ is the subset $\mathcal{L}\mathrm{core}(\nu)$ of $\mathrm{core}(\nu)$ defined by

$$\mathcal{L}\mathrm{core}(\nu) = \mathrm{core}(\nu) \cap \mathrm{span}\{P_1, \dots, P_N\}.$$

Using Lemma 4 and Theorem 11, it is easy to characterize the linear core and to provide bounds for its dimension. All this makes use of linear sets, thus showing their importance for $\mathcal{L}\mathrm{core}(\nu)$.

Proposition 16. *Given a measure game $\nu = g(P) : \Sigma \to \mathbb{R}$, it holds that*

$$\mathcal{L}\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\}, \tag{5}$$

for any linear set A. Moreover,

$$\dim(\mathcal{L}\mathrm{core}(\nu)) \le \dim(R(P)) - \dim(\mathrm{span}\{P(A) : A \in \mathcal{A}\}) \le N - 1. \tag{6}$$

5.1. A linearity theorem

We have introduced the linear core, the subset of the core of a measure game $g(P) : \Sigma \to \mathbb{R}$ that consists of linear combinations of the underlying vector charge P. This part of the core is especially interesting because of its simple form and analytical tractability, and the games whose core and linear core coincide stand out among games in terms of simplicity and tractability. This section is devoted to the study of these games, which we call linear.

Definition 17. A measure game $\nu = g(P) : \Sigma \to \mathbb{R}$ is called linear if $\mathrm{core}(\nu) = \mathcal{L}\mathrm{core}(\nu)$, that is, if $\mathrm{core}(\nu) \subseteq \mathrm{span}\{P_1, \dots, P_N\}$.

To provide a characterization of linear games, we first state a linearity theorem for vector measures that should be of independent interest. The following important class of sets will play a key role.

Definition 18. A set $A \in \Sigma$ is radial if there is a set $E \in \Sigma$ such that, for some $t \in (0, 1)$,

$$P(A) = tP(E) + (1 - t)P(E^{\mathrm{c}}).$$

By the Lyapunov Theorem, radial sets form a significant subset of $R(P)$ and they include the sets called *diagonal* by Epstein and Marinacci (2001), that is, the sets $A \in \Sigma$ such that $P(A) = tP(\Omega)$ for some $t \in (0, 1)$. The next result provides a useful characterization of radial sets in terms of the relative interior of $R(P)$. We omit its simple proof, which is based on the important property of the range $R(P)$ of having the point $2^{-1}P(\Omega)$ as a center of symmetry, that is, $2(2^{-1}P(\Omega)) - x \in R(P)$ for all $x \in R(P)$.[3]

Proposition 19. *Let $P = (P_1, \dots, P_N) : \Sigma \to \mathbb{R}^N$ be a vector charge with each P_i strongly continuous. Then, a set $E \in \Sigma$ is radial if and only if $P(E)$ belongs to the relative interior of $R(P)$.*

We can now state and prove the announced linearity theorem.

Theorem 20. *Let $P = (P_1, \dots, P_N) : \Sigma \to \mathbb{R}^N_+$ be a positive vector charge with each P_i strongly continuous and suppose $m : \Sigma \to \mathbb{R}$ is either a signed measure in $\mathrm{ca}(\Omega)$ or a strongly continuous charge in $\mathrm{ba}(\Omega)$. If there exists a radial set A such that, for all $E \in \Sigma$,*

$$P(E) = P(A) \Rightarrow m(E) = m(A), \tag{7}$$

[3] See, e.g. Bolker (1969), who studies in detail the geometry of $R(P)$.

then

$$m \in \text{span}\{P_1, \dots, P_n\}. \tag{8}$$

If, in addition, it holds that

$$P(E) \geq P(A) \Rightarrow m(E) \geq m(A), \tag{9}$$

then

$$m \in \text{cone}\{P_1, \dots, P_n\}.$$

It is important to note the two key features of this result: (i) the existence of just a single radial set A is required; (ii) no assumption, besides either countable additivity or strong continuity, is made on m. Theorem 20 is the N-dimensional generalization of a uniqueness result of Marinacci (2000), which holds for positive scalar measures P and m. In fact, in the scalar case a set A is radial if and only if $0 < P(A) < P(\Omega)$. Therefore, if there exists a set $A \in \Sigma$ with $0 < P(A) < P(\Omega)$ and such that

$$P(E) = P(A) \Rightarrow m(E) = m(A),$$

whenever $E \in \Sigma$, then $m(E) = (m(\Omega)/P(\Omega))P(E)$ by Theorem 20. When m is positive, this is the uniqueness result of Marinacci (2000). In that paper, however, uniqueness is also proved for lambda systems, while here we only consider σ-algebras.

It can be useful to compare Theorem 20 with the classic result saying that, given any $N+1$ linear functionals $L, L_1, \dots, L_N$ defined on a vector space, it holds $L \in \text{span}\{L_1, \dots, L_N\}$ whenever

$$L_1(x) = \cdots = L_N(x) = 0 \Rightarrow L(x) = 0, \tag{10}$$

for all vectors x (see, e.g. Aliprantis and Border, 1999, p. 207). In our setting, the relevant vector space is $B(\Sigma)$, the space of all bounded Σ-measurable functions, while the linear functionals are the ones naturally associated with $m, P_1, \dots, P_N$. Even though the classic result and Theorem 20 share the same conclusion, that is, $m \in \text{span}\{P_1, \dots, P_n\}$, it is important to notice that our condition (7) is much weaker than condition (10). As a matter of fact, (7) only involves sets, that is, indicators if we view Σ as a subset of the vector space $B(\Sigma)$. Not surprisingly, therefore, Theorem 20 needs additional conditions like the non-atomicity of P and the countable additivity of m, and the proof is altogether different. On the other hand, while the classic result holds for general vector spaces, Theorem 20 only holds in $B(\Sigma)$ as it critically relies on the added structure ensured by this vector space.

As a final remark, observe that Theorem 20 could be also interpreted in a social choice context if we assume that m and each P_i are probability measures representing beliefs. For instance, consider diagonal sets, that in this setting can be viewed as events over which agents have unanimous beliefs, say $P_i(A) = \alpha \in (0, 1)$ for each $i = 1, \dots, N$. By Theorem 20, linear aggregation occurs whenever the aggregator m preserves the agents' unanimous beliefs on some event A, a condition much weaker than the Paretian conditions used in Bayesian aggregation results (cf. Fishburn, 1984 and Mongin, 1995).

5.2. Characterizing linear games

Using Theorem 11 and the just established Theorem 20, we can now provide a simple condition under which a measure game is linear. Recall that a function $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *calm from below* at a point $x_0 \in A$, if there exist a constant L and a radius $\varepsilon > 0$, such that

$$g(x) \geq g(x_0) - L|x - x_0|,$$

for all x in A and $|x - x_0| \leq \varepsilon$ (see, e.g. Rockafellar and Wets, 1997).

Theorem 21. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game and suppose one of the following holds*:

- (i) *P is countably additive and there is a linear set A_* (e.g. $A_* = \emptyset$) such that g is lower semicontinuous at $P(A_*)$ and $P(A_*^{\mathrm{c}})$;*
- (ii) *g is calm from below at 0 and $P(\Omega)$.*

Then, if there exists a linear and radial set, the game ν is linear and

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\}, \tag{11}$$

for each linear sets $A \in \Sigma$. If g is monotone on $R(P)$, then χ can be assumed to be non-negative, i.e. $\chi_i \geq 0$ for all $1 \leq i \leq N$. Finally, if ν is exact the converse holds, that is, a linear and exact measure game has linear and radial sets.

Remarks.

- (i) The core can be empty, that is, in Eq. (11) it may well happen that

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\} = \emptyset.$$

- (ii) The converse does not hold if ν is not exact. In fact, consider the following scalar measure game:

$$\nu(E) = \begin{cases} P(E) & \text{if } P(E) < \frac{1}{2}, \\ P(E)^2 & \text{if } P(E) \geq \frac{1}{2}, \end{cases} \tag{12}$$

with $P(\Omega) = 1$. It is easy to check that $\mathrm{core}(\nu) = \{P\}$. However, there are no radial sets that are linear, i.e. there are no sets A such that $P(A) \in (0, 1)$ and $\nu(A) + \nu(A^{\mathrm{c}}) = 1$.

Conditions (i) and (ii) of Theorem 21 are both very mild requirements. In particular, condition (i) is more demanding on P, which is required to be countable additivity rather than just finite additive, but less on g, which is only required to be lower semicontinuous rather than calm from below.

As to the existence of linear and radial sets, measure games $g(P)$ that are relevant for economic applications typically feature some homogeneity conditions of the function $g : R(P) \to \mathbb{R}$, and these conditions guarantee the existence of many linear and radial sets for the measure game $g(P)$.

For instance, say that the measure game $\nu = g(P) : \Sigma \to \mathbb{R}$ is *radially concave* at E if, for all $t \in (0, 1)$,

$$g(tP(E) + (1-t)P(E^{\mathrm{c}})) \geq tg(P(E)) + (1-t)g(P(E^{\mathrm{c}})). \tag{13}$$

Obviously, ν is radially concave at E if and only if it is radially concave at E^{c}, and ν is radially concave at all sets E in Σ when g is concave.

Definition 22. A measure game $\nu = g(P) : \Sigma \to \mathbb{R}$ is called radially concave if there is some linear set A such that ν is radially concave at A.

For example, since Ω is a linear set, ν is radially concave if, for all $t \in (0, 1)$,

$$g(tP(\Omega)) \geq tg(P(\Omega)),$$

a very mild homogeneity requirement. Another simple case in which ν is radially concave is when the set A such that $P(A) = 2^{-1}P(\Omega)$ is linear. In this case, Eq. (13) is trivially satisfied.

Radial concavity is a weak condition satisfied by many economic TU games. For instance, measure games whose functions $g : R(P) \to \mathbb{R}$ are concave or homogeneous of degree one are radially concave, as well as the measure games that have a function $g : R(P) \to \mathbb{R}$ homogeneous of degree $k < 1$, provided $g(P(\Omega)) \geq 0$. In particular, market games are radially concave, as their function g is homogeneous of degree one.

Radially concave games that have non-empty cores admit many radial and linear sets, and consequently, by Theorem 21, they are linear. This is stated in the next Corollary.

Corollary 23. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a radially concave measure game and suppose one of conditions* (i) *and* (ii) *of* Theorem 21 *holds. Then, the game ν is linear and for each linear set A,*

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\}.$$

Moreover, $\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(2^{-1}P(\Omega))\}$ *provided* $\mathrm{core}(\nu)$ *is non-empty*.

Remark. Interestingly, here $\mathrm{core}(\nu)$ is determined by the superdifferential of g at $2^{-1}P(\Omega)$, the center of symmetry of $R(P)$.

Example. Let $g : \mathbb{R}^N_+ \to \mathbb{R}$ be a concave and positive homogeneous function and assume $P(\Omega) \in \mathbb{R}^N_{++}$. Consider the following two broad classes of functions:

$$g_1(x) = g(x) + h_1(x),$$

$$g_2(x) = g(x)h_2(x),$$

for all $x \in \mathbb{R}^N_+$. If $h_1(tP(\Omega)) = 0$ and $h_2(tP(\Omega)) = 1$ for all $t \geq 0$, then the games $g_1(P)$ and $g_2(P)$ are radially concave. In view of Corollary 23, it is easy to provide conditions under which the cores of these measure games are non-empty. For instance, for the first class it suffices that $\partial h_1(2^{-1}P(\Omega)) \neq \varnothing$, while for the other class it is enough to require that $h_2(x) \in [0, 1]$ for all $x \in \mathbb{R}^N_+$.

In Theorem 21 and in Corollary 23, we only assumed that the real-valued function g was defined on the range $R(P)$. In applications, however, it is often the case that the function g defining the measure game is defined on an open convex subset G containing $R(P)$, for example $\mathbb{R}^N$ itself. In this case, we have two superdifferentials, the one of g restricted to $R(P)$, i.e. $\partial g_{|R(P)}(x)$, and the one that g has relative to the open convex subset G, i.e. $\partial g(x)$. Naturally, $\partial g_{|R(P)}(x)$ is the superdifferential relevant for Theorem 21 and Corollary 23. On the other hand, the superdifferential $\partial g(x)$ may be easier to compute, especially when g is defined on $\mathbb{R}^N$.

The next result can therefore be useful, as it shows that it is possible to use directly $\partial g(P(A))$ when g is concave and A radial.

Proposition 24. *Let $P = (P_1, \dots, P_N) : \Sigma \to \mathbb{R}$ be a strongly continuous vector charge and let $g : G \to \mathbb{R}$ be a concave function, where G is an open convex set containing $R(P)$. For the measure game $\nu = g(P) : \Sigma \to \mathbb{R}$, it holds that*

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^c))\},$$

for each linear and radial set A, and $\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(2^{-1}P(\Omega))\}$ *provided* $\mathrm{core}(\nu)$ *is non-empty.*

Remark. In reading Proposition 24, recall that a bounded concave function $g : R(P) \to \mathbb{R}$ can be extended to a concave function on the whole space $\mathbb{R}^N$ if and only if g is Lipschitz on $R(P)$.

Example (Generalized linear production games). Let us apply the last Proposition to an important class of linear games. Let $a : T \to \mathbb{R}^N$ be a continuous map, where T is a compact metric space, and define a function $g : \mathbb{R}^N \to \mathbb{R}$ by $g(x) = \min_{t \in T} a(t) \cdot x$ for all $x \in \mathbb{R}^N$. Consider the measure game $\nu = g(P)$, which we call a *generalized linear production game*. When T is a finite set and $a(t) \equiv a^t \in \mathbb{R}^N_+$, we have the linear production games of Owen (1975) and Billera and Raanan (1981). Since the function g is concave on $\mathbb{R}^N$, by a standard result in Convex Analysis (see, e.g. Hiriart-Urruty and Lemarechal, 1993), we have

$$\partial g(x) = \mathrm{co}(a(t) : t \in I(x)),$$

where $I(x) = \{t : a(t) \cdot x = g(x)\}$. Consider a diagonal set A with $P(A) = \alpha P(\Omega)$ for some $\alpha \in (0, 1)$. Simple algebra shows that

$$I(P(A)) = I(P(\Omega)) = \{t : a(t) \cdot P(\Omega) = g(P(\Omega))\}.$$

Since each diagonal set is linear, by Proposition 24,

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \mathrm{co}(a(t) : a(t) \cdot P(\Omega) = \nu(\Omega))\}.$$

This includes Corollary 2.7 of Billera and Raanan (1981), which therefore follows from Proposition 24 using standard Convex Analysis.

5.3. Differentiability

Proposition 16 characterized the linear core of a measure game $\nu = g(P) : \Sigma \to \mathbb{R}$ through the superdifferentials of the function $g : R(P) \to \mathbb{R}$, and Theorem 21 provided a simple condition under which the entire core can be characterized in this way. In view of standard subcalculus and in view of Proposition 13, it is natural to wonder what happens when some differentiability is assumed on g, in particular, whether the core shrinks to a singleton.

Proposition 25. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game and suppose one of conditions* (i) *and* (ii) *of* Theorem 21 *holds. If there is a linear and radial set A such that g is differentiable at $P(A)$, then*

$$\text{core}(\nu) = \varnothing \quad \text{or} \quad \text{core}(\nu) = \{\nabla g(P(A)) \cdot P(\cdot)\}.$$

If, in addition, g is differentiable and superdifferentiable at both $P(A)$ and $P(A^c)$, then $\text{core}(\nu) \neq \varnothing$ *if and only if $\nabla g(P(A)) = \nabla g(P(A^c))$.*

Differentiability has therefore a remarkably strong impact on the core: even just assuming that g is differentiable at $P(A)$ forces the core to be at most a singleton.

Example. Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a market game, that is, ν is superadditive and g is homogeneous of degree one. If g is differentiable at $P(\Omega)$, then $\text{core}(\nu) = \{\nabla g(P(\Omega)) \cdot P\}$. In fact, by Proposition 10, $\partial g(P(E)) \neq \varnothing$ for all $E \in \Sigma$. Moreover, all diagonal sets are linear and g is differentiable at all them because it is differentiable at $P(\Omega)$. In particular, $\nabla g(P(A)) = \nabla g(P(\Omega))$ for all diagonal sets. Hence, by Proposition 25, $\text{core}(\nu) = \{\nabla g(P(\Omega)) \cdot P\} = \{\nabla g(P(\Omega)) \cdot P\}$. This result is essentially due to Aumann and Shapley (1974) and plays a key role in their analysis of exchange economies. Interestingly, in our approach this result follows easily from Proposition 25.

Unlike Proposition 25, the next result does not require A to be radial, at the cost of a stronger assumption on the function g.

Corollary 26. *Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game and suppose one of conditions* (i) *and* (ii) *of* Theorem 21 *holds. If there is a linear set A such that ν is radially concave at A and g is differentiable on some neighborhood U of $P(A)$, then*

$$\text{core}(\nu) = \varnothing \quad \text{or} \quad \text{core}(\nu) = \{\nabla g(P(A)) \cdot P(\cdot)\}.$$

Remark. If $g : R(P) \to \mathbb{R}$ is concave and differentiable at $P(A)$, then the corollary holds.[4]

[4] Recall that g can be differentiable at $P(A)$ only if g is defined (or can be extended) on a suitable open subset of $P(A)$.

6. Concluding remarks

1. Theorem 21 generalizes in several ways a well-known result of Billera and Raanan (1981), which establishes that cores of some measure games consist of linear combinations of measures (Corollary 2.6, p. 422).

   First, their result requires the existence of a linear set A such that both $P(A) = 2^{-1}P(\Omega)$ and $\nu(A) = 2^{-1}\nu(\Omega)$. We only require A to be a linear and radial set.

   Second, they require that $\nu \in pNA'$, the supnorm closure of polynomial functions of several non-atomic measures defined on a space isomorphic to $[0, 1]$ with its Borel sets (Aumann and Shapley, 1974, p. 152). This topological structure is crucial for their results, and $g(P) \in pNA'$ if and only if g is continuous on $R(P)$. In contrast, we do not make any topological assumption, and our result holds for any measure game ν.

   Third, their Corollary 2.7 establishes the positivity of the coefficients of the linear combinations for non-atomic linear production games, a special class of measure games whose functions g are monotone. Our Theorem 21, instead, holds for any measure game having a monotone function g.

   Finally, Theorem 21 follows from a subcalculus approach to the core and from a general linearity result for vector measures that put this result in a broader perspective. In particular, Proposition 25 and Corollary 26 are a dividend of this more general approach.

   Notice that Corollary 2.6 of Billera and Raanan (1981) is stated for ν that are not necessarily measure games, while our theory has been developed for measure games. However, it is easy to formulate a similar version of Theorem 21, as follows.

   Proposition 27. *Let A be a linear set of a game ν continuous at $\varnothing$ and at Ω. Assume there exists a positive non-atomic vector measure $P = (P_1, \dots, P_N)$ such that*:

   (i) *A is radial in $R(P)$,*
   (ii) *for all E, $P(E) = P(A) \Rightarrow \nu(E) = \nu(A)$ and $\nu(E^c) = \nu(A^c)$.*

   Then, $\mathrm{core}(\nu) \subseteq \mathrm{span}\{P_1, \dots, P_N\}$.

2. Corollary 23 extends some recent interesting results of Einy et al. (1999). Using different techniques, they prove (Theorem C) a special case of Corollary 23 for measure games whose function $g : R(P) \to \mathbb{R}$ is concave and continuous at $P(\Omega)$, rather than for general radially concave measure games, as we can do on the basis of our generalization of Billera and Raanan (1981).
3. Linear cores are very tractable objects. In fact, it is easy to check that to compute the linear core of a measure game $g(P)$ is enough to solve the following optimization problem in $\mathbb{R}^N$:

$$\min_{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N} \sum_{i=1}^{N} \alpha_i P_i(\Omega),$$

$$\text{s.t.} \sum_{i=1}^{N} \alpha_i x_i \geq g(x_1, \dots, x_N) \quad \text{for all } (x_1, \dots, x_N) \in R(P).$$

This problem is linear and involves finitely many variables—the coefficients $(\alpha_1, \dots, \alpha_N)$—that appear in infinitely many constraint—the inequalities $\sum_{i=1}^{N} \alpha_i x_i \geq g(x_1, \dots, x_N)$ with $(x_1, \dots, x_N) \in R(P)$. Problems of this type are called semi-infinite linear problems and there is a large literature dealing with their theoretical and computational features (see, e.g. Goberna and Lopez, 1998). Since they involve only finitely many variables, computationally they are in general more tractable than standard infinite programs and it is often possible to study them via their finite linear subprograms.

4. In proving the results on measure games of Section 5 we made use of the following result, which might be of independent interest.

Proposition 28. *Let* $\nu = g(P) : \Sigma \to \mathbb{R}$ *be a measure game. Then, all elements in* $\mathrm{core}(\nu)$ *are non-atomic. Moreover, they are strongly continuous provided one of the following conditions holds*:

(i) f *is calm from below at* 0 *and* $P(\Omega)$;
(ii) f *is lower semicontinuous at* 0 *and* $P(\Omega)$, *and* P *is countably additive*.

Appendix A. Proofs

Lemma 1. Given any Σ-measurable simple function $f : \Omega \to \mathbb{R}$, the Choquet integral $\int f \, \mathrm{d}\nu$ is still well defined. Now, let $f, g : \Omega \to \mathbb{R}$ be any two Σ-measurable simple functions. Let $\Sigma_{f,g}$ be the smallest algebra that makes f and g measurable. As $\Sigma_{f,g}$ is finite, there is a (possibly zero) measure m on $\Sigma_{f,g}$ such that $\nu(E) \geq m(E)$ for all $E \in \Sigma_{f,g}$. Hence, $\nu - m$ is a positive convex game on $\Sigma_{f,g}$, and so, by a classic result of Choquet (1953), $\int (f+g) \, \mathrm{d}(\nu - m) \geq \int f \, \mathrm{d}(\nu - m) + \int g \, \mathrm{d}(\nu - m)$. In turn, this obviously implies $\int (f+g) \, \mathrm{d}\nu \geq \int f \, \mathrm{d}\nu + \int g \, \mathrm{d}\nu$. We conclude that the Choquet integral $\int f \, \mathrm{d}\nu$ is a superadditive functional on the vector space $B_0(\Sigma)$ of Σ-measurable simple functions.

Set $C = \{m : m \text{ is finitely additive}, m \geq \nu \text{ and } m(\Omega) = \nu(\Omega)\}$. Since ν is bounded, $C \subseteq \mathrm{ba}(\Omega)$, and so $C = \mathrm{core}(\nu)$. Now, let Σ^* be a subalgebra of Σ on which there is a charge $m^* : \Sigma^* \to \mathbb{R}$ such that $m^* \in \mathrm{core}(\nu_{|\Sigma^*})$. Since $\int f \, \mathrm{d}\nu$ is a superadditive functional on $B_0(\Sigma)$, by the Hahn–Banach Theorem there is a finitely additive extension $m : \Sigma \to \mathbb{R}$ of m^* such that $m \in C$. Then $m \in \mathrm{core}(\nu)$.

Consider now the chain $\{E_i\}_{i \in I}$. Let Σ_J be the algebra generated by a finite subchain $\{E_i\}_{i \in J}$. Let $m_J : \Sigma_J \to \mathbb{R}$ be the, possibly zero, charge on Σ_J such that $\nu(E) \geq m_J(E)$ for all $E \in \Sigma_J$. By well-known results (see, e.g. ?), there exists $m' \in \mathrm{core}(\nu - m_J)$ such that $m'(E_i) = (\nu - m_J)(E_i)$ for all $i \in J$. Hence, there is $m^* \in \mathrm{core}(\nu_{|\Sigma_J})$ such that $m^*(E_i) = \nu(E_i)$ for all $i \in J$. In turn, this implies the existence of an extension $m : \Sigma \to \mathbb{R}$ of m^* such that $m \in C$, and so $m \in \mathrm{core}(\nu)$.

Let $\Lambda_J = \{m \in \mathrm{core}(\nu) : m(E_j) = \nu(E_j) \text{for all } j \in J\}$. Since $\mathrm{core}(\nu)$ is weak*-compact, the set Λ_J is weak*-compact. Moreover, by what we just proved, $\Lambda_J \neq \varnothing$. The collection $\{\Lambda_J\}_{\{J : J \subseteq I \text{ and } |J| < \infty\}}$ has the finite intersection property, and so its overall intersection is non-empty. Let m be an element of such intersection. We have $m \in \mathrm{core}(\nu)$ and $m(E_i) = \nu(E_i)$ for all $i \in I$, as desired.

Proof of Proposition 5. The inclusion $\partial\nu(E) \subseteq \partial\nu(E) + K_{E^c}$ is obvious. As to the opposite inclusion, let $m \in \partial\nu(E) + K_{E^c}$. Then, for suitable $m_1 \in \partial\nu(E)$ and $m_2 \in K_{E^c}$, we have, for all $A \in \Sigma$,

$$\begin{aligned} \nu(E) + m(A) - m(E) &= \nu(E) + m_1(A) + m_2(A) - m_1(E) - m_2(E) \\ &\geq \nu(A) + m_2(A) - m_2(E) \\ &= \nu(A) + m_2(E^c \cap A) - m_2(E \cap A^c) \geq \nu(A), \end{aligned} \qquad \square$$

and so $m \in \partial\nu(E)$.

Proof of Proposition 6. Let ν be exact and let $E \in \Sigma$. By definition, there exists $m \in \text{core}(\nu)$ such that $m(E) = \nu(E)$. Since $m(A) \geq \nu(A)$ for all $A \in \Sigma$, it follows that $m \in \partial\nu(E)$, and so $\partial\nu(E) \cap \text{core}(\nu) \neq \emptyset$. Conversely, suppose that $\partial\nu(E) \cap \text{core}(\nu) \neq \emptyset$ for each $E \in \Sigma$. It is easy to check that $m \in \partial\nu(E) \cap \text{core}(\nu)$ implies $m(E) = \nu(E)$, and so ν is exact. $\square$

Proof of Proposition 7. By Lemma 1, there exists $m \in \text{core}(\nu)$ such that $m(E_1) = \nu(E_1)$ and $m(E_2) = \nu(E_2)$. This immediately implies that $m \in \partial\nu(E_1) \cap \partial\nu(E_2)$, and so $\partial\nu(E_1) \cap \partial\nu(E_2) \neq \emptyset$. As to the converse, suppose that $\partial\nu(E_1) \cap \partial\nu(E_2) \neq \emptyset$ for every $E_1 \subseteq E_2$. Let E and E' be any two sets of Σ. Let $m \in \partial\nu(E \cap E') \cap \partial\nu(E \cup E')$. Then, $\nu(E) \leq \nu(E \cup E') - m(E \cup E') + m(E)$ and $\nu(E') \leq \nu(E \cap E') - m(E \cap E') + m(E')$. By adding up we get $\nu(E) + \nu(E') \leq \nu(E \cup E') + \nu(E \cap E')$, as desired. Hence, ν is convex if and only if $\partial\nu(E_1) \cap \partial\nu(E_2) \neq \emptyset$. $\square$

Proof of Theorem 8. Let $B_1^+(\Sigma) = \{f \in B(\Sigma) : 0 \leq f \leq 1\}$, which are the ideal sets in the terminology of Aumann and Shapley (1974). Given a bounded convex game ν, consider the functional $\nu^* : B(\Sigma) \to \mathbb{R}$ defined by

$$\nu^*(f) = \begin{cases} \int_0^{+\infty} \nu(f \geq t)\, \mathrm{d}t & f \in B_1^+(\Sigma), \\ -\infty & f \notin B_1^+(\Sigma). \end{cases}$$

The integral $\int_0^{+\infty} \nu(f \geq t)\, \mathrm{d}t$ is the Choquet integral of f w.r.t. ν. It is easy to check that, by Lemma 1, $\int_0^{+\infty} \nu(f \geq t)\, \mathrm{d}t$ is a well defined Riemann integral. Again by Lemma 1, it is easy to check that ν^* is a proper concave function on $B(\Sigma)$. Moreover, $\nu^*(1_E) = \nu(E)$ for all $E \in \Sigma$.

Given $m \in \text{ba}(\Omega)$, let $L_m : B(\Sigma) \to \mathbb{R}$ be defined by $L_m(f) = \int f\, \mathrm{d}m$. Let $\partial\nu^*(f)$ be the standard superdifferential of $\nu^* : B(\Sigma) \to \mathbb{R}$ at f. We show that $\partial\nu^*(1_E) = \{L_m : m \in \partial\nu(E)\}$ for all $E \in \Sigma$. Clearly, $\partial\nu^*(1_E) \subseteq \{L_m : m \in \partial\nu(E)\}$ for all $E \in \Sigma$. As to the converse inclusion, let $m \in \partial\nu(E)$. By definition, $\nu(A) - m(A) \leq \nu(E) - m(E)$ for all $A \in \Sigma$. Hence, for all $f \in B_1^+(\Sigma)$,

$$\nu^*(f) - L_m(f) = \int_0^1 (\nu - m)(f \geq t)\, \mathrm{d}t \leq [\nu(E) - m(E)] \int_0^1 \mathrm{d}t,$$

and so $L_m \in \partial\nu^*(1_E)$. This proves the converse inclusion, so that $\partial\nu^*(1_E) = \{L_m : m \in \partial\nu(E)\}$ for all $E \in \Sigma$. With a slight abuse of notation, we write $\partial\nu^*(1_E) = \partial\nu(E)$.

There is $f \in B_1^+(\Sigma)$ in a neighborhood of which (w.r.t. the norm topology) both ν_1^* and ν_2^* are bounded (e.g. $f = \alpha 1_\Omega$ for some $\alpha \in (0, 1)$). Then, by Theorem 20 of Rockafellar (1974), for all $E \in \Sigma$ we have:

$$\begin{aligned}\partial(\nu_1 + \nu_2)(E) &= \partial(\nu_1 + \nu_2)^*(E) = \partial(\nu_1^* + \nu_2^*)(E)\\ &= \partial\nu_1^*(E) + \partial\nu_2^*(E) = \partial\nu_1(E) + \partial\nu_2(E).\end{aligned}$$

□

Proof of Proposition 10. Considering condition (i), the function g admits a concave extension on the entire space $\mathbb{R}^N$. Hence, we can think of g as defined on $\mathbb{R}^N$. Let $x_0 \in R(P)$. Since g is concave on $\mathbb{R}^N$, by a standard result in Convex Analysis (see Theorem 23.4 of Rockafellar (1970)) there is a vector $\chi \in \mathbb{R}^N$ such that $g(x) \le g(x_0) + \chi \cdot (x - x_0)$ for all $x \in \mathbb{R}^N$. Hence, $\chi \cdot P \in \partial\nu(E)$ if $P(E) = x_0$.

Now considering condition (ii), let K and W be, respectively, the cone and subspace generated by the convex set $R(P)$. Since $0 \in R(P)$, $W = K - K$ (see Theorem 2.7 of Rockafellar (1970)). Define the function $g' : K \to \mathbb{R}$ by $g'(\lambda x) = \lambda g(x)$ with $x \in R(P)$ and $\lambda > 0$. The function g' is well-defined and it is superadditive and homogeneous of degree one on K. Define $g'' : W \to \mathbb{R}$ by $g''(w) = \sup\{g'(x) + g'(y) : x, y \in K \text{ and } x - y = w\}$. The function g'' as well is superadditive and homogeneous of degree one on W. Given any $x_0 \in R(P)$, let W_0 be the subspace of W generated by x_0, i.e. $W_0 = \{\alpha x_0 : \alpha \in \mathbb{R}\}$. Define the linear function $L_0 : W_0 \to \mathbb{R}$ by $L_0(\alpha x_0) = \alpha g(x_0)$ for all $\alpha \in \mathbb{R}$. If $\alpha \ge 0$, clearly $L_0(\alpha x_0) = g''(\alpha x_0)$ for all $w \in W_0$. If $\alpha < 0$, we have:

$$L(\alpha x_0) = \alpha g(x_0) = (-\alpha)(-g(x_0)) \ge (-\alpha) g''(-x_0) = g''(\alpha x_0).$$

By the Hahn–Banach Theorem, there exists a linear functional $L : W \to \mathbb{R}$ that extends L_0 on W and such that $L(w) \ge g''(w)$ for all $w \in W$. Since W is a subspace of $\mathbb{R}^N$, there exists a linear functional $L^* : \mathbb{R}^N \to \mathbb{R}$ that extends L on $\mathbb{R}^N$. Let $\chi^* \in \mathbb{R}^N$ such that $L^*(x) = \chi^* \cdot x$ for all $x \in \mathbb{R}^N$. Then, $\chi^* \cdot w \ge g''(w)$ for all $w \in W$ and $\chi^* \cdot x_0 = g(x_0)$. Hence, given any $x \in R(P)$, we have $g(x) - g(x_0) \le \chi^* \cdot x - \chi^* \cdot x_0$, which implies $\chi^* \in \partial g(x_0)$. We conclude that $\partial g(x_0) \ne \varnothing$, as desired. □

Proof of Theorem 11. We first prove that (i) implies (ii). Let $E \in \mathcal{A}$. It is easy to see that $\mathrm{core}(\nu) \subseteq \partial\nu(E) \cap \partial\nu(E^c)$. In fact, for each $m \in \mathrm{core}(\nu)$ it holds that $m \ge \nu$, $m(E) = \nu(E)$, and $m(E^c) = \nu(E^c)$. We now prove the converse inclusion, that is, $\partial\nu(E) \cap \partial\nu(E^c) \subseteq \mathrm{core}(\nu)$. Let $m \in \partial\nu(E) \cap \partial\nu(E^c)$. Since

$$0 = \nu(\varnothing) \le \nu(E) - m(E),$$

we have $m(E) \le \nu(E)$. Moreover, since

$$\nu(\Omega) = \nu(E \cup E^c) \le \nu(E) + m(E^c),$$

we have $m(E^c) \ge \nu(E^c)$. By taking E^c in place of E, a similar argument shows that $m(E) \ge \nu(E)$ and $m(E^c) \le \nu(E^c)$, and we conclude that $m(E) = \nu(E)$ and $m(E^c) = \nu(E^c)$. Finally, each set $B \in \Sigma$ can be written as $B = E \cup F - G$, with $F \cap E = \varnothing$ and $G \subseteq E$. Hence,

$$\nu(B) = \nu(E \cup F - G) \le \nu(E) + m(F) - m(G) = m(E) + m(F) - m(G) = m(B),$$

and so $m \in \mathrm{core}(\nu)$.

Next we prove that (iii) implies (i) when $\mathrm{core}(\nu) \neq \varnothing$. Let $m \in \partial\nu(E) \cap \partial\nu(E^{\mathrm{c}})$. As we have just seen, this implies that $m(E^{\mathrm{c}}) \leq \nu(E^{\mathrm{c}})$ and that $\nu(\Omega) \leq \nu(E) + m(E^{\mathrm{c}})$. Since $\mathrm{core}(\nu) \neq \varnothing$, we have $\nu(E) + \nu(E^{\mathrm{c}}) \leq \nu(\Omega)$. Hence,

$$\nu(\Omega) \leq \nu(E) + m(E^{\mathrm{c}}) \leq \nu(E) + \nu(E^{\mathrm{c}}) \leq \nu(\Omega),$$

and we conclude that $E \in \mathcal{A}$. To complete the proof, observe that (ii) obviously implies (iii) when $\mathrm{core}(\nu) \neq \varnothing$. □

Proof of Proposition 12. Suppose that E is a maximum set for ν, that is, $\nu(E) \geq \nu(A)$ for all $A \in \Sigma$. Then, $\delta\nu(\cdot; E) \in K_E$. In fact, we have:

$$\begin{aligned}
0 &= \lim_{\lambda} \sum_{j=1}^{n_\lambda} |\nu(E \cup F^{j,\lambda} - G^{j,\lambda}) - \nu(E) - \delta\nu(F^{j,\lambda}; E) + \delta\nu(G^{j,\lambda}; E)| \\
&\geq \lim_{\lambda} \sum_{j=1}^{n_\lambda} \nu(E) - \nu(E \cup F^{j,\lambda} - G^{j,\lambda}) + \delta\nu(F^{j,\lambda}; E) - \delta\nu(G^{j,\lambda}; E) \\
&\geq \lim_{\lambda} \sum_{j=1}^{n_\lambda} \delta\nu(F^{j,\lambda}; E) - \delta\nu(G^{j,\lambda}; E) = \delta\nu(F; E) - \delta\nu(G; E),
\end{aligned}$$

and so $\delta\nu(G; E) \geq \delta\nu(F; E)$ for each $F \subseteq E^{\mathrm{c}}$ and $G \subseteq E$. In particular, $\delta\nu(G; E) \geq \delta\nu(\varnothing; E) \geq \delta\nu(F; E)$, and we conclude that $\delta\nu(\cdot; E) \in K_E$. Let $m \in \partial\nu(E)$. By definition, $\nu - m$ is a game with maximum at E. Hence, by what we just proved, $\delta(\nu - m)(\cdot; E) \in K_E$, which implies $\delta\nu(\cdot; E) \in m(\cdot) + K_E$, and so $\delta\nu(\cdot; E) \in \partial\nu(E) + K_E$.

Suppose that $E \in \mathcal{A}$ and that $\mathrm{core}(\nu) \neq \varnothing$. Let $m \in \mathrm{core}(\nu)$. The game $\nu - m$ is a game with maximum at E. Hence, $\delta(\nu - m)(\cdot; E) \in K_E$, which, by Theorem 11 and Proposition 5, implies

$$\delta\nu(\cdot; E) \in \mathrm{core}(\nu) + K_E = \partial\nu(E) \cap \partial\nu(E^{\mathrm{c}}) + K_E \subseteq \partial\nu(E^{\mathrm{c}}) + K_E = \partial\nu(E^{\mathrm{c}}),$$

as desired. □

Proof of Proposition 13. Since A is linear, by Theorem 11 $\mathrm{core}(\nu) \subseteq \partial\nu(A)$ and so $\mathrm{core}(\nu) \neq \varnothing$ implies $\partial\nu(A) \neq \varnothing$. Having established that ν is superdifferentiable at A, we can prove that (i) implies (iv). Let $m \in \mathrm{core}(\nu)$. By Theorem 11, $m \in \partial\nu(A^{\mathrm{c}})$, and so $\nu(A) \leq \nu(A^{\mathrm{c}}) + m(A) - m(A^{\mathrm{c}})$. Moreover, $\delta\nu(\cdot; A) \in \partial\nu(A)$ implies $\nu(A^{\mathrm{c}}) \leq \nu(A) + \delta\nu(A^{\mathrm{c}}; A) - \delta\nu(A; A)$. Adding up, we get $\delta\nu(A; A) - m(A) \leq \delta\nu(A^{\mathrm{c}}; A) - m(A^{\mathrm{c}})$. On the other hand, since $\nu - m$ has a maximum at A, by what we proved in the proof of Proposition 12 we have $\delta(\nu - m)(\cdot; A) \in K_A$, and so

$$\delta\nu(A; A) - m(A) \geq 0 \geq \delta\nu(A^{\mathrm{c}}; A) - m(A^{\mathrm{c}}).$$

All this implies that $\delta\nu(A; A) - m(A) = \delta\nu(A^{\mathrm{c}}; A) - m(A^{\mathrm{c}}) = 0$. Since $A \in \mathcal{A}$ and $m \in \mathrm{core}(\nu)$, we conclude that $\delta\nu(A; A) = \nu(A)$ and $\delta\nu(A^{\mathrm{c}}; A) = \nu(A^{\mathrm{c}})$.

We now show that (iv) implies (ii). Let $m \in \operatorname{core}(\nu)$. Since $\delta(\nu - m)(\cdot; A) \in K_A$, then for each $G \subseteq A$ it holds that $\delta\nu(G; A) \geq m(G)$. Along with $\delta\nu(A; A) = \nu(A) = m(A)$, this implies that $m(G) = \delta\nu(G; A)$ for each $G \subseteq A$. A similar argument shows that $m(F) = \delta\nu(F; A)$ for each $F \subseteq A^c$, and so $m = \delta\nu(\cdot; A)$, which proves that $\operatorname{core}(\nu) = \{\delta\nu(\cdot; A)\}$.

Since (ii) trivially implies (iii), it remains to prove that (iii) implies (i). Since $\operatorname{core}(\nu) \neq \varnothing$ and $A \in \mathcal{A}$, by Theorem 11 $\delta\nu(\cdot; A) \in \operatorname{core}(\nu) = \partial\nu(A) \cap \partial\nu(A^c) \subseteq \partial\nu(A)$.

Suppose that (i) and (iv) hold for some $A \in \mathcal{A}$. By (iv), $\nu(\Omega) = \delta\nu(\Omega; A)$ and together (i) and (iv) imply that, for all $E \in \Sigma$,

$$\nu(E) \leq \delta\nu(E; A) + \nu(A) - \delta\nu(A; A) = \delta\nu(E; A).$$

Hence, $\delta\nu(\cdot; A) \in \operatorname{core}(\nu)$ and so, by what we proved above, $\operatorname{core}(\nu) = \{\delta\nu(\cdot; A)\}$. □

Proof of Proposition 14. We first prove the following Claim.

Claim. Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game and suppose P is countably additive. If there exists a linear set A such that g is lower semicontinuous at $P(A)$ and $P(A^c)$, then $\operatorname{core}(\nu) \subseteq \operatorname{ca}(\Omega)$.

Proof of Claim. If $\operatorname{core}(\nu) = \varnothing$, the claim is trivial. Hence, we assume $\operatorname{core}(\nu) \neq \varnothing$. The proof follows an argument similar to Aumann and Shapley (1974) (p. 173). Let $E_n \uparrow \Omega$ and let $m \in \operatorname{core}(\nu)$. Then $P(E_n \cap A) \uparrow P(A)$ and $P(E_n \cap A^c) \uparrow P(A^c)$. Thanks to the lower semicontinuity at the points $P(A)$ and $P(A^c)$, we can write:

$$\begin{aligned}
\liminf_n m(A \cap E_n) &\geq \liminf_n g(P(A \cap E_n)) \geq g(P(A)) \\
&= g(P(\Omega)) - g(P(A^c)) \geq g(P(\Omega)) - \liminf_n g(P(A^c \cap E_n)) \geq m(\Omega) \\
&\quad - \liminf_n m(A^c \cap E_n) = \limsup_n m(A \cap E_n),
\end{aligned}$$

and so $\lim_n m(A \cap E_n) = m(A)$, as $g(P(A)) = m(A)$. On the other hand, a similar argument shows that $\lim_n m(A^c \cap E_n) = m(A^c)$. Hence, $\lim_n m(E_n) = m(\Omega)$, which implies that m is countably additive. This ends the proof of the Claim. □

By the Claim, $\operatorname{core}(\nu) \subseteq \operatorname{ca}(\Omega)$. It is also easy to check that each $m \in \operatorname{core}(\nu)$ is absolutely continuous w.r.t. P^*. Actually, suppose $P^*(E) = 0$ for some $E \in \Sigma$. Then, $P_i(E) = 0$ for all i. Hence, $m(E) \geq g(P(E)) = g(0) = 0$. On the other hand,

$$m(\Omega) - m(E) = m(E^c) \geq g(P(E^c)) = g(P(\Omega)) = m(\Omega).$$

Therefore, $m(E) \leq 0$, and we conclude that $m(E) = 0$, and $m \ll P^*$.

By a variation of the Lebesgue Decomposition Theorem, there exist measures $\{m_i\}_{i=1}^N$ such that $m_i \ll P_i$ for each $i = 1, \ldots, N$, and $m(E) = \sum_{i=1}^N m_i(E)$ for all E. Moreover, the measures $\{m_i\}_{i=1}^N$ are mutually singular and $\|m\|(E) = \sum_{i=1}^N \|m\|_i(E)$ for all E (see, e.g. Proposition 8.5.1 of Bhaskara Rao and Bhaskara Rao (1983)). By the Radon–Nikodym Theorem, there exists a Σ-measurable vector function $f = (f_1, \ldots, f_N) : \Omega \to \mathbb{R}^N$ such that, for all $E \in \Sigma$, $m(E) = \sum_{i=1}^N \int_E f_i \, \mathrm{d}P_i$.

Proof of Proposition 16. By Lemma 4, $\chi \cdot P \in \partial\nu(A)$ if and only if $\chi \in \partial g(P(A))$. On the other hand, by Theorem 11, $\chi \cdot P \in \text{core}(\nu)$ if and only if $\chi \cdot P \in \partial\nu(A) \cap \partial\nu(A^c)$. Hence, $\chi \cdot P \in \text{core}(\nu)$ if and only if $\chi \in \partial g(P(A)) \cap \partial g(P(A^c))$. This proves (5).

We now prove (6). Assume that $\mathcal{L}\text{core}(\nu) \neq \varnothing$. Set $\dim(R(P)) = n$ and $\dim(\text{span}\{P(A) : A \in \mathcal{A}\}) = k$. Clearly, $k \leq n \leq N$. Let $\{A_i\}_{i=1}^k \subseteq \mathcal{A}$ be such that the vectors $\{P(A_i)\}_{i=1}^k$ be linearly independent. We shall make use of the canonical isomorphism

$$R : \text{span}\{P_1, \dots, P_N\} \to \text{span}\, R(P),$$

defined by $R(\chi \cdot P) = \pi(\chi)$ for all $\chi \in \mathbb{R}^N$, where $\pi : \mathbb{R}^N \to \text{span}R(P)$ is the orthogonal projection. It is easy to check (see Marinacci and Montrucchio, 2002b) that R is well-defined and is a linear isomorphism. Moreover, we have $m = R(m) \cdot P$ for all $m \in \text{span}\{P_1, \dots, P_N\}$.

Hence, given $m \in \mathcal{L}\text{core}(\nu)$, we have $m(\cdot) = R(m) \cdot P(\cdot)$. Consequently, the set $\{R(m) : m \in \mathcal{L}\text{core}(\nu)\} \subseteq \mathbb{R}^N$ belongs to the affine space M defined by the linear equations $\xi \cdot P(A_i) = \nu(A_i)$ for $i = 1, \dots, k$. Since R is an isomorphism, $\dim(\mathcal{L}\text{core}(\nu)) \leq \dim(M)$. The dimension of M is equal to the dimension of the space M_0 defined by the homogeneous linear equations $\xi \cdot P(A_i) = 0$ for $i = 1, \dots, k$. As $M_0 = \text{span}(\{P(A) : A \in \mathcal{A}\})^{\perp}$, we conclude that $\dim(\mathcal{L}\text{core}(\nu)) \leq \dim(M) = n - k$, as desired.

Finally, being $P(\Omega) \neq 0$, $\dim(\text{span}(\{P(A) : A \in \mathcal{A}\})) \geq 1$, which proves the last inequality of (6). $\square$

Proof of Theorem 20. Assume that $m \in \text{ca}(\Omega)$. We first show that $P(E) = 0$ implies $m(E) = 0$ for all $E \in \Sigma$. In fact, consider the sets $E \cap A$ and $E \cap A^c$. We have $P(E \cap A) = P(E \cap A^c) = 0$, and so $P(A - E \cap A) = P(A)$ and $P(A^c - E \cap A^c) = P(A^c)$. By (7), this implies $m(A - E \cap A) = m(A)$ and $m(A^c - E \cap A^c) = m(A^c)$, so that $m(E \cap A) = m(E \cap A^c) = 0$, and we conclude that $m(E) = m(E \cap A) + m(E \cap A^c) = 0$.

Next we show that m is non-atomic. Let $m(E) \neq 0$. By what has been just proved, $P(E) \neq 0$. In particular, set $J = \{1 \leq i \leq N : P_i(E) > 0\}$ and $\bar{P} = \{P_j\}_{j \in J}$. By Lyapunov Theorem there exists a partition E^1, B^1 of E such that $\bar{P}(E^1) = \bar{P}(B^1) = 2^{-1}\bar{P}(E)$. If both $m(E^1) \neq 0$ and $m(B^1) \neq 0$, we are done. Suppose, in contrast, either $m(E^1) = 0$ or $m(B^1) = 0$. W.l.o.g., suppose that $m(E^1) = m(E)$. Again by Lyapunov Theorem, there exists a partition E^2 and B^2 of E^1 such that $\bar{P}(E^2) = \bar{P}(B^2) = (1/2)\bar{P}(E^1)$. If both $m(E^2) \neq 0$ or $m(B^2) \neq 0$, we are done. Suppose, in contrast, that either $m(E^2) = 0$ or $m(B^2) = 0$. W.l.o.g., assume that $m(E^2) = m(E^1)$. Proceeding in this way, either we find a set $B \subseteq E$ such that both $m(B) \neq 0$ and $m(E - B) \neq 0$, or we can construct a chain $\{E^n\}_{n \geq 1}$ such that $\bar{P}(E^n) = 2^{-n}\bar{P}(E)$ and $m(E^n) = m(E)$ for all $n \geq 1$. Hence, being $\bigcap_{n \geq 1} E^n \in \Sigma$, and $\bigcap_{n \geq 1} E^n \subseteq E$, we have $\bar{P}(\bigcap_{n \geq 1} E^n) = 0$ and $m(\bigcap_{n \geq 1} E^n) = m(E) \neq 0$, a contradiction since we have $\bar{P}(\bigcap_{n \geq 1} E^n) = 0$ iff $P(\bigcap_{n \geq 1} E^n) = 0$. Hence, there exists some set $B \subseteq E$ such that both $m(B) \neq 0$ and $m(E - B) \neq 0$, and so m is non-atomic.

Therefore, under both hypotheses on m, m is strongly continuous. Consequently, by the Lyapunov Theorem, the range $R(P, m)$ of $(P, m) : \Sigma \to \mathbb{R}^{N+1}$ is a convex subset of $\mathbb{R}^{N+1}$. Set $W = \text{span}(R(P, m))$ and let

$$R_{P(A)} = \{x \in \mathbb{R} : (P(A), x) \in R(P, m)\},$$

where A is the set of Eq. (7). By Eq. (7), $R_{P(A)} = \{m(A)\}$. Hence, by Theorem 6.8 of Rockafellar (1970), $(P(A), m(A)) \in \mathrm{ri}(R(P, m))$. In turn, this implies that $(0, 1) \notin W$. For, suppose to the contrary that $(0, 1) \in W$. Since $(P(A), m(A)) \in \mathrm{ri}(R(P, m))$, there is $t > 0$ small enough so that $(P(A), m(A)) + t(0, 1) \in R(P, m)$. Since this contradicts Eq. (7), we conclude that $(0, 1) \notin W$.

By a standard separation theorem (see, e.g. Corollary 11.4.2 of Rockafellar (1970)), there is $\pi \in \mathbb{R}^{N+1}$ and $\alpha \in \mathbb{R}$ such that,

$$\pi \cdot y < \alpha < \pi \cdot (0, 1),$$

for all $y \in W$. As $0 \in W$, $\alpha > 0$. Hence, $\pi \cdot (0, 1) > \alpha > 0$ implies $\pi_{N+1} > 0$. Moreover, since W is a vector space, for each $y \in W$ we have $\pi \cdot (\lambda y) < \alpha$ for all $\lambda \in \mathbb{R}$. Clearly, this implies $\pi \cdot y = 0$, for all $y \in W$. Therefore, we have $\pi_{N+1} m(E) + \sum_{i=1}^{N} \pi_i P_i(E) = 0$, for all $(P(E), m(E)) \in R(P, m)$. We conclude that $m \in \mathrm{span}\{P_1, \dots, P_N\}$, with coefficients $\{-(\pi_i/\pi_{N+1})\}_{i=1}^{N}$.

Finally, if Eq. (9) holds, then $\{(x, -1) : x \in \mathbb{R}_+^N\} \cap W = \varnothing$. By now, it is easy to see that, by applying a standard separation result on these two closed and disjoint convex sets, we can find a vector $\pi \in \mathbb{R}^{N+1}$ with $\pi_i/\pi_{N+1} \leq 0$ for all $i = 1, \dots, N$, and such that $\pi_{N+1} m(E) + \sum_{i=1}^{N} \pi_i P_i(E) = 0$. Hence, $m \in \mathrm{cone}\{P_1, \dots, P_N\}$. □

Proof of Theorem 21. Let A be linear and radial. By Theorem 11, $\mathrm{core}(\nu) = \partial\nu(A) \cap \partial\nu(A^{\mathrm{c}})$. Suppose $\mathrm{core}(\nu) \neq \varnothing$. For all $m \in \mathrm{core}(\nu)$ and for all $E \in \Sigma$ we have:

$$g(P(E)) \leq g(P(A)) + m(E) - m(A),$$

$$g(P(E^{\mathrm{c}})) \leq g(P(A^{\mathrm{c}})) + m(E^{\mathrm{c}}) - m(A^{\mathrm{c}}).$$

Hence, $P(E) = P(A)$ implies $m(E) = m(A)$, and so Eq. (7) of Theorem 20 holds. To complete the proof we now prove a Claim.

Claim. Let $\nu = g(P) : \Sigma \to \mathbb{R}$ be a measure game. If g is calm from below at 0 and $P(\Omega)$, there exists $\gamma > 0$ such that $\|m\|(E) \leq \gamma P^*(E)$ for all $E \in \Sigma$ and all $m \in \mathrm{core}(\nu)$.

Proof of Claim. Let $m \in \mathrm{core}(\nu)$. By Theorem 11, $m \in \partial\nu(\Omega)$ and $m \in \partial\nu(\varnothing)$. Since $m \in \partial\nu(\Omega)$,

$$g(P(E)) - g(P(\Omega)) \leq -m(E^{\mathrm{c}}), \tag{A.1}$$

for all $E \in \Sigma$. Moreover, since g is calm from below at $P(\Omega)$, there exists $\gamma > 0$ and $\varepsilon_1 > 0$ such that

$$g(P(\Omega)) - g(P(E)) \leq \gamma_1 |P(E) - P(\Omega)|, \tag{A.2}$$

for all $E \in \Sigma$ such that $|P(E) - P(\Omega)| \leq \varepsilon_1$. Hence, Eqs. (A.1) and (A.2) imply that $m(E^{\mathrm{c}}) \leq \gamma_1 |P(E^{\mathrm{c}})|$ for all $E \in \Sigma$ such that $|P(E^{\mathrm{c}})| \leq \varepsilon_1$. Since this holds for all $E \in \Sigma$, this implies that $m^+(E) \leq \gamma_1 |P(E)|$ for all $E \in \Sigma$ such that $|P(E)| \leq \varepsilon_1$. On the other hand, since $m \in \partial\nu(\varnothing)$, we have $g(P(E)) \leq m(E)$ for all $E \in \Sigma$, and being g calm from below at 0, there exists $\gamma_2 > 0$ and $\varepsilon_2 > 0$ such that $g(P(E)) \geq -\gamma_2 |P(E)|$ for all $E \in \Sigma$

such that $|P(E)| \leq \varepsilon_2$. Hence, $m^-(E) \leq \gamma_2|P(E)|$ for all $E \in \Sigma$ such that $|P(E)| \leq \varepsilon_2$. Setting $\bar{\gamma} = \gamma_1 \vee \gamma_2$ and $\bar{\varepsilon} = \varepsilon_1 \wedge \varepsilon_2$, all this implies that $|m|(E) \leq 2\bar{\gamma}|P(E)|$ for all $E \in \Sigma$ such that $|P(E)| \leq \bar{\varepsilon}$. Since P is positive, there also exists $\gamma > 0$ such that

$$|m|(E) \leq \bar{\gamma}|P(E)| \leq \gamma P^*(E). \tag{A.3}$$

By the strong continuity of the component measures P_i, for each $E \in \Sigma$ there exists a partition $\{E_k\}_{k=1}^K$ of E in Σ such that $P^*(E_k) \leq \bar{\varepsilon}$ for each $k = 1, \dots, K$. Hence, by (A.3),

$$|m|(E) = \sum_{k=1}^{K} |m|(E_k) \leq \gamma \sum_{k=1}^{K} P^*(E_k) = \gamma P^*(E),$$

as desired. This ends the proof of the Claim. □

Summing up, if (i) holds, then, by the Claim proved in Proposition 14, we have $\mathrm{core}(\nu) \subseteq \mathrm{ca}(\Omega)$, while if (ii) holds, then, by the above Claim, all charges in $\mathrm{core}(\nu)$ are strongly continuous. Hence, under both (i) and (ii) we can apply Theorem 20, and we conclude that $m \in \mathrm{span}\{P_1, \dots, P_N\}$ and the game is linear. By Proposition 16,

$$\mathrm{core}(\nu) = \mathcal{L}\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\},$$

for any linear set A. This proves Eq. (11).

If g is monotone, then for all $m \in \mathrm{core}(\nu)$ and for all $E \in \Sigma$ we have:

$$m(E) - m(A) \geq g(P(E)) - g(P(A)) \geq 0,$$

whenever $P(E) \geq P(A)$. Hence, Eq. (9) of Theorem 20 holds and therefore, the vector χ can be chosen to be non-negative.

The only non-trivial part it remains to prove is that exact linear games have radial and linear sets. Given $t \in (0, 1)$, let A be the diagonal set such that $P(A) = tP(\Omega)$. Given $m \in \mathrm{core}(\nu)$, for a suitable $\chi \in \mathbb{R}^N$ we have $m(A) = \chi \cdot P(A) = t\chi \cdot P(\Omega) = t\nu(\Omega)$. Hence, $m(A) = t\nu(\Omega)$ for all $m \in \mathrm{core}(\nu)$ and so, by exactness, $\nu(A) = t\nu(\Omega)$. A similar argument shows that $\nu(A^{\mathrm{c}}) = (1 - t)\nu(\Omega)$, and we conclude that A is linear.

Proof of Corollary 23. In view of Theorem 21, it suffices to prove that radially concave measure games admit radial and linear sets. Let g satisfy (13) for some linear set E, and $\mathrm{core}(\nu) \neq \varnothing$. Suppose first that $P(E) \neq P(E^{\mathrm{c}})$. By the Lyapunov Theorem, for each $\alpha \in (0, 1)$ there exists $E_\alpha \in \Sigma$ such that $P(E_\alpha) = \alpha P(E) + (1-\alpha)P(E^{\mathrm{c}})$. Therefore, using (13),

$$\nu(E_\alpha) = g(P(E_\alpha)) = g(\alpha P(E) + (1 - \alpha)P(E^{\mathrm{c}})) \geq \alpha g(P(E)) + (1 - \alpha)g(P(E^{\mathrm{c}})),$$

$$\nu(E_\alpha^{\mathrm{c}}) = g(P(E_\alpha^{\mathrm{c}})) = g((1 - \alpha)P(E) + \alpha P(E^{\mathrm{c}})) \geq (1 - \alpha)g(P(E)) + \alpha g(P(E^{\mathrm{c}})).$$

Hence, $\nu(E_\alpha) + \nu(E_\alpha^{\mathrm{c}}) \geq \nu(E) + \nu(E^{\mathrm{c}}) = \nu(\Omega) \geq \nu(E_\alpha) + \nu(E_\alpha^{\mathrm{c}})$ because $\mathrm{core}(\nu) \neq \varnothing$, and so each E_α is linear and radial. If $P(E) = P(E^{\mathrm{c}})$, then $P(E) = (1/2)P(\Omega)$ and so it is the center of symmetry of $R(P)$ and it belongs to $\mathrm{ri}(R(P))$. Hence, E itself is linear and radial.

Finally, it is easy to check that $P(\Omega)/2$ is linear. In fact, $P(E_{1/2}) = 2^{-1}P(\Omega)$. □

Proof of Proposition 24. Since g is concave on G, it is Lipschitz over the compact set $R(P)$ (see, e.g. Rockafellar, 1970, Theorem 10.4), and so condition (ii) of Theorem 21 and Corollary 23 holds. Let $\delta(x)$ be the appropriate indicator function of $R(P)$ for our setting, defined by

$$\delta(x) \equiv \delta(x|R(P)) = \begin{cases} 0 & x \in R(P), \\ -\infty & x \notin R(P). \end{cases}$$

Set $\tilde{g}(x) = g(x) + \delta(x)$ for each $x \in G$. Clearly, $\partial\tilde{g}(x) = \partial g_{|R(P)}(x)$ for all $x \in R(P)$. Since g is concave, by a well-known result (see, e.g. Rockafellar (1970) Theorem 23.8), $\partial\tilde{g}(x_0) = \partial g(x_0) + \partial\delta(x_0)$ for all $x_0 \in G$. As well-known, $\partial\delta(x_0) = \{\chi \in \mathbb{R}^N : \chi\cdot(x_0 - x) \leq 0 \text{ for all } x \in R(P)\}$. Let $x_0 \in \mathrm{ri}(R(P))$ and let $w \in W$, where $W = \mathrm{span}(R(P))$. There is $\varepsilon > 0$ such that $x_0 + \varepsilon w \in R(P)$, and so $\chi \cdot w \geq 0$ for all $\chi \in \partial\delta(x_0)$. Since W is a vector subspace, this implies $\chi \cdot w = 0$ for all $\chi \in \partial\delta(x_0)$, which in turn implies that $\partial\delta(x_0) \subseteq W^{\perp}$. Since the converse inclusion is obvious, we conclude that $\partial\delta(x_0) = W^{\perp}$.

Putting everything together, we have $\partial g_{|R(P)}(x_0) = \partial g(x_0) + W^{\perp}$ for all $x_0 \in \mathrm{ri}(R(P))$. Hence, given any $\chi \in \partial g_{|R(P)}(x_0)$, there is $\chi' \in \partial g(x_0)$ such that $\chi \cdot P = \chi' \cdot P$. Since $\partial g(x_0) \subseteq \partial g_{|R(P)}(x_0)$, this implies that $\{\chi\cdot P : \chi \in \partial g_{|R(P)}(P(A))\} = \{\chi\cdot P : \chi \in \partial g(P(A))\}$ for all radial sets A. A simple application of Corollary 23 now completes the proof. □

Proof of Proposition 25. Suppose $\mathrm{core}(\nu) \neq \varnothing$. This implies that $\partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}})) \neq \varnothing$ because, by Theorem 21,

$$\mathrm{core}(\nu) = \{\chi \cdot P : \chi \in \partial g(P(A)) \cap \partial g(P(A^{\mathrm{c}}))\}.$$

Moreover, since $P(A) \in \mathrm{ri}(R(P))$, by a well-known result of Convex Analysis, $\left[\nabla g(P(A)) - \chi\right] \cdot w = 0$ for each $\chi \in \partial g(P(A))$ and each $w \in W$, where $W = \mathrm{span}(R(P))$. Hence, $\nabla g(P(A)) \cdot P = \chi \cdot P$ for each $\chi \in \partial g(P(A))$, and so, by Lemma 4, $\nabla g(P(A)) \cdot P \in \partial\nu(A)$. Since in Epstein and Marinacci (2001) it is proved that $\delta\nu(\cdot;A) = \nabla g(P(A)) \cdot P$, we then have $\delta\nu(\cdot;A) \in \partial\nu(A)$. By Theorem 13, $\mathrm{core}(\nu) = \{\delta\nu(\cdot;A)\}$, as desired. Next, suppose that g is differentiable and superdifferentiable at both $P(A)$ and $P(A^{\mathrm{c}})$. By proceeding as before, it can be shown that $\delta\nu(\cdot;A^{\mathrm{c}}) \in \partial\nu(A^{\mathrm{c}})$. Hence, by Theorem 13, $\mathrm{core}(\nu) \subseteq \{\delta\nu(\cdot;A^{\mathrm{c}})\}$, and so $\mathrm{core}(\nu) \neq \varnothing$ implies $\delta\nu(\cdot;A) = \delta\nu(\cdot;A^{\mathrm{c}})$, i.e. $\nabla g(P(A)) = \nabla g(P(A^{\mathrm{c}}))$. As to the converse, since $\delta\nu(\cdot;A) \in \partial\nu(A)$ and $\delta\nu(\cdot;A^{\mathrm{c}}) \in \partial\nu(A^{\mathrm{c}})$, the equality $\delta\nu(\cdot;A) = \delta\nu(\cdot;A^{\mathrm{c}})$ implies $\partial\nu(A) \cap \partial\nu(A^{\mathrm{c}}) \neq \varnothing$. Then, by Theorem 11, $\mathrm{core}(\nu) \neq \varnothing$. □

Proof of Corollary 26. Since ν is radially concave at A, it is easy to check that all sets A_α, with $\alpha \in (0,1)$, such that $P(A_\alpha) = \alpha P(A) + (1-\alpha)P(A^{\mathrm{c}})$ are linear (see the proof of Corollary 23). For α small enough, $P(A_\alpha) \in U$ and so g is differentiable at $P(A_\alpha)$. Since A_α is a radial set, a simple application of Proposition 25 proves the result. □

Proof of Proposition 28. Let $m \in \mathrm{core}(\nu)$ and E be such that $P(E) = 0$. It is immediately seen that this implies $m(E) = 0$. The argument used in the first part of the proof of Theorem 20 applies. Therefore, m is non-atomic. Let us prove the second statement. Under condition (i), the claim in the proof of Theorem 21 entails that m is strongly continuous. If (ii) holds, the claim of Proposition 14 establishes that $\text{core}(\nu) \subseteq \text{ca}(\Omega)$. Hence, the elements are strongly continuous as they are non-atomic. □

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