Università Commerciale
Luigi Bocconi

MSc. Finance/CLEFIN
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FINANCIAL ECONOMETRICS AND EMPIRICAL FINANCE - MODULE 2

Mock Question 1 (total 16 points, out of 50 from 3 questions)

Time Advised: 22 minutes (for this question)

Question 1.A (13 points)

Write in formal terms an AR(p) model with $p \geq 1$, making sure to explain what each term represents and whether each term is an observable random variable, a latent shock, or a parameter; also explain the economic intuition for the model, if any. What does it mean, both in logical and in statistical terms, that an AR(p) time series process is *stationary*? Assuming stationarity, make sure to discuss what the relevant *population* moments of the process are, also providing a few examples of the corresponding closed-form formulas.

Debriefing:

We expect all sub-questions to be answered but within a well-organized, 12-15 line long reply that will need to fit in the (generous) space provided.

In class we have expressed some doubts as to a rational, efficient-markets hypothesis related explanation for the meaning of AR(p) models in finance, and that represents the expected answer to that part of the question.

Autoregressive Process

- An autoregressive (henceforth AR) process of order p is a process in which the series $\{y_t\}$ is a weighted sum of p past variables in the series $(y_{t-1}, y_{t-2}, \dots, y_{t-p})$ plus a white noise error term, ϵ_t
  - AR(p) models are simple univariate devices to capture the observed Markovian nature of financial and macroeconomic data, i.e., the fact that the series tends to be influenced at most by a finite number of past values of the same series, which is often also described as the series only having a finite memory

Definition **(Autoregressive process)** A p-th order autoregressive process, denoted as AR(p), is a process that can be represented by the p-th order stochastic **difference equation**

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t,$$

where the process of $\{\varepsilon_t\}$ is an IID white noise with mean zero and constant variance σ_ε^2. More compactly, we can write:

$$y_t = \phi_0 + \sum_{j=1}^{p} \phi_j y_{t-j} + \varepsilon_t.$$

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2.2 Autoregressive Processes

Autoregressive process: a p-th order autoregressive process, denoted as $AR(p)$, is a process that can be represented by the p-th order stochastic difference equation

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$$

where the process of $\{\epsilon_t\}$ is an IID white noise with mean zero and constant variance σ_ϵ^2. In a compact form:

$$y_t = \phi_0 + \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t$$

- $AR(p)$ models are simple univariate devices to capture the Markovian nature of financial and macroeconomic data, i.e., the fact that the series tends to be influenced at most by a finite number of past values of the same series.

Lag operator (L): it shifts the time index of a variables regularly sampled over time backward by one unit, e.g. $Ly_t = y_{t-1}$.

Difference operator (Δ): it expresses the difference between consecutive realizations of a time series, e.g $\Delta y_t = y_t - y_{t-1}$.

We can rewrite the AR model using the lag operator

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) y_t = \phi_0 + \epsilon_t$$

In a compact form

$$\phi(L) y_t = \phi_0 + \epsilon_t$$

where $\phi(L)$ is the polynomial of order p, $(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)$.

(Reverse) Characteristic equation: the equation obtained by replacing in the polynomial $\phi(L)$ the lag operator L by a variable λ and setting it equal to zero, i.e., $\phi(\lambda) = 0$.

Root of the polynomial $\phi(L)$: any value of λ which satisfies the polynomial equation $\phi(\lambda) = 0$. It is a determinant of the behavior of the time series: if the absolute value of all the roots of the characteristic equations is higher than one the process is said to be stable.

- A stable process is always weakly stationary.

Stability and Stationarity of AR(p) Processes

- In case of an AR(p), because it is a stochastic difference equation, it can be rewritten as $(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)y_t = \phi_0 + \varepsilon_t$

  or, more compactly, as $\phi(L)y_t = \phi_0 + \varepsilon_t$, where $\phi(L)$ is a polynomial of order p, $(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)$
- Replacing in the polynomial $\phi(L)$ the lag operator by a variable λ and setting it equal to 0, i.e., $\phi(\lambda) = 0$, we obtain the characteristic equation associated with the difference equation $\phi(L)y_t = \phi_0 + \varepsilon_t$
  - A value of λ which satisfies the polynomial equation is called a root
  - A polynomial of degree p has p roots, often complex numbers
- If the absolute value of all the roots of the characteristic equations is higher than one the process is said to be stable
- A stable process is always weakly stationary
  - Even if stability and stationarity are conceptually different, stability conditions are commonly referred to as stationarity conditions

Wold's Decomposition Theorem

Result **(Wold's decomposition)** Every weakly stationary, purely non-deterministic, stochastic process $(y_t-\mu)$ can be written as an infinite, linear combination of a sequence of white noise components:

$$y_t-\mu = \epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \cdots = \sum_{i=0}^{\infty}\psi_i\epsilon_{t-i}$$

- An autoregressive process of order p with no constant and no other predetermined, fixed terms can be ex-pressed as an infinite order moving average process, MA(∞), and it is therefore linear
- If the process is stationary, the sum $\sum_{i=0}^{\infty}\psi_j\epsilon_{t-j}$ will converge
- The (unconditional) mean of an AR(p) model is

$$\mu = \frac{\phi_0}{1-\phi_1-\phi_2-...-\phi_p}$$

  - The necessary and sufficient condition for the mean of an AR(p) process to exist and be finite is that the sum of the AR coefficients is less than one in absolute value, $|\phi_1+\phi_2+...+\phi_p|<1$

Properties:

1. Mean: $\mu = \frac{\phi_0}{1-\phi_1}$ in case of an $AR(1)$
   $\mu = \frac{\phi_0}{1-\phi_1-\phi_2-...-\phi_p}$ in case of an $AR(p)$
2. Variance: $Var[y_t] = \frac{\sigma_\epsilon^2}{1-\phi_1^2}$ in case of an $AR(1)$
   $Var[y_t] = \frac{\sigma_\epsilon^2}{1-\phi_1^2-\phi_2^2-...-\phi_p^2}$ in case of an $AR(p)$

Wold's decomposition: Every weakly stationary, purely non-deterministic, stochastic process $(y_t-\mu)$ can be written as an infinite, linear combination of a sequence of white noise components:

$$y_t-\mu = \sum_{i=0}^{\infty}\theta_j\epsilon_{t-i}$$

- Wold's theorem states that only an autoregressive process of order p with no constant and no other predetermined, fixed terms can be expressed as an infinite order moving average process, $MA(\infty)$.
- This result is useful for deriving the autocorrelation function of an autoregressive process.
- It can be used to check the stationary condition of the mean and the variance of an AR model.

Example:

1. Mean:
   Starting from an $AR(1)$ model

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

   Take the expectation of the model equation

$$E(y_t) = \varphi_0 + \phi_1 E(y_{t-1})$$

   Under the stationarity condition, it must be that

$$E(y_t) = E(y_{t-1}) = \mu$$

   and hence

$$E(y_t) = \mu = \phi_0 + \phi_1\mu$$

which solved for the unknown unconditional mean gives:

$$\mu = \frac{\phi_0}{1-\phi_1}$$

and then $\phi_0 = (1-\phi_1)\mu$.
For μ to be constant and to exist, it must be that $\phi_1 \neq 1$. Substitute ϕ_0 above in the initial model equation and obtain

$$y_t-\mu = \phi_1(y_{t-1}-\mu) + \epsilon_t$$

It must also be the case that

$$y_{t-1}-\mu = \phi_1(y_{t-2}-\mu) + \epsilon_{t-1}$$

and therefore

$$y_t-\mu = \phi_1(\phi_1(y_{t-2}-\mu) + \epsilon_{t-1}-\mu) + \epsilon_t$$

By a process of infinite backward substitution, we obtain

$$y_t-\mu = \epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + ... = \sum_{i=0}^{\infty}\phi_1^i\epsilon_{t-i}$$

Following similar algebraic steps, we can derive also the unconditional mean of an $AR(p)$ model

$$\mu = \frac{\phi_0}{1-\phi_1-\phi_2-...-\phi_p}$$

For μ to be constant and to exist, it must be that

$$|\phi_1+\phi_2+...+\phi_p| < 1$$

2. Variance:
   Starting from an $AR(1)$ model

$$E[(y_t-\mu)^2] = \phi_1^2 E[(y_{t-1}-\mu)^2] + E[\epsilon_t^2]$$

   under the stationarity assumption $E[(y_t-\mu)] = E[(y_{t-1}-\mu)^2] = Var[y_t]$ it becomes

$$Var[y_t] = \phi_1^2 Var[y_t] + \sigma_\epsilon^2$$

   and therefore

$$Var[y_t] = \frac{\sigma_\epsilon^2}{1-\phi_1^2}$$

   provided that ${\phi^2}_1 < 1$.
   Therefore, putting all the conditions derived before together, for an $AR(1)$ model, weak stationarity requires $|\phi_1| < 1$.

   In case of a general $AR(p)$ model the formula of the variance becomes

$$Var[y_t] = \frac{\sigma_\epsilon^2}{1-\phi_1^2-\phi_2^2-...-\phi_p^2}$$

Moments and ACFs of an AR(p) Process

- The (unconditional) variance of an AR(p) process is computed from Yule-Walker equations written in recursive form (see below)
  - In the AR(2) case, for instance, we have
  $$Var[y_t] = \frac{(1-\phi_2)\sigma_\epsilon^2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$
  - For AR(p) models, the characteristic polynomials are rather convoluted – it is infeasible to define simple restrictions on the AR coefficients that ensure covariance stationarity
  - E.g., for AR(2), the conditions are $\phi_1+\phi_2<1$, $\phi_1-\phi_2<1$, $|\phi_2|<1$
- The autocovariances and autocorrelations functions of AR(p) processes can be computed by solving a set of simultaneous equations known as Yule-Walker equations
  - It is a system of K equations that we recursively solve to determine the ACF of the process, i.e., ρ_h for $h = 1, 2, \ldots$
  - See example concerning AR(2) process given in the lectures and/or in the lecture notes
- For a stationary AR(p), the ACF will decay geometrically to zero

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In the general case, covariance stationarity can be checked using the Schur criterion.

Schur criterion: Construct two lower-triangular matrices, $\mathbf{A_1}$ and $\mathbf{A_2}$ of the form:

$$\mathbf{A_1} \equiv \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ -\phi_1 & 1 & & & 0 \\ -\phi_2 & -\phi_1 & & & \\ \ldots & -\phi_2 & & & \\ & \ldots & & & 0 \\ -\phi_{p-1} & -\phi_{p-2} & \ldots & -\phi_1 & 1 \end{pmatrix} \mathbf{A_2} \equiv \begin{pmatrix} -\phi_p & 0 & \ldots & 0 & 0 \\ -\phi_{p-1} & -\phi_p & & & 0 \\ -\phi_{p-2} & -\phi_{p-1} & & & \\ \ldots & -\phi_{p-2} & & & \\ & \ldots & & & 0 \\ -\phi_1 & -\phi_2 & \ldots & -\phi_{p-1} & -\phi_p \end{pmatrix}$$

The $AR(p)$ process is covariance stationary if and only if the matrix $\mathbf{S} = \mathbf{A_1A_1'} - \mathbf{A_2A_2'}$ is positive definite.

The autocovariances and autocorrelations of $AR(p)$ processes can be computed by solving a set of Yule-Walker equations.

Yule-Walker equations for an $AR(2)$ process where $\mu = 0$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$$

We multiply the previous equation by y_{t-s} with $s = 1, 2, \ldots$ and take the expectation of each resulting stochastic difference equation

$$E[y_t y_t] = \phi_1 E[y_{t-1} y_t] + \phi_2 [y_{t-2} y_t] + E[y_t \epsilon_t]$$
$$E[y_t y_{t-1}] = \phi_1 E[y_{t-1} y_{t-1}] + \phi_2 [y_{t-2} y_{t-1}] + E[y_{t-1} \epsilon_t]$$
$$E[y_t y_{t-2}] = \phi_1 E[y_{t-1} y_{t-2}] + \phi_2 [y_{t-2} y_{t-2}] + E[y_{t-2} \epsilon_t]$$
$$E[y_t y_{t-s}] = \phi_1 E[y_{t-1} y_{t-s}] + \phi_2 [y_{t-2} y_{t-s}] + E[y_{t-s} \epsilon_t]$$

By definition

$$E[y_t y_{t-s}] = E[y_{t-k} y_{t-k-s}] = \gamma_s$$

We know that

$$E[\epsilon_t y_t] = \sigma^2 \quad \text{and} \quad E[y_{t-s}\epsilon_t] = 0$$

Therefore

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$$
$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$$
$$\gamma_s = \phi_1\gamma_{s-1} + \phi_2\gamma_{s-2} \quad s \geq 2$$

Divide γ_1 and γ_s by γ_0

$$\rho_1 = \phi_1\rho_0 + \phi_2\rho_1$$
$$\rho_s = \phi_1\rho_{s-1} + \phi_2\rho_{s-2} \quad s \geq 2$$

By construction $\rho_0 = 1$, then

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$

and, consequently, we can solve by recursive substitution ρ_s for any $s \geq 2$.

- The autocorrelation function of an $AR(p)$ will decay geometrically to zero because the leading term will also take the form of powers of sums of the coefficients which need to be restricted to absolute sums that are less than one.

We can use the sample PACF function to identify the order of an $AR(p)$ model. From the definition of PACF

$$y_t = \phi_{0,1} + \phi_{1,1} y_{t-1} + e_{1t}$$
$$y_t = \phi_{0,2} + \phi_{1,2} y_{t-1} + \phi_{2,2} y_{t-2} + e_{2t}$$
$$y_t = \phi_{0,3} + \phi_{1,3} y_{t-1} + \phi_{2,3} y_{t-2} + \phi_{3,3} y_{t-3} + e_{3t}$$

These models are in the form of multiple linear regressions and can be estimated by simple least squares, so that $\hat{\phi}_{j,j}$ is the sample j-order PACF of y_t and should converge to zero for all orders $j > p$.

Invertibility: An invertible MA(q) model can be expressed as an $AR(\infty)$:

$$y_t = \sum_{i=1}^{\infty} \phi_i L^i y_{t-1} + u_t$$

A MA(q) is invertible when the magnitude of all the roots of the MA polynomial exceeds the one.

- The ACF of a MA model has the same shape of the PACF of an AR model, and the PACF of a MA model has the same shape of an AR model.

Question 1.B (2 points)

Using the lag operator L, write an AR(2) process in "lag operator-polynomial" form and discuss how would you go about testing whether the process is stable and hence stationary. Will the resulting stationarity, if verified, be strong or weak? Make sure to explain your reasoning.

Debriefing:

Because if the series is stationary, Wold's decomposition applies, the process is linear and as such weak and strong stationarity are equivalent.
See also material copied below.

Stability and Stationarity of AR(p) Processes

- In case of an AR(p), because it is a stochastic difference equation, it can be rewritten as $(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)y_t=\phi_0+\varepsilon_t$ or, more compactly, as $\phi(L)y_t=\phi_0+\varepsilon_t$, where $\phi(L)$ is a polynomial of order p, $(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)$
- Replacing in the polynomial $\phi(L)$ the lag operator by a variable λ and setting it equal to 0, i.e., $\phi(\lambda)=0$, we obtain the characteristic equation associated with the difference equation $\phi(L)y_t=\phi_0+\varepsilon_t$
  - A value of λ which satisfies the polynomial equation is called a root
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2.2 Autoregressive Processes

Autoregressive process: a p-th order autoregressive process, denoted as $AR(p)$, is a process that can be represented by the p-th order stochastic difference equation

$$y_t=\phi_0+\phi_1 y_{t-1}+\phi_2 y_{t-2}+...+\phi_p y_{t-p}+\epsilon_t$$

where the process of $\{\epsilon_t\}$ is an IID white noise with mean zero and constant variance σ_ϵ^2 . In a compact form:

$$y_t=\phi_0+\sum_{j=1}^{p}\phi_j y_{t-j}+\epsilon_t$$

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Difference operator (Δ): it expresses the difference between consecutive realizations of a time series, e.g $\Delta y_t=y_t-y_{t-1}$.

We can rewrite the AR model using the lag operator

$$(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)y_t=\phi_0+\epsilon_t$$

In a compact form

$$\phi(L)y_t=\phi_0+\epsilon_t$$

where $\phi(L)$ is the polynomial of order p, $(1-\phi_1 L-\phi_2 L^2-...-\phi_p L^p)$.

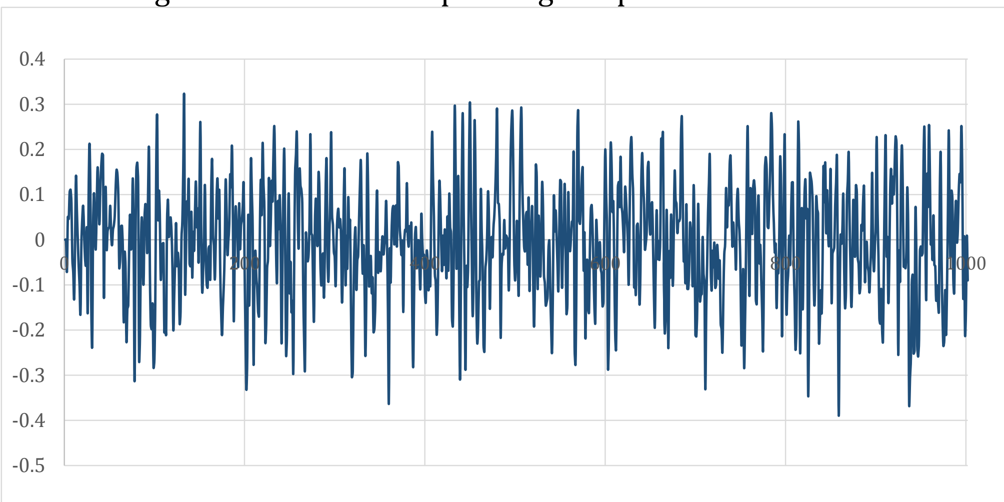
(Reverse) Characteristic equation: the equation obtained by replacing in the polynomial $\phi(L)$ the lag operator L by a variable λ and setting it equal to zero, i.e., $\phi(\lambda)=0$.

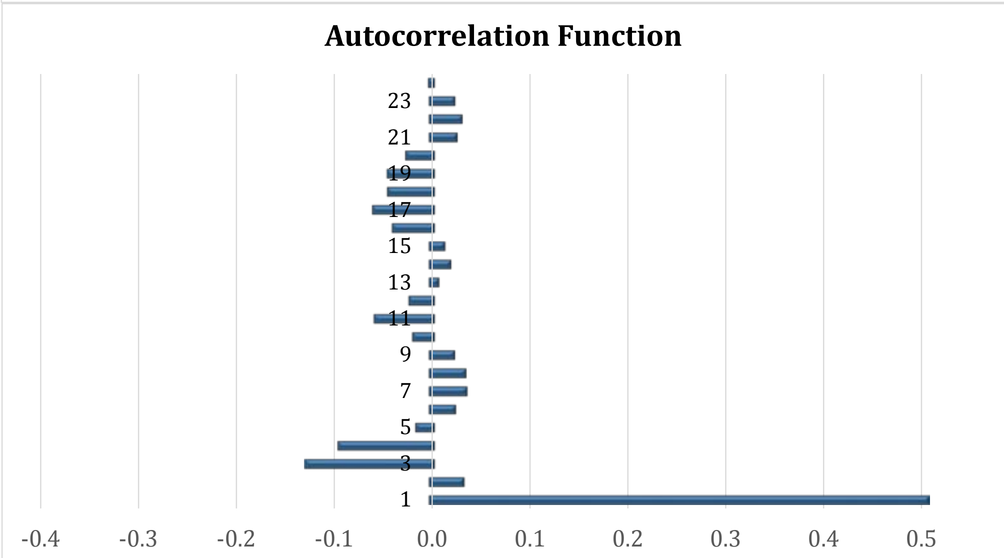
Root of the polynomial $\phi(L)$: any value of λ which satisfies the polynomial equation $\phi(\lambda)=0$. It is a determinant of the behavior of the time series: if the absolute value of all the roots of the characteristic equations is higher than one the process is said to be stable.

- A stable process is always weakly stationary.

Question 1.C (1 point)

Consider the following data and the corresponding sample ACF:





What is the most likely type of ARMA(p, q) process that may have originated this SACF? What other type of information would you be needing in order to make sure of your answer? Make sure to carefully justify your arguments.

Debriefing: The series and the corresponding SACF were generated from 1,000 simulations from an AR(2) process with the following features:

AR(2)	μ	0	**Unconditional**	
	ϕ_1	0.663	**mean**	0
	ϕ_2	-0.335	**Unconditional**	
	σ	0.1	**variance**	0.030

As seen in the lectures (lecture 3, slide 12), this cyclical but fading pattern characterizes stationary AR(2) processes with coefficients of opposite signs. However, one could be really "positive" this SACF comes from an AR(2) only after verifying that the SPACF has the typical behavior of PACF for stationary AR(2) processes: two values statistically significant, followed by no other significant values. Of course, one cannot detect the generating process just on the basis of the SACF (also the confidence intervals were not given), but she could speculate on its likely nature.