Università Commerciale
Luigi Bocconi

MSc. Finance/CLEFIN
2018/2019 Edition

THEORY OF FINANCE – PART 1

Mock Question 2 (total 5 points)

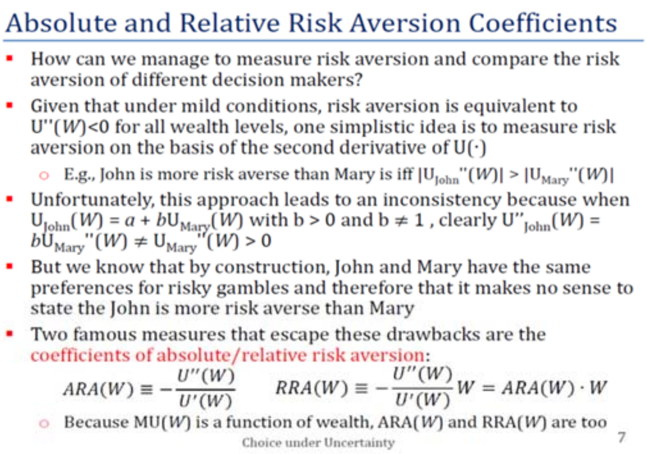
Time Advised: 20-21 minutes (for this question)

Difficulty Level: MEDIUM-HIGH

Question 2.A (3.75 points)

Define the absolute and relative risk aversion functions and explain why, in general, these are functions of an individual's wealth. What are the main economic interpretations/practical applications of ARA(W) and RRA(W)? Make sure to carefully justify your answer and, where necessary, provide examples. For the case of negative exponential utility with $\theta = 2$, use four plots to describe the behavior of ARA(W), RRA(W), the minimum odds $\pi(W; h)$, and the insurance risk premium $\Pi(W; h)$ for a fixed, small fair bet, i.e., h "close to zero".

Debriefing:



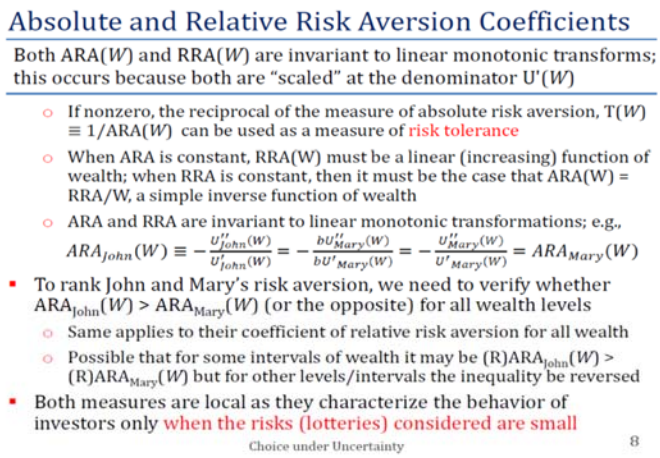
Absolute and Relative Risk Aversion Coefficients

- How can we manage to measure risk aversion and compare the risk aversion of different decision makers?
- Given that under mild conditions, risk aversion is equivalent to $U''(W)<0$ for all wealth levels, one simplistic idea is to measure risk aversion on the basis of the second derivative of $U(\cdot)$
  - E.g., John is more risk averse than Mary is iff $|U_{John}''(W)| > |U_{Mary}''(W)|$
- Unfortunately, this approach leads to an inconsistency because when $U_{John}(W) = a + bU_{Mary}(W)$ with $b > 0$ and $b \neq 1$, clearly $U''_{John}(W) = bU_{Mary}''(W) \neq U_{Mary}''(W) > 0$
- But we know that by construction, John and Mary have the same preferences for risky gambles and therefore that it makes no sense to state the John is more risk averse than Mary
- Two famous measures that escape these drawbacks are the coefficients of absolute/relative risk aversion:

$$ARA(W) \equiv -\frac{U''(W)}{U'(W)} \qquad RRA(W) \equiv -\frac{U''(W)}{U'(W)}W = ARA(W) \cdot W$$

  - Because MU(W) is a function of wealth, ARA(W) and RRA(W) are too

Choice under Uncertainty 7

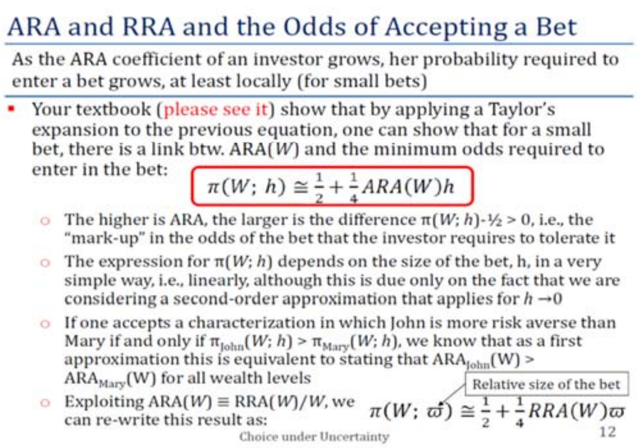


Absolute and Relative Risk Aversion Coefficients

Both ARA(W) and RRA(W) are invariant to linear monotonic transforms; this occurs because both are "scaled" at the denominator U'(W)

  - If nonzero, the reciprocal of the measure of absolute risk aversion, $T(W) \equiv 1/ARA(W)$ can be used as a measure of risk tolerance
  - When ARA is constant, RRA(W) must be a linear (increasing) function of wealth; when RRA is constant, then it must be the case that ARA(W) = RRA/W, a simple inverse function of wealth
  - ARA and RRA are invariant to linear monotonic transformations; e.g.,
  $$ARA_{John}(W) \equiv -\frac{U''_{John}(W)}{U'_{John}(W)} = -\frac{bU''_{Mary}(W)}{bU'_{Mary}(W)} = -\frac{U''_{Mary}(W)}{U'_{Mary}(W)} = ARA_{Mary}(W)$$
- To rank John and Mary's risk aversion, we need to verify whether $ARA_{John}(W) > ARA_{Mary}(W)$ (or the opposite) for all wealth levels
  - Same applies to their coefficient of relative risk aversion for all wealth
  - Possible that for some intervals of wealth it may be $(R)ARA_{John}(W) > (R)ARA_{Mary}(W)$ but for other levels/intervals the inequality be reversed
- Both measures are local as they characterize the behavior of investors only when the risks (lotteries) considered are small

Choice under Uncertainty 8

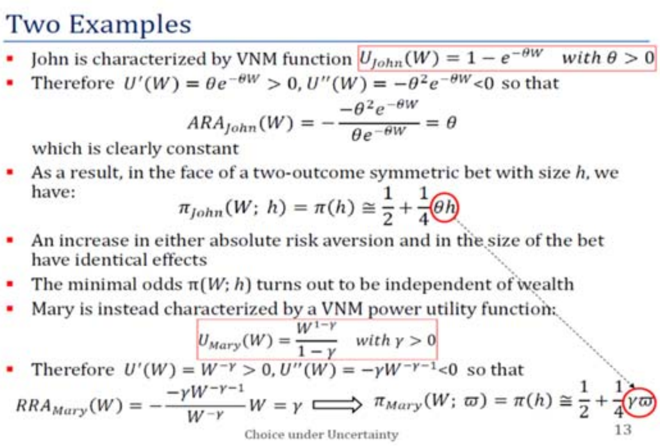


ARA and RRA and the Odds of Accepting a Bet

As the ARA coefficient of an investor grows, her probability required to enter a bet grows, at least locally (for small bets)

- Your textbook (please see it) show that by applying a Taylor's expansion to the previous equation, one can show that for a small bet, there is a link btw. ARA(W) and the minimum odds required to enter in the bet:
$$\pi(W;\, h) \cong \frac{1}{2} + \frac{1}{4}ARA(W)h$$
  - The higher is ARA, the larger is the difference $\pi(W; h)-½ > 0$, i.e., the "mark-up" in the odds of the bet that the investor requires to tolerate it
  - The expression for $\pi(W; h)$ depends on the size of the bet, h, in a very simple way, i.e., linearly, although this is due only on the fact that we are considering a second-order approximation that applies for $h \to 0$
  - If one accepts a characterization in which John is more risk averse than Mary if and only if $\pi_{John}(W; h) > \pi_{Mary}(W; h)$, we know that as a first approximation this is equivalent to stating that $ARA_{John}(W) > ARA_{Mary}(W)$ for all wealth levels
  - Exploiting $ARA(W) \equiv RRA(W)/W$, we can re-write this result as:
  $$\pi(W;\, \varpi) \cong \frac{1}{2} + \frac{1}{4}RRA(W)\varpi$$
  Relative size of the bet

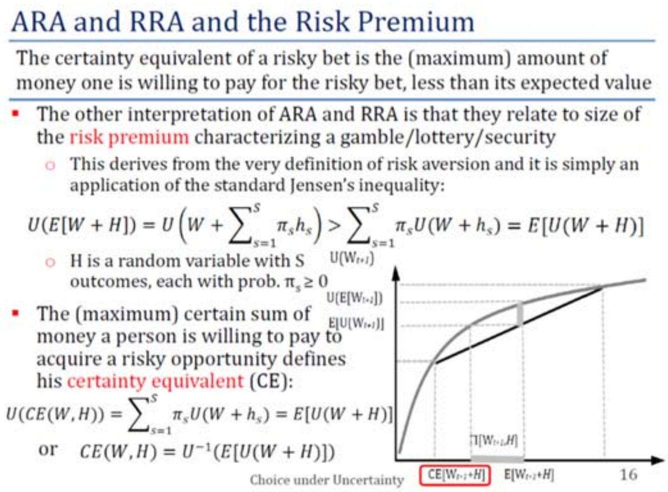
Choice under Uncertainty 12

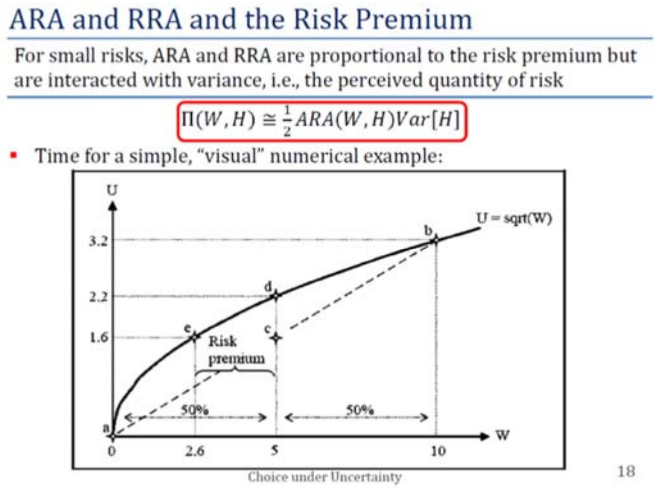


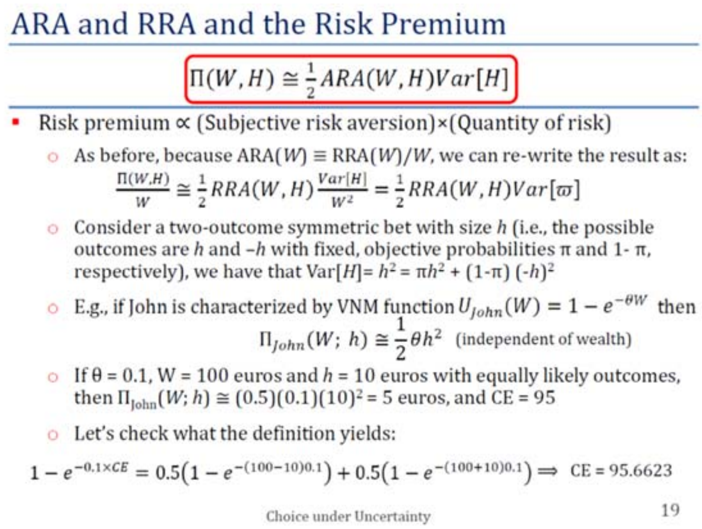
Two Examples

- John is characterized by VNM function $U_{John}(W) = 1 - e^{-\theta W} \quad with\ \theta > 0$
- Therefore $U'(W) = \theta e^{-\theta W} > 0, U''(W) = -\theta^2 e^{-\theta W}<0$ so that
$$ARA_{John}(W) = -\frac{-\theta^2 e^{-\theta W}}{\theta e^{-\theta W}} = \theta$$
which is clearly constant
- As a result, in the face of a two-outcome symmetric bet with size h, we have:
$$\pi_{John}(W;\, h) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\theta h$$
- An increase in either absolute risk aversion and in the size of the bet have identical effects
- The minimal odds $\pi(W; h)$ turns out to be independent of wealth
- Mary is instead characterized by a VNM power utility function:
$$U_{Mary}(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad with\ \gamma > 0$$
- Therefore $U'(W) = W^{-\gamma} > 0, U''(W) = -\gamma W^{-\gamma-1}<0$ so that
$$RRA_{Mary}(W) = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}}W = \gamma \Longrightarrow \pi_{Mary}(W;\, \varpi) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\gamma\varpi$$

Choice under Uncertainty 13







The graphs of ARA(W) = 2, RRA(W) =2W, $\pi(W;h)$ = 0.5 +0.5h (a flat line for fixed h) and $\Pi(W;h) = h^2$ (another flat line for fixed h) are trivial and not supplied here.

Question 2.B (0.75 points)

Mary has an initial wealth of 100 and preferences for wealth described by a power utility function with γ = 6. Write her utility function and check whether it is monotonic increasing and strictly concave. Mary is now facing a small gamble H with stochastic returns

$$\tilde{R}^H = \begin{cases} -0.1 & with\ prob.\ 2/3 \\ \ 0.6 & with\ prob.\ 1/3 \end{cases}$$

Compute the (financial) risk premium that Mary should demand in order to play this lottery.

Debriefing:

As for the first part, if you show that for all values of γ, a power utility function will be strictly increasing and concave, then you are set—exactly as we have done in the lectures. See the slide copied below. As for the second part, this is just a special case of Example 2.4, at p. 50 of the book. Try with different numbers, see below.

Two Examples

- John is characterized by VNM function $U_{John}(W) = 1 - e^{-\theta W} \quad with\ \theta > 0$
- Therefore $U'(W) = \theta e^{-\theta W} > 0, U''(W) = -\theta^2 e^{-\theta W} < 0$ so that
$$ARA_{John}(W) = -\frac{-\theta^2 e^{-\theta W}}{\theta e^{-\theta W}} = \theta$$
which is clearly constant
- As a result, in the face of a two-outcome symmetric bet with size h, we have:
$$\pi_{John}(W;\ h) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\theta h$$
- An increase in either absolute risk aversion and in the size of the bet have identical effects
- The minimal odds $\pi(W; h)$ turns out to be independent of wealth
- Mary is instead characterized by a VNM power utility function:
$$U_{Mary}(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad with\ \gamma > 0$$
- Therefore $U'(W) = W^{-\gamma} > 0, U''(W) = -\gamma W^{-\gamma-1} < 0$ so that
$$RRA_{Mary}(W) = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} W = \gamma \Longrightarrow \pi_{Mary}(W;\ \varpi) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\gamma\varpi$$

Choice under Uncertainty 13

Example 2.4 (continued). Consider again Mary, who is characterized by an isoelastic, power utility function of wealth, $U_{Mary}(W) = \frac{W^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$. In this case, because
$$U = \frac{v^{1-\gamma}}{1-\gamma} \Rightarrow (1-\gamma)U = v^{1-\gamma} \Rightarrow v = [(1-\gamma)U]^{\frac{1}{1-\gamma}}$$
is the inverse function,
$$CER = \frac{\left[(1-\gamma)E\left[U\left((1+\tilde{R}^H)W\right)\right]\right]^{\frac{1}{1-\gamma}}}{W} - 1$$
If we assume that $\gamma = 5$, and considering $\varpi = 0$ and 0.2,
$$E\left[U\left((1+\tilde{R}^H)W\right)\right] = 0.5\frac{100^{-4}}{-4} + 0.5\frac{120^{-4}}{-4} = -1.8528163 \cdot 10^{-9}$$
$$CER = \frac{[(-4)(-1.8528163 \cdot 10^{-9})]^{\frac{1}{-4}}}{100} - 1 = \frac{107.7772}{100} - 1 = 0.077772,$$
Or 7.78%. Clearly, this value is below the expected rate of return of 10% (obtained by equally probability-weighting the 0 and +20% returns). The difference of 2.2228% represents percentage risk premium associated to the risky asset/gamble H.

Because
$$U = \frac{v^{1-\gamma}}{1-\gamma} \Rightarrow (1-\gamma)U = v^{1-\gamma} \Rightarrow v = [(1-\gamma)U]^{\frac{1}{1-\gamma}}$$
is the inverse function,
$$CER = \frac{\left[(1-\gamma)E\left[U\left((1+\tilde{R}^H)W\right)\right]\right]^{\frac{1}{1-\gamma}}}{W} - 1$$
If we assume that $\gamma = 6$, and considering possible realized returns from the bet of -0.1 and 0.6 with the probabilities given above, yields
$$E\left[U\left((1+\tilde{R}^H)W\right)\right] = 0.667\frac{90^{-5}}{-5} + 0.333\frac{160^{-5}}{-5} = -2.321601 \cdot 10^{-11}$$
$$CER = \frac{[(-5)(-2.52679 \cdot 10^{-11})]^{\frac{1}{-5}}}{100} - 1 = \frac{95.43158}{100} - 1 = -0.029382,$$
Or -2.94%. Clearly, this value is below the expected rate of return of 13.31% (obtained by the simple calculation $0.667 \times (-10\%) + 0.333 \times (60\%)$). The difference of 13.31%-(-2.94%) = 16.25% represents percentage risk premium associated to the risky asset/gamble H, which is indeed rather risky (lottery-like) in spite of the positive expected return.

Question 2.C (0.5 points)

For a small, fair gamble with size h, your artificial intelligence system has inferred that that John does not displays a non-monotone minimum odds function $\pi^{John}(W;h)$ (as a function of wealth). Bill is instead characterized by a monotone decreasing function $\pi^{Bill}(W;h)$; Rachel is characterized by a monotone increasing function $\pi^{Rachel}(W;h)$. We know that $\pi^{John}(W;h)$ always (for all wealth levels) exceeds $\pi^{Bill}(W;h)$ and $\pi^{Rachel}(W;h)$. Based on their checking account and credit card transactions as well as on a series of surveys they have filled out over time, the AI system has also established that the VNM utility function of Mary is related to John's by the following relationship:

$$U^{Mary}(W) = -600 + 2U^{John}(W)$$

John, Bill, and Rachel are all risk-averse investors and $\pi^{John}(W;h), \pi^{Bill}(W;h)$, and $\pi^{Rachel}(W;h)$ are everywhere continuous and differentiable. Based on the previous information and assuming the relationship between ARA and minimum odds functions $\pi^{John}(W;h)$ approximately holds, what will your AI system extrapolate from the previous information on the shape of the ARA function of Mary, $ARA^{Mary}(W)$? [*Hint*: probably it is a good idea to try and qualitatively plot the function $ARA^{Mary}(W)$ that the AI system may extrapolate]

Debriefing:

Because Mary's VNM utility is a linear, affine, increasing transformation of John's, we know that $ARA^{Mary}(W) = ARA^{John}(W)$. Therefore, let's think (or let's hope our AI system thinks) about the remaining information:

i. $\pi^{Bill}(W;h) > 1/2$ (from risk-aversion) and decreasing towards ½; from the approximate relationship $\pi^{Bill}(W;h) = \frac{1}{2} + \frac{1}{4}ARA^{Bill}(W)h$, we know that this implies that $ARA^{Bill}(W) > 0$ must be decreasing (and also be eventually convex from continuity and differentiability);

ii. $\pi^{Rachel}(W;h) > 1/2$ and increasing, towards 1 (a probability cannot exceed 1); from the approximate relationship $\pi^{Rachel}(W;h) = \frac{1}{2} + \frac{1}{4}ARA^{Rachel}(W)h$, we know that this implies that $ARA^{Bill}(W) > 0$ must be increasing (and also be eventually concave from continuity and differentiability);

iii. $\pi^{John}(W;h) > \pi^{Bill}(W;h)$ and $\pi^{John}(W;h) > \pi^{Rachel}(W;h)$ which implies, from $ARA^{John}(W) > ARA^{Bill}(W)$ and $ARA^{John}(W) > ARA^{Rachel}(W)$, that John is always (for all wealth levels) more risk averse than Bill and Rachel are.

At this point the only possibly "behavior" for $ARA^{Mary}(W)$ is (qualitatively) as follows: