Università Commerciale
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MSc. Finance/CLEFIN
2018/2019 Edition

THEORY OF FINANCE – PART 1

Mock Question 3 (total 5 points)
Time Advised: 20-21 minutes (for this question)
Difficulty Level: MEDIUM-HIGH

Question 3.A (3.75 points)
Provide a precise definition of mean-variance (MV) preferences, making sure to discuss whether it might make any difference whether such preferences were to relate to the central moments of terminal wealth vs. the first two moments of portfolio returns. What are the models/ assumptions supporting the adoption of a MV framework?

Debriefing:

The Foundations of Mean-Variance Analysis

- One can show that a non-satiated investor with quadratic utility is characterized by an expected utility functional with structure:

$$E[U(W)] = E[W] - \frac{1}{2}\kappa E[W^2] = E[W] - \frac{1}{2}\kappa[Var[W] + (E[W])^2]$$
$$= E[W]\left(1 - \frac{1}{2}\kappa E[W]\right) - \frac{1}{2}\kappa Var[W]$$

  - It explicitly trades off the variance of terminal wealth with its mean because $W < 1/\kappa$ implies that $E[W] < 1/\kappa < 2/\kappa$ which is necessary and sufficient for $(1 - 1/2\kappa E[W])>0$
  - Quadratic utility isn't monotone increasing and may imply ARA,RRA < 0
- More generally, a MV framework is characterized by
  $E[U(W)]=\Gamma(E[W], Var[W])$,
  i.e., by dependence of the VNM functional only on mean and variance
  - If $U(\cdot)$ is quadratic, then $\Gamma(\cdot)$ will be linear in mean and variance
- A MV objective can be justified on grounds other than as the expected value of a quadratic utility function
- There are at least three additional ways of justifying a MV objective

The Foundations of Mean-Variance Analysis

A MV functional, $E[U(W)]=\Gamma(E[W], Var[W])$, can be micro-founded on: (i) quadratic utility, (ii) a Taylor expansion to any general VNM utility $U(\cdot)$, (iii) the EUT when joint return distribution is normal, (iv) directly

- First, a quadratic approximation (i.e., 2nd-order Taylor expansion), see the Appendix for details
- Second, $E[U(W)]=\Gamma(E[W], Var[W])$ may derive from an application of the EUT when the rates of return are described according to a multivariate Normal distribution
  - Normal distributions are characterized entirely by their means (expectations), variances, and covariances;
  - Linear combinations of Normal random variables are also Normal (hence, terminal wealth, or the rate of return on a portfolio of assets with Normally distributed returns, is also Normally distributed)
- Third, often a MV objective is directly assumed, on the grounds that such a criterion is plausible, without recourse to deep assumptions
- Less innocent than it seems, as it implies investors ignore features of the distribution of asset returns besides mean and variance

Mean-Variance of Terminal Wealth or Ptf. Returns?

  - E.g., any skewness in the distribution would be ignored
  - A less obvious feature not captured by just variance, is the thickness of the tails of a distribution; an index of this tendency is the kurtosis
- The problems with MV are not over – normally MV objectives are applied to portfolio returns, $W_{t+1} = (1 + R_{PF,t+1})W_t$ but note that:

$$E_t[W_{t+1}] = E_t[(1 + R_{PF,t+1})]W_t = (1 + E_t[R_{PF,t+1}])W_t$$
$$Var_t[W_{t+1}] = Var_t[(1 + R_{PF,t+1})]W_t^2 = Var_t[R_{PF,t+1}]W_t^2$$

- Therefore, plugging into $E_t[U(W_{t+1})] = E[W_{t+1}](1 - 0.5\kappa E[W_{t+1}]) - 0.5\kappa Var_t[W_{t+1}]$ and dropping $(1 - 0.5\kappa E[W_{t+1}])$ one has:

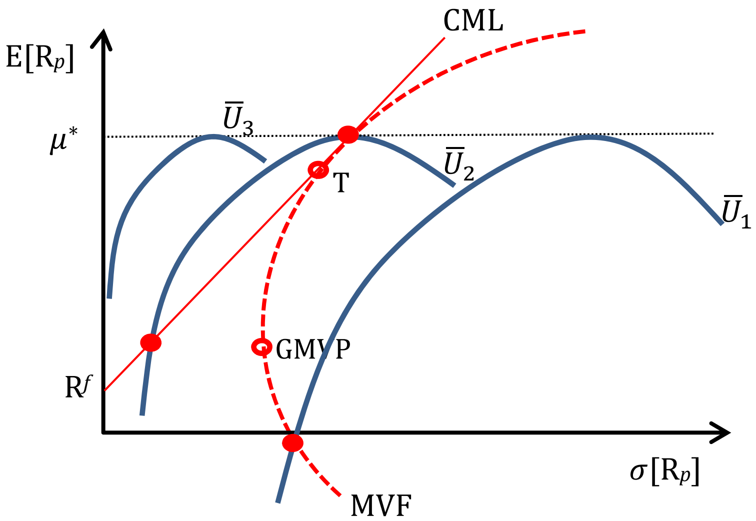
$$E_t[U(W_{t+1})] = (1 + E_t[R_{PF,t+1}])W_t - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t^2$$
$$= \left\{(1 + E_t[R_{PF,t+1}]) - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t\right\}W_t$$
$$\propto E_t[R_{PF,t+1}] - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t,$$

Not the same as: $E_t[R_{PF,t+1}] - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]$

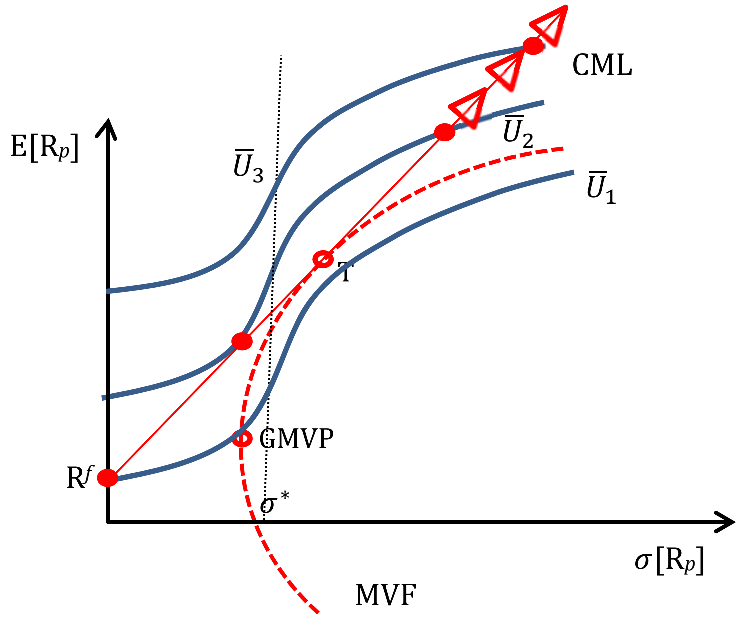
- We call the MV functions that depend on moments of portfolio returns $G(E_t[R_{PF,t+1}], Var_t[R_{PF,t+1}]) = G(\mu_{PF}, \sigma_{PF}^2)$

Question 3.B (0.75 points)
John and Mary are characterized by different mean-variance preferences, in terms of the first two moments of portfolio returns. In particular, John is risk-averse for all wealth levels although his risk aversion declines as (expected) wealth increases, but he (oddly) becomes satiated for very high levels of expected wealth (hence, expected portfolio returns, call it $\mu^* > E[R_T]$ where T is the tangency portfolio). Mary is instead non-satiated at all wealth levels, but while she is increasingly risk-averse for low risk (call it σ^*), she becomes decreasingly risk-averse above some volatility of portfolio returns (again, σ^*). Assume all other ingredients of optimal portfolio selection in a MV framework are typical, as seen in the lectures. First, plot in two distinct graphs the maps of indifference curves of John and Mary. Second, in the same plots locate—if it exists—the optimal portfolio that John and Mary ought to select. Carefully explain why you have plotted the indifferences curves with the shapes you have selected and why the optimal portfolios exist or fail to exist.

Debriefing:
In the case of John, his indifferences curves are concave because (this is especially relevant for expected portfolio returns below μ^*) his risk-aversion is decreasing in expected wealth and hence in expected returns (recall that given the relationship between expected terminal wealth and expected returns, to write about the former or the latter is equivalent, in a static framework). More interestingly, his indifference curves will be monotone increasing up to μ^* and then turn monotone decreasing, because as John suffers from increasing risk, being satiated, the only way to keep him indifferent is by *decreasing* his expected portfolio return and terminal wealth. Because of their odd and concave shape the existence of an optimum portfolio is not guaranteed or trivially established (but on this point, there are many possible arguments and all adequately supported claims will be acceptable, see for instance the three red dots in the picture).



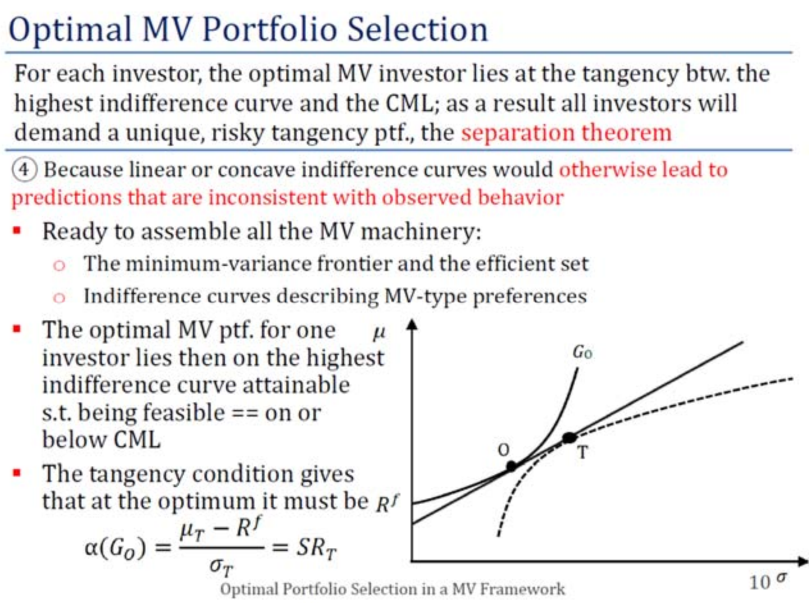
In the case of Mary instead, we are facing monotone increasing (because she is non-satiated), S-shaped indifferences curves that are at first convex (because initially Mary is increasingly risk averse) and then become concave (above σ^*, when she becomes decreasingly risk-averse). Also in this case, it is possible to have multiple optimal portfolios even though the eventual concavity of the indifferences curves should lead to the conclusion that the optimal portfolio implies infinite leverage (see the arrows)



Question 3.C (0.5 points)

John is a non-satiated, risk-averse investor (with increasing risk aversion as a function of wealth and the risk of wealth) that maximizes a standard mean-variance objective that depends on the moments of portfolio returns. You know that he invests 50% of his wealth in the riskless asset and 50% in the tangency portfolio so that *his optimal* portfolio is characterized by a standard deviation of 12%. Moreover, you know that in correspondence to John's optimal portfolio, the slope of the highest achievable indifference curve is equal to 0.5 and that the expected return of the tangency portfolio is 14%. Based on this data, compute the risk-free rate under which John is selecting his portfolio.

Debriefing:



Because $E[R_T]$ =14% and $\sigma[R_p] = \omega\sigma[R_T] = 12\%$ with $\omega = 0.5$, it must be that $\sigma[R_T] = \sigma[R_T - R^f] = 24\%$. Based on the information on the slope of the tangent (i.e., highest, given convexity) indifference curve, we also know that:

$$\frac{E[R_T] - R^f}{\sigma[R_T]} = \frac{14\% - R^f}{24\%} = 0.5 \Longrightarrow R^f = 2\%.$$