The Economic Value of Derivatives and Structured Products in Long-Horizon, Dynamic Asset Allocation

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Abstract

This note reviews key models, methods, and results concerning the issue of whether and how derivatives and portfolios thereof (called structured investment products) may generate an increase in ex-ante welfare to long-run portfolio optimizers. Simple examples show that relatively plain combinations of derivatives (such asymmetric straddles) have the potential to incresase risk-adjusted performance by important amounts, easily in excess of 200 basis points per year. The implied positions in derivatives are often modest but they affect the exposure to the underlying risky portfolio in important ways.

1. Introduction

Derivatives trading is the world's biggest business. Yet, the practice as well as academic research on dynamic asset allocation typically exclude derivatives from investment portfolios.[1] Part of the problem is that the portfolio optimization methods that are currently used, like Markowitz's mean-variance model, are ill-suited to handle options. First, the distribution of option returns departs significantly from normality and therefore cannot be described by means and variances alone. Second, the short history of options returns available severely limits the estimation of their complex (predictive) return distribution. For example, we have data for Standard & Poor's 500 options only since 1993, which may not long enough to estimate the moments of their return distribution with sufficient precision. Third, there are high transaction costs in option markets. On average, at-the-money (ATM) options have a 5% relative bid-ask spread, while out-of-the-money (OTM) options have relative bid-ask spreads of 10%. These high transaction costs are often (and, we shall argue, incorrectly) estimated to be so high as to bar any plausible increase in ex-ante, risk-adjusted portfolio performance to be interesting enough to be taken into serious consideration in practice.

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[1]Mutual funds' usage of derivatives is limited, see Koski and Ponti (1999). Mutual funds generally face legal constraints in terms of short-selling, borrowing and derivatives usage. Most hedge funds use derivatives, but they represent only a small part of their holdings (see Chen, 2011, for empirical evidence).

In a complete market setting, of course, such an exclusion can also be justified by th fact that derivative securities are redundant (e.g., Black and Scholes, 1973; Cox and Ross, 1976).[2] When the completeness of the market breaks down, however—either because of infrequent trading or the presence of additional sources of uncertainty, such as stochastic volatility and price jumps—it becomes suboptimal to exclude derivatives from portfolio decisions. Optimal portfolio choice certainly belongs to one of the most extensively studied problems in finance. Merton (1969, 1971, 1973) considers continuous time economies in which individuals dynamically adjust portfolio positions in order to maximize expected utility. Although portfolio selection is of major concern to asset managers, it has taken some time until there were attempts to include derivatives into dynamic portfolio optimization. This might be partially explained by the fact that initially, derivatives were seen as redundant securities which could be replicated by implementing a dynamic trading strategy in stocks and bonds. The standard Black-Scholes option pricing model supports this view. However, more advanced option pricing models take additional risk factors such as stochastic volatility into account. In these models, markets are incomplete and derivatives are no longer replicable by stocks and bonds alone. Instead they provide opportunities to earn additional risk premiums.

In this paper, we study optimal investment strategies in incomplete markets that give investor access not only to bond and stock markets but also to the derivatives market. The problem is at first solved in closed form when trading is continuous, transaction costs, taxes and other frictions are ignored, and investors derive utility only from their terminal wealth. Derivatives extend the risk and return trade-offs associated with stochastic volatility and price jumps. Therefore derivatives increase the (ex-ante) risk-adjusted performance achievable by optimizing a portfolio of securities. Equivalently, derivatives are a vehicle to achieve the optimal exposure to the fundamental risk factors in an economy, e.g., to diffusions and jump components.

The introduction of derivatives thus always increases the investor's utility in a partial equilibrium model. In this paper we therefore estimate also the extent of such improvements. To assess the portfolio improvement from participating in the derivatives market, we compare the certainty-equivalent wealth of two utility-maximizing investors with and without access to the derivatives market, which can be thought of as a kind of an additional (riskless) rate of return earned by the investor with access to derivatives. We calibrate the parameters of a stochastic volatility (later jumps are added too) model to those reported by empirical studies on the *FTSE MIB* index and option markets. Our results show that the improvement from including derivatives is driven mostly by the *myopic component*. Under normal market conditions and a conservative (or positive) estimates of the volatility-risk premium, the improvement in certainty equivalent wealth for an investor with relative risk aversion of *three* is about *4*%per year, which becomes higher when the market becomes more volatile (as

[2]Financial markets are said to be complete (in the an Arrow-Debreu sense) if it is possible to construct a portfolio of securities at a point in time which guarantees a specific payoff in a specific state of nature at some future date. The notion of dynamic completeness is the natural extension of this idea to dynamic trading strategies. See Harrison and Kreps (1979) and Duffie and Huang (1985) for a more detailed discussion.

captured by volatility of volatility).

Specifically, we adopt an empirically realistic model for the aggregate stock market that incorporates two types of risk factors: diffusive price shocks and volatility risks. Later, also jump risks are introduced. Taking this market condition as given, we solve the dynamic asset allocation problem (Merton, 1971) of a power-utility investor whose investment opportunity set includes not only the usual riskless bond and risky stock, but also derivatives on the stock. What makes derivatives valuable in such a setting of multiple risk factors is that the stock and bond alone cannot provide independent exposure to each and every risk factor. For example, the risky stock by itself can only provide a "package deal" of risk exposures: one unit each to diffusive and jump risks and none to volatility risk. With the help of derivatives, however, this "package deal" can be broken down into its two individual components. For example, an at-the-money option, being highly sensitive to market volatility, provides exposure to volatility risk. The market incompleteness that makes derivatives valuable in our setting also makes the pricing of such derivatives not unique. When we introduce derivatives to complete the market, say one at-the-money, we need to make additional assumptions on the volatility-risk and jump-risk premia implicit in such derivatives. Once such assumptions are made and the derivatives are introduced, the market is complete.[3]

In the first part of our paper, we assume the presence of diffusive and volatility risks only. Here the dynamic asset allocation problem is solved in closed form. Our results can be interpreted in three steps. First, we solve the investor's optimal wealth dynamics. Second, we find the exposure to the three risk factors that supports the optimal wealth dynamics. Finally, we find the optimal positions in the risky stock and the derivative securities that achieve the optimal exposure to the risk factors. Instrumental to the final step is the ability of the derivative securities to complete the market. When derivatives can be used to hedge volatility risk, the optimal weight on the derivative security depends explicitly on how sensitive the chosen derivative is to volatility. There are two economically different sources from which the need to access the volatility risk arises. Acting myopically, the investor participates in the derivatives market simply to take advantage of the risk-and-return trade-off provided by volatility risk. Under the assumption of a positive compensation of volatility risk, buying volatility (which under some conditions is equivalent to assigning a positive weight to a positive Vega structured product) improves the ex-ante risk-return trade-off of the optimal portfolio. Moreover, acting non-myopically, the investor holds derivatives to further exploit the time-varying nature of the investment opportunity set, which, in our setting, is driven exclusively by stochastic volatility. As the volatility becomes more persistent, this non-myopic demand for derivatives grows.

As for the choice among alternative derivatives or structured option baskets, in the first part of

[3]By exogenously specifying the market prices of risk factors, our analysis is of a partial-equilibrium nature. In fact, this is very much the spirit of the asset allocation problem: a small investor takes prices (both risks and returns) as given and finds for himself the optimal trading strategy. By the same token, when we quantify the improvement for including derivatives, we are addressing the improvement in certainty-equivalent wealth for this very investor, not the welfare improvement of the society as a whole. The latter would require an equilibrium treatment.

our paper, in a complete market and with continuous trading, an investor can achieve the overall optimal future payoff profile by investing in the stock, the risk-free bond, and sufficiently many (maybe even infinitely many) derivatives. In this setup, the exact payoff profiles of the derivatives do not matter, and the investor is indifferent between trading plain-vanilla options or more exotic contracts. There would thus be nothing special about retail plain vanilla derivatives vs. structured products.

In reality, however, trading takes place in discrete time, and markets are in general incomplete. Furthermore, retail investors are hindered from implementing the optimal portfolio strategies due to too high minimum investment amounts, high transaction costs or margin requirements, short-selling restrictions, and maybe also a lack of knowledge. In this situation, the characteristics of the available derivatives may actually matter a lot. A retail investor will then benefit from a derivative whose payoff profile is equal (or close) to her optimal payoff profile and which is offered by an issuer who is better than himself able to implement the corresponding replication strategy. In the second part of the paper, we therefore tackle the case of an incomplete market, in which there are diffusive, volatility, and jump risks and in which the investor solves a buy-and-hold portfolio problem. In this case, the resulting optimal payoff profile is in general highly complicated and does not only depend on observable stock prices, but also on the paths of state variables like stochastic volatility. Furthermore, such an optimal profile becomes specific to the investor under consideration. Due to these reasons, these optimal payoff profiles will not be traded in the market. Nevertheless, financial institutions might design contracts that at least approximate these optimal payoffs and help the investors to come closer to their first best. If structured investment products (henceforth, SPS, such as investment certificates) exist, they should increase the utility of the investor significantly, and there should be a significant demand for them.

In our work we take a perspective inspired after the problem of long-run investors. In particular, the long lasting turmoil in the financial markets since 2008 has emphasized the challenges that pension funds face in developing strategies which focus on "(...) delivering an adequate target pension with a high degree of probability (...)" (see Blake, Cairns, and Dowd, 2008). In this paper, we therefore investigate whether the use of equity- and volatility-based derivative instruments may help the pension industry in meeting such challenges, by considering the design problem for optimal derivative strategies in the context of strategic asset allocation and for long-horizon investors. To do so, it is important to use asset pricing models which incorporate risks such as stochastic volatility and jumps (in the second part of this paper). Moreover, while traditional asset allocation focuses on diversification across asset classes, we propose an allocation strategy which emphasizes diversification across risk premia (i.e., across underlying risk factors) instead. The possibility to earn volatility and jump risk premia may be particularly interesting for investors with a long time horizon. Focusing on a long investment horizon, Littermann (2011) argues that pension funds may be more capable to bear volatility and jump risks than other financial institutions since volatility is mean-reverting over longer time horizons and jumps will hurt the objectives of short-term investors much more than those of long-term investors. Therefore the latter have a comparative advantage over the former investors and there is an opportunity for trades that may lead to an improvement in risk-adjusted performances.

The rest of the paper is organized as follows. Section 1 presents a rather stylized and yet realistic model of the dynamic of asset returns for the simplified case of one risky portfolio (stock), one riskless bond, and one SPS of varying complexity. Such a model represents in fact the state of the art in the derivative pricing and portfolio choice literatures. Section 2 describes the strategic asset allocation problem solved by an investor with a horizon T who maximizes her expected power utility and who can continuously trade, including the chance to resort to SPS with maturity $\tau \leq T$. Section 3 examines the ex-ante, risk-adjusted welfare gains deriving from holding SPS positions. Section 4 extends the set up in Section 2 to incomplete markets and applies numerical methods to solve the portfolio problem. In this case, the specific SPS that is made available to an investor, will affect the ex-ante welfare gains caused by the availability of the derivatives. Section 5 concludes by discussing several directions for extensions and additional research.

2. **The Model**

The primitive securities in this economy are a riskless bond that pays a constant rate of interest r, and a risky stock that represents the aggregate equity market.[4] To capture the empirical features that are important in time series data on the aggregate stock market, we assume the following dynamics for the (price) return process dS/S of the risky stock:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r + \eta V_t)dt + \sqrt{V_t}dB_t \\ dV_t &= \kappa(\bar{\upsilon} - V_t)dt + \sigma\sqrt{V_t}(\rho dB_t + \sqrt{1-\rho^2}dZ_t), \end{aligned}$$

where B and Z are standard Brownian motions assumed to be independent. The instantaneous variance process V is a stochastic process with long-run mean $\bar{\upsilon} > 0$, mean-reversion rate $\kappa > 0$, and volatility coefficient $\sigma \geq 0$. This formulation of stochastic volatility (Heston, 1993), allows the diffusive price shock B to enter the volatility dynamics via the constant coefficient ρ that introduces correlation between price and volatility shocks. Finally, η is a constant coefficient capturing the risk premium associated to diffusive risk, B. The single risky asset represents an equity index and although our analysis could also be performed in a multi-asset framework we will use this one single representative risky portfolio throughout the paper.

In addition to investing in the risky stock and the riskless bond, the investor is also given the chance to include derivatives in her portfolio. The relevant derivative securities are those that serve to expand the dimension of risk-and-return trade-offs for the investor. More specifically for our setting,

[4] In reality interest rate risk must of course be mitigated separately (for example using interest rate swaps) but we shall ignore this additional risk for the time being.

such derivatives are those that provide differential exposures to the two fundamental risk factors in the economy. For concreteness, we consider the class of derivatives whose time t price O_t depends on the underlying stock price S_t and the stock volatility V_t through the generic function $O_t = g(S_t, V_t)$. Letting τ be its time to expiration, this particular derivative is defined by its payoff structure at the time of expiration. For example, a derivative with a linear payoff structure $g(S_\tau, V_\tau) = S_\tau$ is the stock itself, and it must be that $g(S_t, V_t) = S_t$ at all times. On the other hand, for some strike price $K > 0$, a derivative with the non-linear payoff structure $g(S_\tau, V_\tau) = (S_\tau - K)^+$ is a European-style call option, while that with $g(S_\tau, V_\tau) = (K - S_\tau)^+$ is a European-style put option. Unlike our earlier example of the linear contract, the pricing relation $g(S_t, V_t)$ at $t < \tau$ is not uniquely defined in these two cases from the information contained in the risky stock only. In other words, by including multiple sources of risks in a non-trivial way, the market is incomplete with respect to the risky stock and riskless bond.

The market can be completed once we introduce enough non-redundant derivatives $O_t^{(i)} = g^{(i)}(S_t, V_t)$ for $i = 1, 2, ..., N$. Alternatively, we can introduce a specific pricing kernel to price all of the risk factors in this economy, and consequently any derivative securities. These two approaches are equivalent. In incomplete markets the prices of derivatives are not unique. They depend on the pricing of diffusion and volatility factors. That is, the particular specification of the N derivatives that complete the market is linked uniquely to a pricing kernel $\{\pi_t,\, 0 \le t \le \tau\}$ such that:

$$O_t^{(i)} = \frac{1}{\pi_t} E_t[\pi_{\tau_i} g^{(i)}(S_{\tau_i}, V_{\tau_i})]$$

where τ_i is the time to expiration for the ith derivative security. In this paper, we choose the latter approach and start with the following parametric pricing kernel:[5]

$$d\pi_t = -\pi_t r dt - \pi_t \eta \sqrt{V_t} dB_t - \pi_t \xi \sqrt{V_t} dZ_t \qquad (\pi_0 = 1). \tag{1}$$

The following parametric specification of the price dynamics for the ith derivative security is consistent with the pricing kernel in (1):

$$dO_t^{(i)} = rO_t^{(i)} dt + (g_S^{(i)} S_t + \sigma\rho g_V^{(i)})(\eta V_t dt + \sqrt{V_t} dB_t) + \sigma\sqrt{1-\rho^2} g_V^{(i)} (\xi V_t dt + \sqrt{V_t} dZ_t)$$

where $g_S^{(i)}$ and $g_V^{(i)}$ measure the sensitivity of the ith SPS price to infinitesimal changes in the stock price and volatility, respectively, i.e., they are the partial derivatives of the payoff function vs. its two arguments (one may generically call them "delta" and "Vega"). A derivative with non-zero g_S provides exposure to the diffusive price shock B; a derivative with non-zero g_V provides exposure to additional volatility risk Z. This specific parametric form has the advantage of having two parameters (η and ξ) to separately price the two risk factors in the economy.[6]

[5] To verify that this parametric pricing kernel is a valid pricing kernel, which rules out arbitrage opportunities involving the riskless bond, the risky stock, and any derivative securities, one can apply Ito's lemma and show that $\pi_t \exp(r_t)$, $\pi_t S_t$, and $\pi_t O_t$ are local martingales. For the special case of constant volatility, this pricing kernel can be mapped to the equilibrium result of Naik and Lee (1990). Letting γ be the relative risk-aversion coefficient of the representative agent, the coefficient for the diffusive-risk premium is $\eta = \gamma$.

[6] One can show that the model-implied variance risk premium parameter is simply $\sigma\rho\eta$. Due to a negative correlation

3. **The Asset Allocation Problem**

The investor starts with positive wealth W_0. Given the opportunity to invest in the riskless asset (a money market account, which earns interest at the constant rate r), the risky stock and the derivative securities, he chooses, at each time t, to invest a fraction ϕ_t of her wealth in the stock S_t, and a fractions ψ_t in the derivative security O_t. The investment objective is to maximize the expected utility of her terminal wealth W_T, where $T \geq \tau$,[7]

$$\max_{\phi_t,\, \psi_t} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{2}$$

(subject to the law of motion of prices and volatility) where $\gamma > 0$ is the relative risk-aversion coefficient of the investor, and where the wealth process satisfies the self-financing condition:

$$\frac{dW_t}{W_t} = rdt + \theta_t^B(\eta V_t dt + \sqrt{V_t}dB_t) + \theta_t^Z(\xi V_t dt + \sqrt{V_t}dZ_t).$$

Here θ_t^B and θ_t^Z are defined as

$$\theta_t^B \equiv \phi_t + \psi_t\left(g_S\frac{S_t}{O_t} + \sigma\rho g_V\frac{1}{O_t}\right) \qquad \theta_t^Z \equiv \sigma\sqrt{1-\rho^2}\psi_t g_V\frac{1}{O_t}. \tag{3}$$

Utility functions with constant relative risk aversion guarantee that the terminal wealth will never go negative since the absolute risk aversion will go to infinity when wealth levels approach zero. Effectively, by taking positions ϕ_t and ψ_t on the risky assets, the investor invests θ_t^B in the diffusive price shock B and θ_t^Z in the additional volatility risk Z. Except for adding derivative securities to the investor's opportunity set, the investment problem is the standard Merton's (1971) problem. Interestingly, the maturities of the chosen derivatives do not have to match the investment horizon T. For example, it might be hard for an investor with a ten-year investment horizon to find an option with a matching maturity. For the purpose of choosing the optimal portfolio weights at time t, what matters is the choice of derivative securities O_t at that time, not the future choice of derivatives. This is true as long as, at each point in time in the future, there exist non-redundant derivative securities to complete the market. Moreover, portfolio rebalancing occurs in continuous time, in the sense that a solution consists of the process followed by the weights, $\{\phi_s,\, \psi_s,\ t \leq s \leq T\}$. Branger, Breuer, and Schlag (2007) show that the rebalancing frequency is a major determinant of the utility gain that can be realized by investing in options, and that the investor should rebalance her portfolio at least monthly.

Following Merton (1971), we define the indirect utility function by

$$J(t, w, v) = \max_{\{\phi_s,\, \psi_s,\ t \leq s \leq T\}} E\left[\left.\frac{W_T^{1-\gamma}}{1-\gamma}\right| W_t = w, V_t = v\right]$$

between the price process S and the variance process V,the resulting variance risk premium parameter has a sign opposite to η,

[7]An imperfect alternative to long-dated options is a carefully managed sequence of shorter-term options. We derive a dynamic trading strategy consisting of a sequence of overlapping options contracts that will yield the same investment profile as the optimal buy-and-hold strategy.

which, by the principle of optimal stochastic control, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\max_{\phi_t,\,\psi_t}\Big\{J_t + W_t J_W(r_t + \theta_t^B \eta V_t + \theta_t^Z \xi V_t) + \frac{1}{2}W_t^2 J_{WW} V_t[(\theta_t^B)^2 + (\theta_t^Z)^2] + \kappa(\bar{v} - V_t)J_V + \frac{1}{2}\sigma V_t J_{VV} + \sigma V_t W_t J_{WV}[\rho\theta_t^B + \sqrt{1-\rho^2}\theta_t^Z]\Big\} = 0.$$

where J_{xy} denotes the (second, possibly mixed) partial derivative with respect to x and y. To solve the HJB equation, we first solve the optimal positions on the risk factors θ_t^B and θ_t^Z and then transform them back via the linear relation (3) to the optimal positions in the risky assets. This transformation is feasible as long as the chosen derivatives are non-redundant in the following sense (see the proof in Liu and Pan, 2003): At any time t, the derivative security O_t is non-redundant if $g_V \neq 0$. Effectively, this non-redundancy condition guarantees market completeness with respect to the chosen derivative security, the risky stock, and the riskless bond.

At this point, assuming that there are non-redundant derivatives available for trade at any time $t < T$, for given wealth W_t and volatility V_t, the solution to the HJB equation is given by

$$J(t, w, v) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp[\gamma h(T-t) + \gamma H(T-t)V_t],$$

where $h(\cdot)$ and $H(\cdot)$ are time-dependent coefficients that are independent of the state variables. That is, for any τ,

$$\begin{aligned}
h(\tau) &= \frac{2\kappa\bar{v}}{\sigma^2}\ln\left(\frac{2k_2\exp((k_1+k_2)\tau/2)}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)}\right) + \frac{1-\gamma}{\gamma}r\tau,\\
H(\tau) &= \frac{\exp(k_2\tau)-1}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)}\frac{1-\gamma}{\gamma^2}(\eta^2+\xi^2).\\
k_1 &\equiv \kappa - \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2}\right)\sigma \qquad k_2 \equiv \sqrt{k_1^2 - \delta\sigma^2}.
\end{aligned}$$

The optimal portfolio weights on the risk factors B and Z are then given by:

$$\hat{\theta}_t^B = \frac{\eta}{\gamma} + \sigma\rho H(T-t) \qquad \hat{\theta}_t^Z = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2}H(T-t)$$

Transforming the $\hat{\theta}_t^B$s to the optimal portfolio weights on the risky assets, we have:[8]

$$\begin{aligned}
\hat{\phi}_t &= \frac{\eta}{\gamma} - \frac{\xi\rho}{\gamma\sqrt{1-\rho^2}} - \left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H(T-t)\right)\frac{g_S}{g_V}S_t\\
\hat{\psi}_t &= \left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H(T-t)\right)\frac{O_t}{g_V}.
\end{aligned}$$

The optimal derivative position is inversely proportional to g_V/O_t which measures the volatility exposure for each dollar invested in the derivative security. Intuitively, the more "volatility exposure

[8]Note that it is always possible to derive the exposures to the risk factors, given the portfolio weights. The other direction, however, i.e. the calculation of the portfolio weights which result in a given set of exposures, is only possible in a complete market. In an incomplete market, there are some exposures which cannot be attained.

per dollar" a derivative security provides, the more effective it is as a vehicle to hedge or invest in volatility risk. Hence a smaller portion (in absolute value) of the investor's wealth needs to be invested in this derivative security. The demand for derivatives—or the need for volatility exposure—arises for two different reasons. First, a myopic investor finds the derivative security attractive because, as a vehicle to volatility risk, it could potentially expand the dimension of risk-and-return trade-offs. This myopic demand for derivatives is reflected in the first term of $\hat{\psi}_t$. For example, negatively priced volatility risk ($\xi < 0$) makes short positions in volatility attractive, inducing investors to sell derivatives with positive "volatility exposure per dollar." The opposite occurs when $\xi > 0$. Moreover, the least risk-averse investors are more aggressive in taking advantage of the risk and return trade-off through investing in derivatives.

Second, for an investor who acts non-myopically, there is a benefit in derivative investments even when the myopic demand declines because of a zero volatility–risk premium ($\xi = 0$). This non-myopic demand for derivatives is reflected by the second term of the expression of $\hat{\psi}_t$, $H(T-t)(O_t/g_V)$. In our setting, the Sharpe ratio of the option return is driven exclusively by stochastic volatility. In fact, it is proportional to volatility. This implies that a higher realized option return at one instant is associated with a higher Sharpe ratio (better risk-return trade-off) for the next-instant option return. That is, a good outcome is more likely to be followed by another good outcome. By the same token, a bad outcome in the option return predicts a sequence of less attractive future risk-return trade-offs. An investor with relative risk aversion $\gamma < 1$ is particularly averse to sequences of negative outcomes because her utility is unbounded from below. On the other hand, an investor with $\gamma > 1$ benefits from sequences of positive outcomes because her utility is unbounded from above. As a result, they act quite differently in response to this temporal uncertainty and this is visible from the different sign of the coefficient

$$H(\tau) = \frac{\exp(k_2\tau) - 1}{2k_2 + (k_1 + k_2)(\exp(k_2\tau) - 1)} \frac{1-\gamma}{\gamma^2}(\eta^2 + \xi^2),$$

depending on whether $\gamma \lesseqgtr 1$. The investor with $\gamma > 1$ takes a short position in volatility so as to hedge against temporal uncertainty, while the one with $\gamma < 1$ takes a long position in volatility so as to speculate on the temporal uncertainty. Indeed, it is easy to verify that $H(T-t)$, the driving force of this non-myopic term, is strictly positive for investors with $\gamma < 1$, strictly negative for investors with $\gamma > 1$, and zero for the log-utility investor.

The hedging demand of the investor arises from the impact of V_t on the compensation per unit of risk, as e.g. discussed in Munk (2004) or Munk and Sørensen (2004). The intuition is as follows. For small V_t, the expected return is low, and the risk of a large negative return and thus a low terminal wealth is comparably high. This induces the investor to shift wealth from states with high V_t to states with low V_t. On the other hand, for high V_t, the expected return is high. This induces the investor to shift wealth to states with a high V_t to grasp these good investment opportunities. Taken together, her ultimate hedging demand depends on the trade-off between these two opposite effects.

For $\gamma > 1$, the utility function of an investor is unbounded from below, but bounded from above. He cares more about states with low V_t since this implies a higher probability of losses due to a lower expected return. Thus, her hedge is to take a short position in V_t. In line with this intuition, the marginal indirect utility of this investor is decreasing in V_t, which formally follows from the fact that H is non-positive. A low volatility thus corresponds to a high indirect utility, implying a bad state.[9]

Given that volatility risk exposure is taken care of by the derivative holdings, the "net" demand for stock should simply be linked to the risk-and-return trade-off associated with diffusive price risk. Focusing on the first term of $\hat{\phi}_t$, this is indeed true. Specifically, the demand for stock is proportional to the attractiveness of the stock and inversely proportional to the investor's risk aversion, η/γ. The interaction between the derivative security and its underlying stock, however, complicates the optimal demand for stocks. For example, by holding a call option, one effectively invests a fraction g_S—typically referred to as the "delta" of the option—in the underlying stock. The last term in $\hat{\phi}_t$ is there to correct for this "delta" effect. In addition, there is also a "correlation" effect that originates from the negative correlation between volatility and price shocks, typically referred to as the leverage effect (Black, 1976). Specifically, a short position in the volatility automatically involves long positions in the price shock, and equivalently, the underlying stock. The second term in $\hat{\phi}_t$ appears to correct this "correlation" effect.

As a benchmark, a "no-derivatives" (ND) investor solves the same investment problem as that in (2) with the additional constraint that he cannot invest in the derivatives market, $\hat{\psi}_t = 0$. At any time t, the indirect utility of a "no-derivatives" investor with a year-T investment horizon is

$$\begin{aligned}
J^{ND}(\tau, W_t, V_t) &= \frac{W_t^{1-\gamma}}{1-\gamma} \exp[\gamma h^{ND}\tau + \gamma H^{ND}\tau V_t] \\
h^{ND}(\tau) &= \frac{2\kappa\bar{v}}{\sigma^2(\rho^2 + \gamma(1-\rho^2))} \ln\left(\frac{2k_2 \exp((k_1+k_2)\tau/2)}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)}\right) + \frac{1-\gamma}{\gamma} r\tau \\
H^{ND}(\tau) &= \frac{\exp(k_2\tau)-1}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)} \frac{1-\gamma}{\gamma^2}\eta^2 \\
k_1 &\equiv \kappa - \frac{1-\gamma}{\gamma}\eta\sigma\rho \qquad k_2 \equiv \sqrt{k_1^2 - \frac{1-\gamma}{\gamma^2}\eta\sigma^2(\rho^2 + \gamma(1-\rho^2))}.
\end{aligned}$$

Trivially, in this case the optimal weight given to the stock is $\hat{\phi}_t^{ND} = \eta/\gamma$, which is the standard expression with the price of diffusive risk divided by the coefficient of relative risk aversion.

3.1. *Empirical Characterization of the Optimal Strategies*

To examine the empirical properties of our results, we fix a set of base-case parameters for our current model. Specifically, for one-factor volatility risk, we set the long-run mean at $\bar{v} = (13\%)^2$, the rate of mean reversion at $\kappa = 5$, and the volatility coefficient at $\sigma = 0.25$. The correlation between price and volatility risks is set at $\rho = -0.40$. Given the well established empirical property of the equity

[9] Driessen and Maenhout (2007) have shown that empirically, improvements by including derivatives are driven mostly by the myopic component.

risk premium, calibrating the market price of the Brownian shocks B is straightforward. Specifically, setting $\eta = 4$ and coupling it with the base-case value of $\bar{v} = (13\%)^2$ for the long-run mean of volatility, we have an average equity risk premium of 6.76% per year. We fix the risk-free rate a *3.5%*. The properties of the market price of volatility risk, however, are not as well established. In part because volatility is not a directly tradable asset, there is less consensus on reasonable values for its market price. Empirically, however, there is strong support that volatility risk is priced (see, e.g., Chernov and Ghysels, 2000, Pan, 2002, Benzoni, 1998, and Bakshi and Kapadia, 2003, report that volatility risk is negatively priced).[10] Instead of calibrating the volatility-risk premium coefficient ξ to the existing empirical results, however, we will allow this coefficient to vary in our analysis so as to get a better understanding of how different levels and signs of the volatility risk premium could a effect the optimal investment decision. However, differently from Liu and Pan (2003) and the papers that have followed from their efforts, we also allow the volatility risk premium to become positive.

Using this set of base-case parameters, particularly the risk-and-return trade-off implied by the data, we now provide some quantitative examples of optimal investments in the S&P 500 index and options. Specifically, we consider the following "equity protection" structured product (henceforth, EPSP), based on purchasing an asymmetric straddle option that bets long on volatility:

$$EPC_t = g(S_t, V_t; K_1, K_2, \tau) = \alpha_1 p(S_t, V_t; K_1, \tau) + \alpha_2 c(S_t, V_t; K_2, \tau) \qquad K_2 > K_1$$

where $c(S_t, V_t; K, \tau)$ is the pricing formula for a standard European call option with strike K and time-to-maturity τ and $p(S_t, V_t; K, \tau)$ is the corresponding formula for a put. Initially, we set $\alpha_1 = 4$ and $\alpha_2 = 1$.

Figure *1* indicates that *the demand for derivatives is driven mainly by the myopic component.* In particular, when the volatility-risk premium is set to zero, the non-myopic demand for straddles is basically zero percent of the total wealth for an investor with relative risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. In contrast, when we set $\xi = -6$, which is a conservative estimate for the volatility-risk premium, the optimal portfolio weight in the EPSP *increases to 54%* for the same investor.[11] The quantitative effect of the non-myopic component can best be seen by varying the investment horizon (bottom left panel) or the volatility persistence (bottom right panel). Consider an investor with $\gamma = 3$ who would like to hedge against temporal uncertainty by taking short positions in volatility. The bottom left panel shows that *as we increase the investment horizon, this intertemporal hedging demand increases.* Similarly, the bottom right panel shows that *as we decrease the persistent*

[10] At an intuitive level, a negative volatility risk premium could be supported by the fact that aggregate market volatility is typically high during recessions. A short position in volatility, which loses value when volatility becomes high during recessions, is therefore relatively more risky than a long position in volatility, requiring an additional risk premium. However, it remans true that a positive risk premium appears to be a case that is easily imagined.

[11] For example, Coval and Shumway (2001) report that zero-beta at-the-money straddle positions produce average losses of approximately 3% per week. This number roughly corresponds to $\xi = -12$. Using volatility-risk premium to explain the premium implicit in option prices, Pan (2002) reports a total volatility-risk premium that translates to $\xi = -10$.

level of the volatility by increasing the mean-reversion rate κ, there is less benefit in taking advantage of the intertemporal persistence, hence a reduction in intertemporal hedging demand.

As the market becomes more volatile, the costs of the EPSP *increases*, but the volatility sensitivity of such a basket of derivatives *decreases*. In fact, the EPSP provides *less* "volatility exposure per dollar" as market volatility increases. To achieve the optimal volatility exposure, *more* needs to be invested in the EPSP, hence the *increase* in $|\hat{\psi}_t|$ with the market volatility. As the volatility of volatility increases, the risk-and-return trade-off on volatility risk becomes less attractive, hence the *decrease* in magnitude of the EPSP position with increasing "vol of vol". Finally, the optimal strategy with varying risk aversion γ is as expected: less risk-averse investors *are more aggressive* in their risky investment strategies.

Lakonishok et al. (2006) using a unique option data set provide detailed descriptive statistics on the purchased and written open interest and open buy and sell volume of several classes of investors. They establish that written option positions are more common than purchased positions: for both calls and puts, non-market maker investors (mostly full service brokerage clients) in aggregate have more written than purchased open interest. These findings appear consistent with the result that it is rather difficult to generate significant optimal long positions in both call and put options when the volatility risk premium is negative. A large number of life cycle portfolio choice papers now document that a large group of young wage earners, once their income streams and stock market returns are reasonably calibrated to data, will desire to invest 100% of their savings in the stock market (see, e.g., Curcuru, Heaton, Lucas and Moore, 2004). From a normative perspective, long call positions may help this class of agents since call options achieve the desired leverage without costly margin borrowing.

Although most of the derivative products suggested in this paper are frequently traded Over The Counter (OTC) as well as on exchanged-traded platforms, some of the products may be substantially less liquidly available than their underlying equity portfolio. Plain-vanilla equity derivatives (single or basket index puts and calls) are more liquidly available, although for large investors, transaction programs may take a substantial amount of time before they are completed, unless one is willing to accept significant implementation costs—yielding lower efficiency gains. Once derivative products become part of the overall strategy, investors should also manage liquidity risk in trading derivatives, i.e., the risk that the intended strategy can not be maintained in the future when the derivative contracts expire. As a second class of potential implementation issues, it may be cumbersome to align the derivative strategy with other parts of the existing portfolio. The benchmark may not be available in a single derivative contract, unless one is willing to trade OTC basket options tailored towards the investor's specific equity portfolio.

4. **Ex-ante, Risk-Adjusted Portfolio Improvements**

Suppose that market volatility is V_t and consider an investor with initial wealth W_t and investment horizon T years, who takes advantage of the derivatives market. One can show that her optimal expected utility is

$$J(\tau, W_t, V_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp[\gamma h(\tau) + \gamma H(\tau) V_t],$$

where $\tau \equiv T - t$. Hence optimal expected utility is independent of the specific derivative contract chosen by the investor. This makes intuitive sense, because in our setting the market is complete in the presence of the derivative security. Letting $\tilde{W}_t$ be the investor's certainty-equivalent wealth, defined by

$$\frac{\tilde{W}_t^{1-\gamma}}{1-\gamma} = J(\tau, W_t, V_t),$$

so that

$$\tilde{W}_t = W_t \exp\left(\frac{\gamma}{1-\gamma} h(\tau) + \frac{\gamma}{1-\gamma} H(\tau) V_t\right).$$

The indirect utility for an investor with no access to the derivatives market is instead

$$\tilde{W}_t^{ND} = W_t \exp\left(\frac{\gamma}{1-\gamma} h^{ND}(\tau) + \frac{\gamma}{1-\gamma} H^{ND}(\tau) V_t\right).$$

At this point, to quantify the portfolio improvement from including derivatives, we adopt the following measure:

$$\tilde{\mathcal{R}}^W \equiv \frac{\ln \tilde{W}_t - \ln \tilde{W}_t^{ND}}{T},$$

which measures the portfolio improvement in terms of the annualized, continuously compounded return in certainty-equivalent wealth. Because the investor has constant relative risk aversion, $\tilde{\mathcal{R}}^W$ does not depend on her initial wealth W_t. Following Liu and Pan (2003), one can prove that for an investor with $\gamma \neq 1$, the portfolio improvement from including derivatives is strictly positive. For an investor with log-utility ($\gamma = 1$), the improvement is strictly positive if $\xi \neq 0$ and zero otherwise. The improvement from including derivatives is closely linked to the demand for derivatives. For a myopic investor with log-utility, the demand for derivatives arises from the need to exploit the risk-and-return trade-off provided by volatility risk. When the volatility-risk premium is zero, there is no myopic demand for derivatives. Consequently, there is no benefit from including derivatives. However, in the presence of a non-myopic demand for derivatives, the realized welfare gain for a non-myopic investor is strictly positive regardless of the value of ξ.

To provide a quantitative assessment of the ex-ante risk-adjustment performance improvement, we again use the base-case parameters described in Section 2. The results are summarized in Figure *2*. Focusing first on the top right panel, we see that the welfare improvement is very sensitive to how volatility risk is priced. Under normal market conditions with a conservative estimate of the volatility–risk premium $\xi = 4$, our results show that the welfare improvement from including derivatives is about *4.1%* per year in certainty-equivalent wealth for an investor with risk aversion

$\gamma = 3$. As the investor becomes less risk averse and more aggressive in taking advantage of the derivatives market, the improvement from including derivatives becomes even higher.

We can further evaluate the relative importance of the myopic and non-myopic components of portfolio improvement by setting $\xi = 0$. The welfare improvement from non-myopic trading in derivatives is *as low as 0.02%* per year. This is consistent with our earlier result: the demand for derivatives *is driven mostly by the myopic component.* The non-myopic component of the risk-adjusted portfolio gain is further examined in the bottom panels of Figure *2* as we vary the investment horizon and the persistence of volatility. Intuitively, as the investment horizon T increases, or as the volatility shock becomes more persistent, the benefit of the derivative security as a hedge against temporal uncertainty becomes *more pronounced.* Hence there is an increase in portfolio improvement. Finally, from the middle two panels, we can also see that *when market volatility increases, or when the volatility of volatility increases, there is more to be gained from investing in the derivatives market.*

Once derivatives allow us to take a direct exposure to volatility without exposure to the (direction of) stock prices, the investment opportunity set is enlarged and volatility can then be considered to be an asset class on its own. Investors thus can make trade-offs among distinct risk factors according to their preferences. Driessen and Maenhout (2007) show that constant relative risk aversion investors always find it optimal to short OTM puts, and only with distorted probability assessments are they able to obtain positive weights for puts using cumulative prospect theory and anticipated utility.

5. **An Extension to an Incomplete Market Setting**

It is widely acknowledged that the continuous-time framework in which most of modern finance has been developed is an approximation to reality—it is currently impossible to trade continuously, and even if it were possible, market frictions would render continuous trading infinitely costly. Consequently, any practical implementation of continuous-time asset allocation policies invariably requires some discretization in which the investor's portfolio is rebalanced only a finite number of times, typically at equally spaced time intervals, with the number of intervals chosen so that the discrete asset-allocation policy 'approximates' the optimal continuous-time policy in some metric.

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6. **Discussion, Conclusion, and Further Extensions**

Similarly to Liu and Pan (2003), it would be interesting to extend our exercise to assess the role of SPS as a vehicle to disentangle jump risk from diffusive risk in complete markets, where analytical results are available.[12] In this setting, the relative importance of jump risk vs. diffusive risk would represent the economic driving force behind any empirical findings. In fact, the empirical evidence

[12]Recently, Branger, Schlag, and Schneider (2008) have further examined the impact of jumps in volatility on optimal portfolio choice, also dealing with the case in which nonlinear payoff securities may be used to hedge such risk.

from the US option market suggests that, for investors with a reasonable range of risk aversion, jump risk is compensated more highly than diffusive risk (Pan, (2002)) and this may create strong portfolio tilts towards deep out-of-money options and similar structured payoff functions. Intuitively, this is because in contrast to diffusive risk, which can be controlled via continuous trading, the sudden, high-impact nature of jump risk takes away the investor's ability to continuously trade out of a leveraged position to avoid negative wealth. As a result, without access to derivatives, the investor may avoid taking too leveraged a position in the risky stock (Liu et al. 2003). The same investor would nevertheless feel freer to make choices when the worst-case scenarios associated with jump risk can be taken care of by trading derivatives. For instance, this could be accomplished by taking a larger position in the risky stock and buying deep out-of-the-money put options to hedge out the negative jump risk.

Recently, Branger and Brauer (2008) have investigated the ex-ante economic gains deriving from adding SPS(i.e., investment certificates, of the type of EPSPs but also discount certificates and turbos, see their paper for detailed descriptions) to a portfolio that contains not only stocks and bonds, but also a fixed number of plain vanilla options. Their results are not very different from the case in which the investor is not allowed any access to derivatives and this shows how the complex features of structured products may be crucial in portfolio management applications. In the ideal case of a complete market and with continuous trading, the investor should carry a positive exposure to stock diffusion risk, a negative exposure to volatility risk, and a negative exposure to jump risk. If she can only trade the stock and the money market account and she is restricted to discrete trading, there is no exposure to volatility risk, a lower exposure to stock diffusion risk, and a slightly higher exposure to jump risk. With two derivatives, the investor would be able to match the initial exposures to the diffusion risks and to one jump size from the ideal case. However, the exposure to volatility risk and also to diffusion risk is significantly lower. When the investor can use only one structured product, she has two objectives. She tries to come as close as possible to the optimal initial exposure given by the ideal case. She thus looks for derivatives that offer her a certain (optimal) relation between stock diffusion risk, volatility risk and jump risk.[13]

A different problem is when derivatives in incomplete markets are used to move an expected utility maximizer closer the optimal strategy, i.e., to the portfolio rule that yields the payoff profile that maximizes expected utility. It is well known that under certain conditions, complex financial instruments such as options and other derivative securities can be replicated by sophisticated dynamic trading strategies involving simpler securities such as stocks and bonds. The essence of such a delta-hedging argument is the ability to actively manage a portfolio continuously through time, and to do so in a self-financing manner, i.e., with no cash inflows or outflows after the initial investment, so that the portfolio's value tracks the value of the derivative security without error at each point in time,

[13]For instance, guaranteed certificates have a positive exposure to volatility risk. Given that the volatility risk premium may be negative, however, the investor may desire a negative exposure to earn the premium. He would thus rather want to take a short position in guarantee certificates.

until the maturity date of the derivative. Haugh and Lo (2001) consider the reverse implications of this correspondence by constructing an optimal portfolio of complex securities at a single point in time to mimic the properties of a dynamic trading strategy. Specifically, they focus on investment policies that arise from standard dynamic optimization problems in which an investor maximizes the expected utility of her end-of-period wealth, and construct a buy-and-hold portfolio of stocks, bonds and options at the start of the investment horizon that will come closest to the optimal dynamic policy, by defining "closest" in three distinct ways (expected utility, mean-squared error of terminal wealth, and utility-weighted mean-squared error of terminal wealth). They find that under certain conditions, a buy-and-hold portfolio consisting of just a few options is an excellent substitute for considerably more complex dynamic investment policies.[14] This effect is particularly strong for highly risk tolerant investors under no short-sale constraints, as these investors use long positions in call options to increase their risk exposure by exploiting the leverage that options imply. Haugh and Lo also propose an approximate grid algorithm to optimize the structure of the SPS they advocate (in the form of a portfolio of plain vanilla calls).

It would be also interesting to explicitly consider the downside risk constraints of pension funds at short term horizons. In periods when the liquidity of certain assets is reduced, derivative contracts may become valuable instruments to generate payoffs which cannot otherwise be obtained. Moreover, pension funds are subject to stochastic, unanticipated withdrawals. Cui, Oldenkamp, and Vellekoop (2013) model an expected utility maximization problem with a displaced CRRA utility function under a threshold value which guarantees that the fund can never lose more than a percentage φ of its value in a period.[15] Using an extended asset menu with equity and variance derivatives, they find that an optimal portfolio that includes derivatives not only markedly improves welfare in an expected utility framework, but also improves along most (and in some cases all) other evaluation criteria which are frequently used by pension funds, such as the 2.5% quantile of the funding ratio and the expected shortfall at that level (i.e., Value-at-risk and Tail-Value-at-Risk) as well as measures of income security, which is represented by the probability of reaching specific return targets.

The optimal portfolio in Cui, Oldenkamp, and Vellekoop (2013) optimally loads on equity risk premium, volatility risk premium and jump risk premium by holding a long position in equity and a short position in variance derivatives.[16] It also contains a long position in the OTM put and a short position in the OTM call, which resembles a so-called *collar* strategy. The portfolio loads on volatility

[14] Merton (1995) first noted that, in the presence of transactions costs, derivative securities may be an efficient way to implement optimal dynamic investment policies.

[15] Utility functions with constant relative risk aversion guarantee that the terminal wealth will never go negative since the absolute risk aversion will go to infinity when wealth levels approach zero. In reality, pension funds would use a higher threshold than zero for the absolute minimum of wealth they will allow in their optimization programme.

[16] Their variance derivative is the floating leg of a *variance swap* which generates a payoff which depends on the average realized variance of the risky asset over the investment horizon. As such it is purely linked to the stochastic volatility process and not directly to the asset price, although there is an indirect exposure due to the correlation between the stock price and its volatility.

risk via variance derivatives because these give investors a controlled exposure to this risk whereas options' sensitivity to volatility risk depends on the remaining time to maturity and the stock price path. When variance derivatives are present in the portfolio one can further enhance the retirement income security by a short position in calls (60% of stocks) and a long position in puts (8%, hence not necessarily in equal amounts). The use of short calls to help pay for the purchase of puts is based on the intuition that to improve the chances of achieving a desired income target in pension plans, upside potential has to be relinquished if no extra external funding is available. The combination of an out-of-the-money put and a shorted out-of-the-money call is used to transfer probability mass from the investment portfolio's rates of return for relatively extreme economic scenarios (where the stock price increases or decreases dramatically) to more moderate ones. The direct protection offered by put options is suboptimal.

Hsuhku (2007) and Tan (2009) study a problem in which investors are assumed to derive utility from consumption, which means that spending off cumulated wealth is admissible. This is what pension funds do off cumulated wealth as a result of withdrawals. Interestingly, Tan finds that when there is no non-capital income, the utility cost of not being able to add long call or put positions is generally small, while when there is non-capital income in the form of wage income reasonably calibrated, the same cost for long call option positions can be substantial while that for long put option positions is close to zero. Hsuhku assumes instead that investors have Duffie and Epstein's (1992) continuous-time recursive preferences that allow to separate an investor's elasticity of intertemporal substitution in consumption from the coefficient of relative risk aversion. When he introduces non-redundant derivative securities written on the risky stock in an incomplete financial market, the derivative provides differential exposures to stochastic volatility and make the markets complete. The derivative securities can also supplement the deficient hedging ability of the intertemporal hedging component of the risky stock, because of the nonlinear nature of derivative payoffs.

The work by Faias and Santa Clara (2011) has simulated in real time realized, out-of-sample risk-adjusted returns from investing in the riskless asset, the S&P 500 index, and four constant moneyness, one-month option contracts (one ATM call, one ATM put, a 5% OTM call, and a 5% OTM put option) under power utility, similarly to what we have proposed in our paper, but assuming a flexible process for index returns. Hence they adopt a simulation approach to portfolio optimization similar to Cui, Oldenkamp, and Vellekoop's (2013) incorporating realistic transaction costs. They find that investors could have obtained high Sharpe ratios and positive certainty equivalents over a January 1996 - October 2010 sample. The best strategy yields a Sharpe ratio of 0.50. This compares well with the Sharpe ratio of the market in the same period of 0.13. Several strategies also present positive skewness and low excess kurtosis, which is not achievable simply by shorting individual options. Their strategies load significantly on all four options and are almost delta-neutral. On average, we hold long positions of ATM puts and OTM calls and short positions of OTM puts. The holdings of ATM call options change the most over time. It would be interesting to extend their exercise to structured investment products, when the resulting payoff function is made endogenous to some extent.

In this paper, for simplicity, we have used only European call options in our buy-and-hold strategies. A natural extension is to include more complex derivatives, perhaps with path dependences such as knock-out or average-rate options. This extension may be especially relevant in the presence of predictable drifts and volatility, when portfolios are restricted to be of the buy-and-hold type, so that the first-best cannot be attained (see the discussion in Haugh and Lo, 2001). Finally, more work remains to be done to facilitate the implementation of ALM (asset-liability management) strategies which focus on allocations in terms of risk premia instead of particular assets.

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