MSc. Finance & CLEFIN A.A. 2013/2014

Prep Course in Statistics – Prof. Massimo Guidolin

A Few Review Questions and Problems concerning, Hypothesis Testing and Confidence Intervals

SUGGESTION: try to approach the questions first, without looking at the answers. Occasionally the questions/problems explicitly send you to check out concepts and definitions that have not explicitly discussed during the prep course. In this case—besides Casella and Berger's book—Wikipedia may be very useful, as always.

1. (Likelihood Ratio Test for a Normal Population Mean when the Variance is Unknown) Suppose X_1, ..., X_n is a random sample from a $N(\mu, \sigma^2)$ but an experimenter is interested only in inferences about μ, such as performing the one-sided mean test H_o: $\mu \leq \mu_0$ vs. H_1: $\mu > \mu_0$. In this case that the parameter σ^2 is a nuisance parameter. Derive the LRT and show that the test based on LRT($\mathbf{x}$) has the same structure as a test that defines the rejection region on the basis of a Student's t statistic.

Solution. The LRT statistic is (in this case, taking the max or the sup delivers the same value function, ask yourself why):

$$LRT(\mathbf{x}) = \frac{\max_{\{\mu,\sigma^2:\mu\leq\mu_0,\ \sigma^2\geq 0\}} L(\mu,\sigma^2;\mathbf{x})}{\max_{\{\mu,\sigma^2:\sigma^2\geq 0\}} L(\mu,\sigma^2;\mathbf{x})} = \frac{\max_{\{\mu,\sigma^2:\mu\leq\mu_0,\ \sigma^2\geq 0\}} L(\mu,\sigma^2;\mathbf{x})}{L(\hat{\mu}^{ML},\hat{\sigma}^2_{ML};\mathbf{x})},$$

where $\hat{\mu}^{ML}$ and $\hat{\sigma}^2_{ML}$ are the well-known ML estimators. At this point, if $\hat{\mu} \leq \mu_0$ then the restricted maximum is the same as the unconstrained one and numerator and denominator will be identical; otherwise, the restricted maximum is $L(\mu_0, \sigma_0^2; \mathbf{x})$, where $\sigma_0^2 \equiv n^{-1}\sum_{i=1}^n (x_i - \mu_0)^2$. Therefore:

$$LRT(\mathbf{x}) = \begin{cases} 1 & \bar{x} < \mu_0 \\ \frac{L(\mu_0,\sigma_0^2;\mathbf{x})}{L(\hat{\mu}^{ML},\hat{\sigma}^2_{ML};\mathbf{x})} = \frac{(\sigma_0^2)^{-n/2}}{(\hat{\sigma}^2)^{-n/2}} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} & \bar{x} \geq \mu_0 \end{cases},$$

because we know that $\hat{\mu}^{ML} = \bar{x}$. At this point, note that

$$\begin{aligned} \hat{\sigma}^2 &= \frac{n-1}{n}S^2 \\ \sigma_0^2 &= n^{-1}\sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu_0)^2 = n^{-1}\sum_{i=1}^n (x_i - \bar{x})^2 + n^{-1}\sum_{i=1}^n (\bar{x} - \mu_0)^2 + \\ &\quad +2n^{-1}\sum_{i=1}^n (x_i - \bar{x})(\bar{x} - \mu_0) \end{aligned}$$

$$
\begin{aligned}
&= n^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2+(\bar{x}-\mu_0)^2+2n^{-1}(\bar{x}-\mu_0)\sum_{i=1}^{n}(x_i-\bar{x})\\
&= \frac{n-1}{n}S^2+(\bar{x}-\mu_0)^2.
\end{aligned}
$$

As a result, because

$$
LRT(\mathbf{x})=\begin{cases}1 & \bar{x}<\mu_0\\ \left(\frac{\frac{n-1}{n}S^2}{\frac{n-1}{n}S^2+(\bar{x}-\mu_0)^2}\right)^{n/2} & \bar{x}\geq\mu_0\end{cases}=\begin{cases}1 & \bar{x}<\mu_0\\ \left(\frac{\frac{n-1}{n}}{\frac{n-1}{n}+\frac{(\bar{x}-\mu_0)^2}{S^2}}\right)^{n/2} & \bar{x}\geq\mu_0\end{cases},
$$

the rejection region $LRT(\mathbf{x})<c$ implies that if $\bar{x}<\mu_0$ rejection is impossible, as one would expect, while if $\bar{x}\geq\mu_0$,

$$
\frac{\frac{n-1}{n}}{\frac{n-1}{n}+\frac{(\bar{x}-\mu_0)^2}{S^2}}<c^{-\frac{n}{2}}\Longrightarrow\frac{(\bar{x}-\mu_0)^2}{S^2/n}>(n-1)\left(c^{\frac{n}{2}}-1\right)
$$

which will be the case when $(\bar{x}-\mu_0)^2/S^2$ is sufficiently large exceeding an increasing function of the fixed c. Now note that

$$
\begin{aligned}
\Pr\left(\bar{x}>\mu_0+t_{n-1,\alpha}\frac{S}{\sqrt{n}}\middle|\,\mu=\mu_0\right) &= \Pr\left(\underbrace{\frac{\bar{x}-\mu_0}{S/\sqrt{n}}}_{T_{n-1}}>t_{n-1,\alpha}\middle|\,\mu=\mu_0\right)\\
&= \Pr\left(\frac{(\bar{x}-\mu_0)^2}{S^2/n}>t^2_{n-1,\alpha}\middle|\,\mu=\mu_0\right)=\alpha,
\end{aligned}
$$

which has the structure as the rejection region determined above.

The lesson of this exercise is: Besides being an extension of an example analyzed during the lecture, this problem shows that the structure of LRTs is independent—for a given and fixed c—from the specific statistical assumptions that are made.

2. (Computing and Optimizing a Normal Power Function) Let X_1, ..., X_n be a random sample from a $N(\mu,\sigma^2)$ population, σ^2 known. An LRT of H_o: $\mu\leq\mu_0$ versus H_1: $\mu.\mu_0$ is a test that rejects H_o if

$$
\mathrm{LRT}\equiv\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}>c.
$$

The constant $c\in[0,1]$ can be any positive number.

(a) Compute the power function of this LRT.

(b) Suppose the experimenter wishes to have a maximum Type I Error probability of 0.1. Suppose, in addition, the experimenter wishes to have a maximum Type II Error probability of 0.2 if $\mu\geq\mu_0+\sigma$. Show how to choose c and n to achieve these goals.

Solution. (a) From the definition of power function,

$$
\begin{aligned}
\beta(\mu) &= \Pr(\bar{X}\in R)=\Pr\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}>c\right)=\Pr\left(\frac{\bar{x}-\mu+\mu-\mu_0}{\sigma/\sqrt{n}}>c\right)\\
&= \Pr\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}+\frac{\mu-\mu_0}{\sigma/\sqrt{n}}>c\right)=\Pr\left(\underbrace{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}_{Z\sim N(0,1)}>c-\frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)\\
&= \Pr\left(Z>c-\frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right).
\end{aligned}
$$

As μ increases from $-\infty$ to $+\infty$, it is easy to see that this normal probability increases from 0 to 1. Therefore, it follows that $\beta(\mu)$ is an increasing function of μ, with

$$\begin{aligned}
\lim_{\mu\to-\infty}\beta(\mu) &= \lim_{\mu\to-\infty}\Pr\left(Z > c - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right) = \Pr\left(Z > c+\infty\right) = 0 \\
\lim_{\mu\to+\infty}\beta(\mu) &= \lim_{\mu\to-\infty}\Pr\left(Z > c - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right) = \Pr\left(Z > c-\infty\right) = 1.
\end{aligned}$$

(b) As noted above, the power function of a LRT test in which one rejects if $\bar{x}-\mu_0/(\sigma/\sqrt{n}) > c$ is

$$\beta(\mu) = \Pr\left(Z > c - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right).$$

Because $\beta(\mu)$ is increasing in μ, so that solutions to the equations below are unique, the requirements will be met if

$$\beta(\mu_0) = 0.1 \text{ and } \beta(\mu_0+\sigma) = \underbrace{1-0.2}_{\beta(\mu_0)=1-\text{ type II error}} = 0.8.$$

By choosing $c = 1.28$, we achieve $\beta(\mu_0) = \Pr(Z > 1.28) = 0.1$, regardless of n. Now we wish to choose n so that $\beta(\mu_0+\sigma) = \Pr(Z > 1.28-\sqrt{n}) = 0.8$. However, we know that $\Pr(Z > -0.84) = 0.8$, so that setting $1.28-\sqrt{n} = -0.84$ and solving for n, yield $n = 4.49$. Of course n must be an integer. So choosing $c = 1.28$ and $n = 5$ yield a test with error probabilities controlled as specified by the experimenter.

3. (UMP Binomial Test Derived from Neyman-Pearson's Lemma) Notice that the conditions of Neyman-Pearson's lemma for simple hypotheses may be re-written as:

$$\begin{cases} \mathbf{x}\in R & \text{if } \frac{f(\mathbf{x};\theta_1)}{f(\mathbf{x};\theta_0)} > k \\ \mathbf{x}\in R^c & \text{if } \frac{f(\mathbf{x};\theta_1)}{f(\mathbf{x};\theta_0)} \le k \end{cases} \quad \text{for some } k\ge 0 \text{ and } \alpha = \Pr{}_{\theta\in\Theta_o}(\mathbf{x}\in R),$$

assuming $f(\mathbf{x};\theta_0)\neq 0$. Let $X\sim Bi(2,\theta)$. Use Neyman-Pearson lemma to show that in the test of H_o: $\theta = 1/2$ vs. H_1: $\theta = 3/4$, the test that rejects the null hypothesis if $X = 2$ is the UMP test at size $\alpha = 0.25$.

Solution. Calculating the ratios of PMFs featured in the lemma, we have that for the three possibile realizations of X:

$$\begin{aligned}
\frac{f(0;\theta_1)}{f(0;\theta_0)} &= \frac{\binom{2}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^2}{\binom{2}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^2} = \frac{1/16}{1/4} = \frac{1}{4} \\
\frac{f(1;\theta_1)}{f(1;\theta_0)} &= \frac{\binom{2}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^1}{\binom{2}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1} = \frac{3/16}{1/4} = \frac{3}{4} \\
\frac{f(2;\theta_1)}{f(2;\theta_0)} &= \frac{\binom{2}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^0}{\binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0} = \frac{9/16}{1/4} = \frac{9}{4}.
\end{aligned}$$

If we choose $(3/4) < k < (9/4)$, the Neyman-Pearson Lemma says that the test that rejects the null hypothesis if $X = 2$ and otherwise fails to reject the null of $\theta = 1/2$ is the UMP test at size $\alpha = \Pr(X = 2; \theta = 1/2) = 0.25$. Note that the proof really consists of showing the existence of such a $k \geq 0$, and $k^* \in (\frac{3}{4}, \frac{9}{4})$ satisfies this condition.

4. (Deriving Normal One-Sided Confidence Bounds by Inversion) Let X_1, ..., X_n be a random sample from a $N(\mu, \sigma^2)$ population. Construct a $1 - \alpha$ upper confidence bound for μ, that is, we want a confidence interval of the form $C(\mathbf{x}) = (-\infty, U(\mathbf{x})]$.

Solution. To obtain such an interval by inversion, consider the one-sided tests (plural because as we change μ_0 we derive different tests) of H_o: $\mu = \mu_0$ versus H_1: $\mu < \mu_0$. Note that we use the specification of H_1 t determine the form of the confidence interval: H_1 specifies "large" values of μ_0, so the confidence set will contain "small" values, values less than a bound; thus, we will get an upper confidence bound. The size α LRT of H_o versus H_1 rejects H_o if

$$\frac{\bar{X} - \mu_0}{S/\sqrt{n}} < -t_{n-1,\alpha}.$$

Thus the non-rejection (abusing language, one may say acceptance) region for this test is:

$$\left\{\mathbf{x} : \bar{X} \geq \mu_0 - t_{n-1,\alpha}\frac{S}{\sqrt{n}}\right\}$$

and this occurs with probability $1 - \alpha$. Inverting this expression to find an inequality that expresses μ as a function of sample statistics, we have:

$$\Pr\left\{\mu_0 \leq \bar{X} + t_{n-1,\alpha}\frac{S}{\sqrt{n}}\right\} = 1 - \alpha,$$

which means that the desired one-sided confidence bound is $(-\infty, \bar{X} + t_{n-1,\alpha}S/\sqrt{n}]$.

5. (Confidence Interval for the Sample Variance) Let X_1, ..., X_n be a random sample from a $N(\mu, \sigma^2)$ population. Construct $1 - \alpha$ confidence intervals for σ^2 and σ, respectively.

Solution. Because we know from lecture 1 that

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

A confidence interval is defined by a choice of a and b ($a < b$) such that

$$\begin{aligned}
\Pr\left\{a \leq (n-1)\frac{S^2}{\sigma^2} \leq b\right\} &= \Pr\left\{l \leq \chi^2_{n-1} \leq u\right\} \\
&= \Pr\left\{\chi^2_{n-1,1-\alpha/2} \leq \chi^2_{n-1} \leq \chi^2_{n-1,\alpha/2}\right\} = 1 - \alpha
\end{aligned}$$

where $\chi^2_{n-1,1-\alpha/2}$ and $\chi^2_{n-1,\alpha/2}$ are the critical values under the chi-square the leave $\alpha/2$ probabilities in both tails. This choice splits the probability equally, putting $\alpha/2$ in each tail of the distribution (but the χ^2_{n-1} distribution, however, is a skewed distribution and it is not immediately clear that an equal probability split is optimal for a skewed distribution). We can invert this set to find

$$\Pr\left\{\frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)S^2} \leq \frac{1}{\sigma^2} \leq \frac{\chi^2_{n-1,\alpha/2}}{(n-1)S^2}\right\} = \Pr\left\{\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right\},$$

or equivalently

$$\Pr\left\{\sqrt{\frac{(n-1)}{\chi^2_{n-1,\alpha/2}}}S \leq \sigma \leq \sqrt{\frac{(n-1)}{\chi^2_{n-1,1-\alpha/2}}}S\right\}.$$

6. (MLEs and LRT-Based Tests in the Case of a Pareto Distribution) Let X_1, ..., X_n be a random sample from a Pareto population with PDF

$$f(x;\theta,v) = \frac{\theta v^\theta}{x^{\theta+1}} I_{[v,+\infty)}(x) \qquad \theta > 0,\ v > \theta \quad I_{[v,+\infty)}(x) = \begin{cases} 1 & \text{if } x \geq v \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the MLEs of θ and v.

(b) Show that the LRT of H_o: $\theta = 1$ and v unknown, versus H_1: $\theta \neq 1$ and v unknown, has critical region of the form $\{\mathbf{x} : \Lambda(\mathbf{x}) \leq c_1,\ \Lambda(\mathbf{x}) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$\Lambda(\mathbf{x}) = \ln\left[\frac{\prod_{i=1}^n x_i}{(\min_i x_i)^n}\right] > 0$$

(*Hint*: note that by construction $\prod_{i=1}^n x_i/(\min_i x_i)^n > 1$).

Solution. (a) The log-likelihood is

$$\begin{aligned}
\ln L(\theta,v;\mathbf{x}) &= \ln \prod_{i=1}^n \frac{\theta v^\theta}{x_i^{\theta+1}} I_{[v,+\infty)}(x_i) = \begin{cases} \ln\left\{\theta^n v^{\theta n} \prod_{i=1}^n \left(\frac{1}{x_i}\right)^{\theta+1}\right\} & \text{if } v \leq \min_i x_i \\ -\infty & \text{if } v > \min_i x_i \end{cases} \\
&= \begin{cases} n\ln\theta + \theta n \ln v + (\theta+1)\ln\prod_{i=1}^n \frac{1}{x_i} & \text{if } v \leq \min_i x_i \\ -\infty & \text{if } v > \min_i x_i \end{cases} \\
&= \begin{cases} n\ln\theta + \theta n \ln v - (\theta+1)\ln\prod_{i=1}^n x_i & \text{if } v \leq \min_i x_i \\ -\infty & \text{if } v > \min_i x_i \end{cases}.
\end{aligned}$$

For any value of , this is an increasing function of v for $v \leq \min_i x_i$. So both the restricted and unrestricted MLEs of are $\hat{v}^{MLE} = \min_i x_i$. This is a typical example of an MLE that follows the definition of lecture 2, but that does not correspond to an interior (global) maximum of the log-likelihood function and that as such is NOT obtained by solving a first-order condition. To find the MLE of θ, for $\hat{v}^{MLE} = \min_i x_i$ set

$$\frac{\partial \ln L(\theta, \hat{v}^{MLE};\mathbf{x})}{\partial\theta} = \frac{n}{\theta} + n\ln\min_i x_i - \ln\prod_{i=1}^n x_i$$

to zero, to obtain:

$$\frac{n}{\hat{\theta}} + n\ln\min_i x_i - \ln\prod_{i=1}^n x_i = 0 \Longrightarrow \hat{\theta}^{MLE} = \frac{n}{\ln\left[(\prod_{i=1}^n x_i)/(\min_i x_i)^n\right]} = \frac{n}{\Lambda(\mathbf{x})}.$$

Moreover,

$$\frac{\partial^2 \ln L(\hat{\theta}^{MLE}, \hat{v}^{MLE};\mathbf{x})}{\partial\theta^2} = -\frac{n}{(\hat{\theta}^{MLE})^2} = -\frac{n^2}{\Lambda(\mathbf{x})} < 0$$

(as $\Lambda(\mathbf{x}) > 0$) for all possible realizations of the sample, so that $\hat{\theta}^{MLE}$ indeed represents an (interior) MLE.

(b) Under H_o, the MLE of is $\hat{\theta} = 1$, and the MLE of is still $\hat{v} = \min_i x_i$. So the likelihood ratio statistic is

$$
\begin{aligned}
LRT(\mathbf{x}) &= \frac{\exp[n \ln 1 + n \ln \min_i x_i - (1+1)\ln \prod_{i=1}^n x_i]}{\exp\left[n \ln \frac{n}{\Lambda(\mathbf{x})} + \frac{n}{\Lambda(\mathbf{x})} n \ln \min_i x_i - \left(\frac{n}{\Lambda(\mathbf{x})} + 1\right) \ln \prod_{i=1}^n x_i\right]} \\
&= \frac{\frac{(\min_i x_i)^n}{\left(\prod_{i=1}^n x_i\right)^2}}{\left(\frac{n}{\Lambda(\mathbf{x})}\right)^n \frac{[(\min_i x_i)^n]^{\frac{n}{\Lambda(\mathbf{x})}}}{\left(\prod_{i=1}^n x_i\right)^{\frac{n}{\Lambda(\mathbf{x})}}} \prod_{i=1}^n x_i} = \frac{\frac{e^{-\Lambda(\mathbf{x})}}{\prod_{i=1}^n x_i}}{\left(\frac{n}{\Lambda(\mathbf{x})}\right)^n e^{-\Lambda(\mathbf{x}) \cdot \frac{n}{\Lambda(\mathbf{x})}} \prod_{i=1}^n x_i} \\
&= \frac{e^{-\Lambda(\mathbf{x})}}{\left(\frac{n}{\Lambda(\mathbf{x})}\right)^n e^{-n}} = \left(\frac{\Lambda(\mathbf{x})}{n}\right)^n e^{-\Lambda(\mathbf{x})+n}.
\end{aligned}
$$

Note now that

$$
\begin{aligned}
\frac{\partial \ln LRT(\mathbf{x})}{\partial \Lambda(\mathbf{x})} &= \frac{\partial}{\partial \Lambda(\mathbf{x})} \left\{ n \ln \left(\frac{\Lambda(\mathbf{x})}{n}\right) - \Lambda(\mathbf{x}) + n \right\} \\
&= \frac{n}{\Lambda(\mathbf{x})} - 1,
\end{aligned}
$$

so that $LRT(\mathbf{x})$ is increasing for $\Lambda(\mathbf{x}) \leq n$ and decreasing if $\Lambda(\mathbf{x}) > n$. Thus, $\Lambda(\mathbf{x}) \leq c$ is equivalent to $\Lambda(\mathbf{x}) \leq c_1$ or $\Lambda(\mathbf{x}) \geq c_2$, for appropriately chosen constants c_1 and c_2.

7. (A Simple Application of the CLT) For a random sample X_1, ..., X_n of Bernoulli(p) variables, it is desired to test H_o: $p = 0.49$ versus H_1: $p = 0.51$. Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about .01. Use a test function that rejects H_o if $\sum_{i=1}^n x_i$ is large.

Solution. The CLT tells us that

$$
Z = \frac{\sum_{i=1}^n x_i - np}{\sqrt{np(1-p}}
$$

is approximately $N(0,1)$. For a test that rejects H_o when $\sum_{i=1}^n x_i \geq c$, we need to find c and n to satisfy

$$
\Pr\left(Z > \frac{c - n(0.49)}{\sqrt{n \cdot 0.49 \cdot 0.51}}\right) = 0.01 \text{ and } \Pr\left(Z > \frac{c - n(0.51)}{\sqrt{n \cdot 0.49 \cdot 0.51}}\right) = 0.99.
$$

We thus want

$$
\frac{c - n(0.49)}{\sqrt{n \cdot 0.49 \cdot 0.51}} \simeq 2.33 \text{ and } \frac{c - n(0.51)}{\sqrt{n \cdot 0.49 \cdot 0.51}} \simeq -2.33.
$$

Solving these equations, gives $n = 13,567$ and $c = 6,783.5$.

8. (Abusing the Size α in Practical Applications) One very striking abuse of α size levels is to choose them after seeing the data and to choose them in such a way as to force rejection (or acceptance) of a null hypothesis. To see what the true Type I and Type II error probabilities of such a procedure are, calculate size and power of the following two trivial tests:

(a) Always reject H_o, no matter what data are obtained (equivalent to the practice of choosing the α size level to force rejection of H_o).

(b) Always accept H_o, no matter what data are obtained (equivalent to the practice of choosing the α size level to force acceptance of H_o).

Solution. (a)

$$\begin{aligned} Size &= \Pr(\text{reject } H_0|H_0 \text{ is true}) = 1) \Longrightarrow \text{Type I error} = 1. \\ Power &= \Pr(\text{reject } H_0|H_1 \text{ is true}) = 1) \Longrightarrow \text{Type II error} = 0. \end{aligned}$$

(b)

$$\begin{aligned} Size &= \Pr(\text{reject } H_0|H_0 \text{ is true}) = 0) \Longrightarrow \text{Type I error} = 0. \\ Power &= \Pr(\text{reject } H_0|H_1 \text{ is true}) = 0) \Longrightarrow \text{Type II error} = 1. \end{aligned}$$

References

Casella, G., and R.L., Berger, *Statistical inference*, Duxbury Press, 2002.