Risk Measures: Rationality and Diversification*

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Abstract

When there is uncertainty about interest rates (typically due to either illiquidity or defaultability of zero coupon bonds) the cash-additivity assumption on risk measures becomes problematic. When this assumption is weakened, to cash-subadditivity for example, the equivalence between convexity and the diversification principle no longer holds. In fact, this principle only implies (and it is implied by) quasiconvexity.

For this reason, in this paper quasiconvex risk measures are studied. We provide a dual characterization of quasiconvex cash-subadditive risk measures and we establish necessary and sufficient conditions for their law invariance. As a byproduct, we obtain an alternative characterization of the actuarial mean value premium principle.

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1 Introduction

Risk assessment is a fundamental activity for both regulators and agents in financial markets. The problem of a formal definition of a risk measure and of the economic and mathematical properties that it should satisfy has been heating the debate since the seminal papers of Artzner, Delbaen, Eber, and Heath (1997, 1999) on *coherent risk measures*.

In the last ten years there has been a flourishing of methodological proposals, mathematical extensions, and variations on this topic. The *convex monetary risk measures* of Föllmer and Schied (2002, 2004) and Frittelli and Rosazza Gianin (2002) are especially interesting in terms of economic content and mathematical tractability among the generalizations of coherent risk measures. Moreover, these measures naturally appear in pricing and hedging problems in incomplete markets, as shown, for example, by El Karoui and Quenez (1997), Carr, Geman, and Madan (2001), Frittelli and Rosazza Gianin (2004), Staum (2004), Filipović and Kupper (2008), and Jouini, Schachermayer, and Touzi (2008).

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A risk measure is a decreasing function ρ that associates to a future risky position X the minimal reserve amount $\rho(X)$ that should be collected today to cover risk X. The leading examples are solvency capital requirements imposed by supervising agencies to insurance companies and financial institutions. Decreasing monotonicity is a minimal *rationality* requirement imposed on the agencies: higher losses require higher reserves.

Convex monetary risk measures have the additional requirement of being convex and cash-additive.[1] As pointed out by El Karoui and Ravanelli (2008), cash-additivity fails as soon as there is any form of uncertainty about interest rates; for example when the risk-free asset is illiquid or inexistent.[2] For this reason, they suggest to replace cash-additivity with cash-subadditivity, and, maintaining convexity, they provide a representation result for *convex cash-subadditive risk measures*, together with several examples arising from applications.

This paper starts from the observation that once cash-additivity is replaced with the economically sounder assumption of cash-subadditivity, *convexity* should be replaced by *quasiconvexity* in order to maintain the original interpretation in terms of *diversification*. Although convexity is generally regarded as the mathematical translation of the fundamental principle "diversification cannot increase risk," literally this principle means

"if positions X and Y are less risky than Z, so it is any diversified position $\lambda X + (1-\lambda) Y$ with λ in $(0,1)$."

Using a measure of risk ρ, this statement translates into

"$\rho(X), \rho(Y) \leq \rho(Z)$ implies $\rho(\lambda X + (1-\lambda) Y) \leq \rho(Z)$ for all λ in $(0,1)$,"

which is equivalent to convexity under the cash-additivity assumption, while in general (also under cash-subadditivity) it only corresponds to quasiconvexity.[3]

From a financial viewpoint, the passage from convexity to quasiconvexity is conceptually very important. It allows a complete disentangling between the diversification principle, which is arguably the central pillar of risk management, and the assumption of liquidity of the riskless asset, which is an abstract (still very useful and popular) simplification. The economic counterpart of quasiconvexity of risk measures is quasiconcavity of utility functions (that is, convexity of preferences), which is classically associated to uncertainty aversion in the economics of uncertainty (see, e.g., Debreu, 1959, and Schmeidler, 1989). Uncertainty aversion, namely

"if X and Y are preferred to Z, so it is any mixture $\lambda X + (1-\lambda) Y$ with λ in $(0,1)$,"

is one of the soundest empirical findings in situations where agents ignore the probabilistic model that underlines the economic phenomenon they are facing (for example, it has been recently indicated by Caballero and Krishnamurthy, 2008, as one of the possible causes behind the 2008 crisis).

For this reason, in this paper we study *quasiconvex cash-subadditive risk measures* on an L^∞ space.[4] We show in Section 3 that these measures take the form

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} R\left(\mathbb{E}_Q(-X), Q\right), \tag{1}$$

[1]See Section 2 for details and formal definitions.

[2]Black (1972) is one of the first contributions that casted doubts on liquidity and existence of riskless assets.

[3]See Föllmer and Schied (2008), Proposition 1, and Example 2.

[4]The extension to the general L^p case is studied in Appendix A.

where $\mathcal{M}_{1,f}$ is the set of (finitely additive) probabilities and $R : \mathbb{R} \times \mathcal{M}_{1,f} \to [-\infty, \infty]$ is an upper semicontinuous quasiconcave function that is increasing and nonexpansive in the first component and such that $\inf_{t \in \mathbb{R}} R(t, \cdot)$ is constant. The function R is unique.

Convex monetary risk measures correspond to the separable specification

$$R(t, Q) = Dt - \alpha(DQ) \tag{2}$$

for some constant $D \in (0, 1]$, while convex cash-subadditive risk measures correspond to

$$R(t, Q) = \sup_{c \in [0,1]} (ct - \alpha(cQ)), \tag{3}$$

where $\alpha(\cdot)$ is the Fenchel conjugate of $\rho(-\cdot)$.[5]

Representation (1) is not only general enough to capture most of the risk measures introduced in the literature, but it also has a very natural interpretation: $R(t, Q)$ is the reserve amount required today, under the probabilistic scenario Q, to cover an expected loss t in the future. Since there is uncertainty about probabilistic scenarios, the supervising agency follows the most cautious approach, that is, it requires the maximum reserve. The evaluations $R(t, Q)$ keep two factors into account, the expected loss t and the plausibility of scenario Q, assessed by the supervising agency. As the special cases (2) and (3) show (see again the discussion in El Karoui and Ravanelli, 2008), the separability of these two risk factors is lost as soon as risky positions and reserve amounts cannot be expressed in the same numeraire in an unambiguous way. This loss of separability becomes even clearer if, inspired by (2), one sets $a(t, Q) = t - R(t, Q)$ and rewrites (1) as

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q(-X) - a(\mathbb{E}_Q(-X), Q)\} \tag{4}$$

where the "penalty function" $a(\mathbb{E}_Q(-X), Q)$ now depends both on the probabilistic scenario and on the expected loss of the position (rather than on the probabilistic scenario alone).

It is important to observe that, while the results on convex and cash-subadditive measures build on classic convex duality, our results build on the quasiconvex monotone duality developed in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b). Specifically, the techniques developed there are the main tool for our analysis in the L^∞ case. We extend them in Appendix A where the general L^p case ($1 \leq p \leq \infty$) is considered. As a consequence, there is also a substantial difference between the mathematics that underlies our results and that used in the study of convex risk measures.

In view of the importance of law-invariance with respect to a given probability measure P, in Section 5 we characterize quasiconvex risk measures that satisfy this property and we show that in this case the quantile representation

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 q_{-X}(s)\, q_{\frac{dQ}{dP}}(s)\, ds, Q\right) \tag{5}$$

holds. This result extends those of Chong and Rice (1971), Kusuoka (2001), Föllmer and Schied (2004), Dana (2005), Frittelli and Rosazza Gianin (2005), and Leitner (2005) from the domain of convex analysis to that of quasiconvex analysis. Technically speaking, this is one of the most substantive contributions of the present paper.

As a byproduct, in Subsection 5.1 we characterize the risk measures that agree with the actuarial *mean value premium principle* (see Rotar, 2007), that is, the measures of the form

$$\rho(X) = \ell^{-1}(\mathbb{E}_P(\ell(-X))), \tag{6}$$

[5] More in general, there is a natural correspondence between the properties of ρ and those of R as shown in Section 4.

where ℓ is a strictly increasing and convex loss function. Though in a static setting, this result is in the spirit of a very recent one of Kupper and Schachermayer (2008) and it builds on the classic Nagumo-Kolmogorov-de Finetti Theorem.[6] Interestingly, Proposition 12 shows that for this class of functions

$$R(t,Q) = t - L(-t;Q,P), \tag{7}$$

where $L(-t;Q,P)$ is the generalized distance between probability measures induced by ℓ, introduced by Bellini and Frittelli (2002) in the context of minimax martingale measures. The closing Subsection 5.2 proposes maxima of risk measures of the form (6) as emerging from the agreement of a group of supervising agencies, and studies their properties.

We conclude by observing that, as it happens with coherent risk measures and maxmin expected utility preferences (Gilboa and Schmeidler, 1989), or convex monetary risk measures and variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), also quasiconvex risk measures have a decision theoretic foundation: the uncertainty averse preferences we recently studied in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008a) to which we refer the interested reader for details.

2 Preliminaries

Let $(\Omega,\mathcal{A},P)$ be a probability space and $L^\infty(\Omega,\mathcal{A},P)$ be the space of bounded random variables.[7] Its topological dual $L^\infty(\Omega,\mathcal{A},P)^*$ is isometrically isomorphic to the space of all bounded finitely additive set functions on $\mathcal{A}$ that are absolutely continuous with respect to P (see, e.g., Yosida, 1980, Ch. IV.9).

The positive unit ball of $L^\infty(\Omega,\mathcal{A},P)^*$ is denoted by $\mathcal{M}_{1,f}(\Omega,\mathcal{A},P)$ and coincides with the set of finitely additive probabilities that are absolutely continuous with respect to P; in particular, $\mathcal{M}_1(\Omega,\mathcal{A},P)$ is the subset of $\mathcal{M}_{1,f}(\Omega,\mathcal{A},P)$ consisting of all its countably additive elements. For this reason, given $X\in L^\infty(\Omega,\mathcal{A},P)$ and $\mu\in L^\infty(\Omega,\mathcal{A},P)^*$, we indifferently write: $\mu(X)$, $\int X d\mu$, or even $\mathbb{E}_\mu(X)$ if $\mu\in\mathcal{M}_{1,f}(\Omega,\mathcal{A},P)$. The specification of the probability space $(\Omega,\mathcal{A},P)$ is often omitted and we just write L^∞ and $\mathcal{M}_{1,f}$.

Unless otherwise stated, $L^\infty(\Omega,\mathcal{A},P)$ is endowed with its norm topology, $L^\infty(\Omega,\mathcal{A},P)^*$ is endowed with its weak* topology, and its subsets with the relative weak* topology. Product spaces are endowed with the product topology.

We consider one period of uncertainty $\{0,T\}$. The elements of L^∞ represent payoffs at time T of financial positions held at time 0.

A **risk measure** is a decreasing function $\rho : L^\infty \to [-\infty,\infty]$.

As anticipated in the introduction, $\rho(X)$ is interpreted as the minimal reserve amount that should be collected today to cover future risk X. Decreasing monotonicity is justified by the fact that smaller losses cannot require greater reserves.

Given a (deterministic) discount factor $D\in(0,1]$, the function ρ is a *monetary* risk measure if, in addition, it satisfies:

Cash-additivity $\rho(X-m)=\rho(X)+Dm$ for all $X\in L^\infty$ and $m\in\mathbb{R}$.

This condition is interpreted in the following way "when m dollars are subtracted from the future position the present capital requirement is augmented by the same discounted amount Dm." In fact, investing Dm in a risk-free manner offsets the certain future loss m.

[6] See, Nagumo (1930), Kolmogorov (1930), de Finetti (1931), as well as Hardy, Littlewood, and Pólya (1934).

[7] Equalities and inequalities among random variables hold almost surely with respect to P.

Cash-additivity is a controversial assumption both from a theoretical and practical viewpoint. For, D is the price of a non-defaultable zero coupon bond available on the market at time 0 with maturity T and face value 1: existence and liquidity of such an asset is not an innocuous assumption and, as observed by El Karoui and Ravanelli (2008), any form of uncertainty in interest rates is sufficient to make the cash-additivity assumption too stringent. For example, in case of illiquidity, D may well depend on the amount m of purchased assets.

These considerations lead to the following relaxed version of cash-additivity, which only takes into account the time value of money:

Cash-subadditivity $\rho(X-m) \leq \rho(X) + m$ for all $X \in L^\infty$ and $m \in \mathbb{R}_+$.

The meaning of this condition is "when m dollars are subtracted from a future position the present capital requirement cannot be augmented by more than m dollars." This is a much more compelling assumption than cash-additivity since it just relies on the fact that an additional reserve of m dollars surely covers the additional loss of the same amount.[8]

As discussed in the introduction, the risk diminishing effect of diversification is usually translated by:

Convexity $\rho(\lambda X + (1-\lambda) Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in L^\infty$ and $\lambda \in (0,1)$.

But, it actually corresponds to the much weaker:

Quasiconvexity $\rho(\lambda X + (1-\lambda) Y) \leq \max\{\rho(X), \rho(Y)\}$ for all $X, Y \in L^\infty$ and $\lambda \in (0,1)$.

The next simple proposition shows that convexity is equivalent to quasiconvexity for monetary risk measures.[9] Clearly, this is not the case for cash-subadditive risk measures.[10] In reading the result, recall that a function $\rho : L^\infty \to [-\infty, \infty]$ is *nonexpansive* (Lipschitz continuous with constant 1) if $\rho(Y) \leq \rho(X) + \|X-Y\|$ for all $X, Y \in L^\infty$.

Proposition 1 *Let ρ be a risk measure.*

(a) If ρ is cash-additive, then it is convex if and only if it is quasiconvex.

(b) ρ is cash-subadditive if and only if it is nonexpansive.

In both cases, ρ is either finite valued or identically $\pm\infty$.

Proof. (a) is essentially known (see, e.g., Gilboa and Schmeidler, 1989, Lemma 3.3, or Marinacci and Montrucchio, 2004, Corollary 4.2). Next we prove (b). If $\rho : L^\infty \to \mathbb{R}$ is nonexpansive then $\rho(X-m) \leq \rho(X) + 1\|X-(X-m)\|$ for all $X \in L^\infty$ and all $m \in \mathbb{R}_+$, that is, $\rho(X-m) \leq \rho(X) + m$. Conversely, for all $X, Y \in L^\infty$, $X - Y \leq \|X-Y\|$, then $X - \|X-Y\| \leq Y$, monotonicity and cash-subadditivity imply $\rho(Y) \leq \rho(X - \|X-Y\|) \leq \rho(X) + \|X-Y\|$, as wanted. ■

Next example shows how the illiquidity of the risk-free asset naturally generates quasiconvex cash-subadditive risk measures that are neither convex nor cash-additive.

[8] Notice that cash-subadditivity is equivalent to require $\rho(X+m) \geq \rho(X) - m$ for all $X \in L^\infty$ and all $m \in \mathbb{R}_+$. In fact, it implies $\rho(X) = \rho(X+m-m) \leq \rho(X+m) + m$, and the converse is proved in the same way. In particular, our definition is equivalent to that of El Karoui and Ravanelli (2008).

[9] Clearly, the above definition of quasiconvexity and the one we reported in the introduction are equivalent.

[10] See Example 2 below.

Example 2 *Let $\emptyset \subsetneq C \subsetneq L^\infty$ be the set of future positions considered acceptable by the supervising agency, and assume that C is convex and $C + L_+^\infty \subseteq C$. For all $m \in \mathbb{R}$ denote by $v(m)$ the price at time 0 of m dollars at time T and define, as in Artzner, Delbaen, Eber, and Heath (1999),*

$$\rho_{C,v}(X) = \inf\{v(m) : X + m \in C\} \quad \forall X \in L^\infty.$$

If $v(m) = Dm$ with $D \in (0,1]$ then $\rho_{C,v}$ is a (finite valued) convex monetary risk measure.[11] *The linearity of v is precisely the assumption that fails when zero coupon bonds with maturity T are illiquid. Still it remains sensible to assume that $v : \mathbb{R} \to (-\infty, \infty]$ is increasing and $v(0) = 0$.*

Provided v is also upper semicontinuous, we have

$$\rho_{C,v}(X) = v(\inf\{m \in \mathbb{R} : X + m \in C\}) = v(\rho_{C,\mathrm{id}}(X)) \quad \forall X \in L^\infty,$$

where $\mathrm{id} : \mathbb{R} \to (-\infty, \infty]$ is the identity. Moreover, since $\rho_{C,\mathrm{id}}$ is a convex monetary risk measure, then for any nonexpansive v that is not convex, $\rho_{C,v}$ is a quasiconvex cash-subadditive risk measure that is neither convex nor cash-additive.

Finally, $\mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ denotes the class of functions $R : \mathbb{R}\times\mathcal{M}_{1,f} \to [-\infty, \infty]$ that are upper semicontinuous, quasiconcave, increasing in the first component, with $\inf_{t\in\mathbb{R}} R(t,Q) = \inf_{t\in\mathbb{R}} R(t,Q')$ for all $Q, Q' \in \mathcal{M}_{1,f}$. Moreover, $\mathcal{R}_1(\mathbb{R}\times\mathcal{M}_{1,f})$ is the subset of $\mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ consisting of functions R that are nonexpansive in the first component, that is, $R(t',Q) \le R(t,Q) + |t - t'|$ for all $t, t' \in \mathbb{R}$ and all $Q \in \mathcal{M}_{1,f}$.

3 Representation

We are now ready to state and prove our first representation result.

Theorem 3 *A function $\rho : L^\infty \to [-\infty, \infty]$ is a quasiconvex cash-subadditive risk measure if and only if there exists $R \in \mathcal{R}_1(\mathbb{R}\times\mathcal{M}_{1,f})$ such that*

$$\rho(X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{8}$$

The function $R \in \mathcal{R}_1(\mathbb{R}\times\mathcal{M}_{1,f})$ for which (8) holds is unique and satisfies

$$R(t,Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_{1,f}. \tag{9}$$

Recall that ρ is a quasiconvex cash-subadditive risk measure if and only if it is a quasiconvex and nonexpansive risk measure. The next lemma characterizes quasiconvex and upper semicontinuous risk measures.

Lemma 4 *A function $\rho : L^\infty \to [-\infty, \infty]$ is a quasiconvex upper semicontinuous risk measure if and only if there exists $R \in \mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ such that*

$$\rho(X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{10}$$

The function $R \in \mathcal{R}_0(\mathbb{R}\times\mathcal{M}_{1,f})$ for which (10) holds is unique and satisfies

$$R(t,Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_{1,f}. \tag{11}$$

Proof. Notice that L^∞ is a normed Riesz space with order unit I_Ω, $\mathcal{M}_{1,f}$ is the positive unit ball of its topological dual, and $-\rho$ is a quasiconcave, lower semicontinuous, and monotone increasing function.

[11] See, for example, Föllmer and Schied (2004, Ch. 4).

The statement then follows from Lemma 8 and Theorem 3 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b). ■

Proof of Theorem 3. It only remains to show that ρ is cash-subadditive if and only if R is nonexpansive in the first component.

Suppose ρ is cash-subadditive, then, for all $(t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}$ and $m \in \mathbb{R}_+$,

$$\begin{aligned} R(t+m, Q) &= \inf\{\rho(X) : \mathbb{E}_Q(-X) = t+m\} = \inf\{\rho(X) : \mathbb{E}_Q(-(X+m)) = t\} \\ &= \inf\{\rho(Y-m) : \mathbb{E}_Q(-Y) = t\} \leq \inf\{\rho(Y) + m : \mathbb{E}_Q(-Y) = t\} = R(t, Q) + m. \end{aligned}$$

Therefore, for all $t, t' \in \mathbb{R}$ and $Q \in \mathcal{M}_{1,f}$, $t' \leq t + |t-t'|$ and monotonicity of R in the first component imply

$$R(t', Q) \leq R(t + |t-t'|, Q) \leq R(t, Q) + |t-t'|,$$

as wanted.

Conversely, if R is nonexpansive in the first component, then, for all $(t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f}$ and $m \in \mathbb{R}_+$,

$$R(t+m, Q) \leq R(t, Q) + |t-(t+m)| = R(t, Q) + m.$$

Moreover, for all $X \in L^\infty$, there is $Q' \in \mathcal{M}_{1,f}$ such that $\rho(X-m) = R(\mathbb{E}_{Q'}(-(X-m)), Q')$. Therefore,

$$\begin{aligned} \rho(X-m) &= R(\mathbb{E}_{Q'}(-(X-m)), Q') = R(\mathbb{E}_{Q'}(-X) + m, Q') \leq R(\mathbb{E}_{Q'}(-X), Q') + m \\ &\leq \max_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) + m = \rho(X) + m, \end{aligned}$$

as wanted. ■

In particular, denoting by $\alpha(\cdot)$ the Fenchel conjugate of $\rho(-\cdot)$,[12] a quasiconvex cash-subadditive risk measure ρ is convex if and only if

$$R(t, Q) = \sup_{c\in[0,1]} (ct - \alpha(cQ)) \quad \forall (t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f},$$

thus obtaining the result of El Karoui and Ravanelli (2008). Moreover, ρ is cash-additive if and only if

$$R(t, Q) = Dt - \alpha(DQ) \quad \forall (t, Q) \in \mathbb{R}\times\mathcal{M}_{1,f},$$

which corresponds to the well known characterization of convex monetary risk measures.[13]

Maintaining the interpretation of $R(t, Q)$ as the reserve amount required today to cover an expected loss t in the future under the probabilistic scenario Q, the above relations corroborate the claim of El Karoui and Ravanelli (2008) that the passage to cash-subadditivity is the most parsimonious way of taking into account interest rate uncertainty and a supervising agency that is averse to such uncertainty.

4 Additional Properties

In this section we further investigate the correspondence between the properties of ρ and those of R.

[12] That is $\alpha(\mu) = \sup_{X\in L^\infty(\Omega,\mathcal{A},P)} (\mu(X) - \rho(-X))$ for all $\mu \in L^\infty(\Omega, \mathcal{A}, P)^*$.

[13] For more details on the relations between convex duality and quasiconvex monotone duality, see Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b).

4.1 Subadditivity, Positive Homogeneity, and Star-Shapedness

Here we analyze some of the most common properties that risk measures, directly or indirectly, have been required to satisfy. We already discussed cash-additivity and convexity as well as their consequences.

The first property, introduced by Artzner, Delbaen, Eber, and Heath (1999), is very natural and suggests that the overall risk is controlled by controlling the risk of each position:

Subadditivity $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in L^{\infty}$.

The second property, still introduced by the same authors, is:

Positive Homogeneity $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \in (0, \infty)$.

Positive homogeneity was early realized to be controversial in terms of liquidity.[14] Finally, we consider:

Star-Shapedness $\rho(\lambda X) \geq \lambda\rho(X)$ for all $X \in L^{\infty}$ and $\lambda \in [1, \infty)$.

This property seems to be the sensible weakening of positive homogeneity imposing that capital requirements increase more than linearly if the position is magnified by a factor greater than 1.[15]

Next proposition characterizes the above properties of ρ in terms of properties of R. From a financial perspective it confirms the intuition that R represents scenario-dependent reserves while ρ represents their synthesis. In order to avoid indeterminacies, real valued risk measures are considered.

Proposition 5 *Let $\rho : L^{\infty} \to \mathbb{R}$ be an upper semicontinuous and quasiconvex risk measure. Then:*

(a) ρ is subadditive if and only if $R(\cdot, Q)$ is subadditive for all $Q \in \mathcal{M}_{1,f}$.

(b) ρ is positively homogeneous if and only if $R(\cdot, Q)$ is positively homogeneous for all $Q \in \mathcal{M}_{1,f}$.

(c) ρ is star-shaped if and only if $R(\cdot, Q)$ is star-shaped for all $Q \in \mathcal{M}_{1,f}$.

Proof. (a) Suppose ρ is subadditive, and let $t, t' \in \mathbb{R}$ and $Q \in \mathcal{M}_{1,f}$. Then,

$$\begin{aligned}
R(t+t', Q) &= \inf\{\rho(X) : \mathbb{E}_Q(-X) = t + t'\} \\
&\leq \inf\{\rho(Y+Z) : \mathbb{E}_Q(-Y) = t \text{ and } \mathbb{E}_Q(-Z) = t'\} \\
&\leq \inf\{\rho(Y) + \rho(Z) : \mathbb{E}_Q(-Y) = t \text{ and } \mathbb{E}_Q(-Z) = t'\} \\
&= \inf\{\rho(Y) : \mathbb{E}_Q(-Y) = t\} + \inf\{\rho(Z) : \mathbb{E}_Q(-Z) = t'\} \\
&= R(t, Q) + R(t', Q).
\end{aligned}$$

Conversely, assume $R(\cdot, Q)$ is subadditive for all $Q \in \mathcal{M}_{1,f}$, and let $X, Y \in L^{\infty}$. Then, by (10), there exists $\bar{Q} \in \mathcal{M}_{1,f}$ such that

$$\begin{aligned}
\rho(X+Y) &= R\left(\mathbb{E}_{\bar{Q}}(-X-Y), \bar{Q}\right) \leq R\left(\mathbb{E}_{\bar{Q}}(-X), \bar{Q}\right) + R\left(\mathbb{E}_{\bar{Q}}(-Y), \bar{Q}\right) \\
&\leq \max_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) + \max_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(-Y), Q) = \rho(X) + \rho(Y).
\end{aligned}$$

(b) Suppose ρ is positively homogeneous, and let $t \in \mathbb{R}$, $\lambda \in (0, \infty)$, and $Q \in \mathcal{M}_{1,f}$. Then,

$$\begin{aligned}
R(\lambda t, Q) &= \inf\{\rho(X) : \mathbb{E}_Q(-X) = \lambda t\} = \inf\left\{\rho(X) : \mathbb{E}_Q\left(-\frac{X}{\lambda}\right) = t\right\} \\
&= \inf\{\rho(\lambda Y) : \mathbb{E}_Q(-Y) = t\} = \inf\{\lambda\rho(Y) : \mathbb{E}_Q(-Y) = t\} \\
&= \lambda \inf\{\rho(Y) : \mathbb{E}_Q(-Y) = t\} = \lambda R(t, Q).
\end{aligned}$$

[14] The fact that an additional liquidity risk may arise if a position is multiplied by a large factor is indeed one of the motivations leading to the introduction of convex risk measures (see Föllmer and Schied, 2002).

[15] Notice that this property is equivalent to: $\rho(\lambda X) \leq \lambda\rho(X)$ for all $\lambda \in (0, 1]$.

Conversely, assume that $R(\cdot,Q)$ is positively homogeneous for all $Q \in \mathcal{M}_{1,f}$, and let $X \in L^{\infty}$ and $\lambda \in (0,\infty)$. Then,

$$\rho(\lambda X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-\lambda X), Q) = \lambda \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) = \lambda\rho(X).$$

(c) Suppose ρ is star-shaped, and let $t \in \mathbb{R}$, $\lambda \in [1,\infty)$, and $Q \in \mathcal{M}_{1,f}$. Then,

$$\begin{aligned} R(\lambda t, Q) &= \inf\{\rho(X) : \mathbb{E}_Q(-X) = \lambda t\} = \inf\left\{\rho(X) : \mathbb{E}_Q\left(-\frac{X}{\lambda}\right) = t\right\} \\ &= \inf\{\rho(\lambda Y) : \mathbb{E}_Q(-Y) = t\} \geq \inf\{\lambda\rho(Y) : \mathbb{E}_Q(-Y) = t\} \\ &= \lambda \inf\{\rho(Y) : \mathbb{E}_Q(-Y) = t\} = \lambda R(t,Q). \end{aligned}$$

Conversely, assume that $R(\cdot,Q)$ is star-shaped for all $Q \in \mathcal{M}_{1,f}$, and let $X \in L^{\infty}$ and $\lambda \in [1,\infty)$. Then,

$$\rho(\lambda X) = \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-\lambda X), Q) \geq \lambda \max_{Q\in\mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) = \lambda\rho(X).$$

■

Similar considerations hold for cash-additivity and convexity.[16]

We conclude this section by providing an alternative characterization of coherent risk measures (i.e. risk measures that are: cash-additive, subadditive, and positively homogeneous). The following proposition can be also proved by standard convex analysis, but the proof presented here seems shorter and elegant.

Proposition 6 *Let $\rho : L^{\infty} \to \mathbb{R}$ be a risk measure. The following conditions are equivalent:*

(i) ρ is coherent;

(ii) ρ is subadditive, star-shaped, and $\rho(-\mathrm{I}_{\Omega}) \leq -\rho(\mathrm{I}_{\Omega}) = D \in (0,1]$.

Proof. (i) implies (ii) is trivial. First, by subadditivity $\rho(0) \geq 0$. Then,

$$0 \geq \rho(\mathrm{I}_{\Omega}) + \rho(-\mathrm{I}_{\Omega}) \geq \rho(\mathrm{I}_{\Omega} - \mathrm{I}_{\Omega}) = \rho(0) \geq 0$$

it follows that $\rho(0) = 0$ and $-\rho(-\mathrm{I}_{\Omega}) = \rho(\mathrm{I}_{\Omega}) = -D \in (0,1]$.

For all $X \in L^{\infty}$, the function $\rho_X : \mathbb{R}_+ \to \mathbb{R}$ defined by $\rho_X(t) = \rho(tX)$, for all $t \in \mathbb{R}_+$, is star-shaped (since ρ is), hence it is superadditive, and it is subadditive (since ρ is), thus it is additive. Moreover, since $\rho_X(t)$ is star-shaped, then $\frac{\rho_X(t)}{t}$ is increasing on $(0,\infty)$, and $\rho_X(t) = t\frac{\rho_X(t)}{t}$ has at most countably many discontinuity points on $(0,\infty)$. By Corollary 9 of Aczél and Dhombres (1989, Ch. 2), ρ_X is linear, therefore ρ is positively homogeneous. Thus, ρ is sublinear and Lipschitz continuous of rank D. Finally, cash-additivity is proved.

By Proposition 5 and since ρ is quasiconvex, then $R(\cdot,Q)$ is subadditive and positively homogeneous for all $Q \in \mathcal{M}_{1,f}$. Finally, let

$$\mathcal{Q} = \{Q \in \mathcal{M}_{1,f} : R(1,Q) > -\infty\}.$$

If $Q \in \mathcal{Q}$, by subadditivity $R(1,Q) + R(-1,Q) \geq R(0,Q) = 0$ (where the latter equality descends from monotonicity and upper semicontinuity of $R(\cdot,Q)$). This, in turn, implies that $R(-1,Q) > -\infty$. By positive

[16] Cash-additivity of ρ translates into $R(t-m,Q) = R(t,Q) - Dm$ for all $t \in \mathbb{R}$, $m \in \mathbb{R}$, and $Q \in \mathcal{M}_{1,f}$. Convexity of ρ corresponds to convexity of R in the first component; see Corollary 5 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b).

homogeneity of $R(\cdot, Q)$, it follows that $R(\cdot, Q)$ is finite. Moreover, since $\rho(X) \geq R(\mathbb{E}_Q(-X), Q)$ for all $X \in L^\infty$,

$$D = \rho(-\mathrm{I}_\Omega) \geq R(1, Q) \geq -R(-1, Q) \geq -\rho(\mathrm{I}_\Omega) = D$$

and positive homogeneity of $R(\cdot, Q)$ yields $R(t, Q) = Dt$ for all $t \in \mathbb{R}$. If $Q \notin \mathcal{Q}$ then $R(1, Q) = -\infty$. By monotonicity and positive homogeneity, it follows that $R(t, Q) = -\infty$ for all $t \in \mathbb{R}$.

Lemma 4 guarantees $\rho(X) = D \max_{Q \in \mathcal{Q}} \mathbb{E}_Q(-X)$ for all $X \in L^\infty$, proving the statement. ■

4.2 Continuity from Below (and Above)

Next proposition shows that, as in the special cases of convex cash-additive and cash-subadditive risk measures, the possibility of replacing finitely additive probabilities with countably additive probabilities in the variational representation (8), and indeed in (10), corresponds to the following continuity requirement:

Continuity from below $X_n \nearrow X$ implies $\rho(X_n) \to \rho(X)$ for all $X_n, X \in L^\infty$.

Proposition 7 *Let $\rho : L^\infty \to [-\infty, \infty]$ be a quasiconvex upper semicontinuous risk measure. The following conditions are equivalent:*

(i) ρ is continuous from below;

(ii) $R(t, Q) = \inf_{X \in L^\infty} \rho(X)$ for all $(t, Q) \in \mathbb{R} \times (\mathcal{M}_{1,f} \setminus \mathcal{M}_1)$.

In this case,

$$\max_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(-X), Q) = \max_{Q \in \mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty. \tag{12}$$

Proof. Define $\inf_{L^\infty} \rho = \inf_{X \in L^\infty} \rho(X)$. Consider the following condition:

(iii) $\{Q \in \mathcal{M}_{1,f} : R(t, Q) \geq m\} \subseteq \mathcal{M}_1$ for all $m \in (\inf_{L^\infty} \rho, +\infty]$ and all $t \in \mathbb{R}$.

We show that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(i) implies (iii). Let $t \in \mathbb{R}$, $m \in (\inf_{L^\infty} \rho, +\infty]$, and $Q' \in \{Q \in \mathcal{M}_{1,f} : R(t, Q) \geq m\}$. Since $m > \inf_{L^\infty} \rho$, there exist $X \in L^\infty$ such that $\rho(X) < m$ and $x \geq X$ in $\mathbb{R}$ such that $\rho(x) \leq \rho(X) < m$. If $E_n \searrow \emptyset$ in $\mathcal{A}$,[17] then $x - k\mathrm{I}_{E_n} \nearrow x$ in L^∞ for each $k > 0$. Continuity from below guarantees that there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$

$$m > \rho(x - k\mathrm{I}_{E_n}) = \max_{Q \in \mathcal{M}_{1,f}} R(\mathbb{E}_Q(k\mathrm{I}_{E_n} - x), Q) = \max_{Q \in \mathcal{M}_{1,f}} R(kQ(E_n) - x, Q).$$

If $kQ'(E_{n'}) - x \geq t$ for some $n' \geq N_k$, since R is increasing, it follows that

$$\max_{Q \in \mathcal{M}_{1,f}} R(kQ(E_{n'}) - x, Q) \geq R(kQ'(E_{n'}) - x, Q') \geq R(t, Q') \geq m,$$

which is absurd. Then $kQ'(E_n) - x < t$ for all $n \geq N_k$, hence

$$Q'(E_n) < \frac{x + t}{k} \quad \forall n \geq N_k,$$

thus $\lim_{n \to \infty} Q'(E_n) \leq k^{-1}(x + t)$. Since this is the case for each $k > 0$, then $\lim_{n \to \infty} Q'(E_n) = 0$ and $Q' \in \mathcal{M}_1$.

[17] That is E_n is a decreasing and vanishing sequence of elements of $\mathcal{A}$.

(iii) implies (ii). Clearly, for all $(t,Q)\in\mathbb{R}\times\mathcal{M}_{1,f}$, $R(t,Q)=\inf\{\rho(X):\mathbb{E}_Q(-X)=t\}\geq\inf_{L^\infty}\rho$. If, per contra, there exists $(t_0,Q_0)\in\mathbb{R}\times(\mathcal{M}_{1,f}\setminus\mathcal{M}_1)$ such that $R(t_0,Q_0)>\inf_{L^\infty}\rho$, then, setting $m_0=R(t_0,Q_0)$, by (iii) it follows that

$$Q_0\in\{Q\in\mathcal{M}_{1,f}:R(t_0,Q)\geq m_0\}\subseteq\mathcal{M}_1,$$

a contradiction.

(ii) implies (i). Let $\{X_n\}_{n\geq 1}$ be a sequence in L^∞ such that $X_n\nearrow X_0\in L^\infty$. For each $n\geq 0$, define $\gamma_n:\mathcal{M}_{1,f}\to[-\infty,+\infty]$ by

$$\gamma_n(Q)=R(\mathbb{E}_Q(-X_n),Q)\quad\forall Q\in\mathcal{M}_{1,f}.$$

Each γ_n is weak* upper semicontinuous, and the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is decreasing. If $Q\in\mathcal{M}_1$, then $\mathbb{E}_Q(-X_n)\searrow\mathbb{E}_Q(-X_0)$, by the Levi Monotone Converge Theorem, and so, since $R(\cdot,Q)$ is upper semicontinuous and increasing on $\mathbb{R}$, $\lim_{n\to\infty}R(\mathbb{E}_Q(-X_n),Q)=R(\mathbb{E}_Q(-X_0),Q)$; else if $Q\notin\mathcal{M}_1$, then $R(\mathbb{E}_Q(-X_n),Q)=\inf_{L^\infty}\rho$ for all $n\geq 0$. Conclude that $\{-\gamma_n\}_{n\in\mathbb{N}}$ pointwise converges and so Γ-converges to $-\gamma_0$ (see, e.g., Dal Maso, 1993, Rem. 5.5). By Theorem 7.4 of Dal Maso (1993), $\min_{Q\in\mathcal{M}_{1,f}}-\gamma_n(Q)\to\min_{Q\in\mathcal{M}_{1,f}}-\gamma_0(Q)$, that is $-\rho(X_n)\to-\rho(X_0)$.

Finally, we show that (ii) implies (12). If X is such that $\rho(X)=\inf_{L^\infty}\rho$, then for all $Q\in\mathcal{M}_{1,f}$, by Lemma 4,

$$\rho(X)\geq R(\mathbb{E}_Q(-X),Q)=\inf\{\rho(Y):\mathbb{E}_Q(-Y)=\mathbb{E}_Q(-X)\}\geq\inf{}_{L^\infty}\rho=\rho(X).$$

Therefore the maximum in (10) is attained at each Q in $\mathcal{M}_{1,f}$, thus at each Q in $\mathcal{M}_1$. Else if $\rho(X)>\inf_{L^\infty}\rho$, by (ii), the maximum in (10) cannot be attained on $\mathcal{M}_{1,f}\setminus\mathcal{M}_1$, thus it is attained on $\mathcal{M}_1$. ■

Next, notice that continuity from below implies norm upper semicontinuity for a risk measure ρ. Indeed:

Proposition 8 *A risk measure ρ is continuous from below (resp., above) if and only if it is upper (resp., lower) semicontinuous with respect to bounded pointwise convergence.*

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be a bounded sequence in L^∞ that pointwise converges to X. Set $Y_n=\inf_{k\geq n}X_k$ for all $n\in\mathbb{N}$. Then, we have that $X_n\geq Y_n$ for all $n\in\mathbb{N}$ and $Y_n\nearrow X$. Monotonicity and continuity from below imply

$$\limsup{}_n\,\rho(X_n)\leq\lim{}_{n\to\infty}\rho(Y_n)=\rho(X).$$

Conversely, if $X_n\nearrow X$, then monotonicity of ρ delivers $\rho(X)\leq\liminf_n\rho(X_n)$, while upper semicontinuity with respect to bounded pointwise convergence delivers $\limsup_n\rho(X_n)\leq\rho(X)$. ■

Moreover, continuity from below and norm lower semicontinuity imply continuity with respect to bounded pointwise convergence, provided ρ is quasiconvex. Formally:

Proposition 9 *Let $\rho:L^\infty\to[-\infty,\infty]$ be a quasiconvex risk measure. The following conditions are equivalent:*

(i) ρ is continuous from below and norm lower semicontinuous;

(ii) ρ is continuous with respect to bounded pointwise convergence.

Proof. Clearly, (ii) and Proposition 8 yield (i). By Proposition 8, to prove the converse it is sufficient to show that ρ is continuous from above. Let $X_n\searrow X$. By monotonicity, $\rho(X_n)$ is increasing and $\lim_{n\to\infty}\rho(X_n)\leq$

$\rho(X)$. Assume, per contra, strict inequality holds. Then $\{X_n\}_{n\in\mathbb{N}}$ is contained in $\{\rho < c\}$ for some $c < \rho(X)$. The assumptions on ρ guarantee that $\{\rho \le c\}$ is nonempty, convex, norm closed, and

$$\{\rho \le c\} \subseteq \bigcap_{i\in I} [Q_i \ge b_i],$$

where $\{(b_i, Q_i) : i \in I\} = \{(b, Q) \in \mathbb{R} \times \mathcal{M}_1 : [Q \ge b] \supseteq \{\rho \le c\}\}$. As to the converse inclusion, let $Y \notin \{\rho \le c\}$. By a Separating Hyperplane Theorem, there exist $b \in \mathbb{R}$, $\varepsilon > 0$, and $Q \in L^\infty(\Omega, \mathcal{A}, P)^* \setminus \{0\}$ such that

$$\{\rho \le c\} \subseteq [Q \ge b] \text{ and } Y \in [Q < b - \varepsilon].$$

Monotonicity allows to assume $Q \in \mathcal{M}_{1,f}$.[18] If $Q \notin \mathcal{M}_1$, then $R(t, Q) = \inf_{X\in L^\infty} \rho(X) \le \rho(X_1) < c$ for all $t \in \mathbb{R}$. For $t = -b + \varepsilon$, this implies

$$c > R(-b + \varepsilon, Q) = \inf\{\rho(Z) : \mathbb{E}_Q(Z) = b - \varepsilon\}.$$

Then $\rho(Z') < c$ for some $Z' \in [Q = b - \varepsilon]$, which is absurd since $\{\rho \le c\} \subseteq [Q \ge b]$. Summing up, if $Y \notin \{\rho \le c\}$ there are $b \in \mathbb{R}$ and $Q \in \mathcal{M}_1$ such that $[Q \ge b] \supseteq \{\rho \le c\}$ and $Y \notin [Q \ge b]$. Thus, $\{\rho \le c\}^c \subseteq \left(\bigcap_{i\in I} [Q_i \ge b_i]\right)^c$.

Finally, $\{X_n\}_{n\in\mathbb{N}} \subseteq \{\rho \le c\}$ implies $\mathbb{E}_{Q_i}(X_n) \ge b_i$ for all $n \in \mathbb{N}$ and $i \in I$. By the Monotone Convergence Theorem, $\mathbb{E}_{Q_i}(X) \ge b_i$ for all $i \in I$, then $\rho(X) \le c$ which contradicts $c < \rho(X)$. ■

5 Law-invariance

In this section we consider a continuous from below quasiconvex risk measure

$$\rho(X) = \max_{Q\in\mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^\infty.$$

In the study of law-invariance it is useful to consider some important stochastic orders. The *convex order* $\succsim_{cx}$ is defined on L^1 by

$$X \succsim_{cx} Y \text{ if and only if } \mathbb{E}_P(\phi(X)) \ge \mathbb{E}_P(\phi(Y))$$

for all convex $\phi : \mathbb{R} \to \mathbb{R}$. The *increasing convex order* $\succsim_{icx}$ and *second order stochastic dominance* $\succsim_{ssd}$ are defined analogously by replacing convex functions by increasing convex functions and increasing concave functions, respectively. Notice that $X \succsim_{icx} Y$ if and only if $-X \precsim_{ssd} -Y$ and that the three preorders share the same symmetric part $\sim$, which is the *identical distribution* with respect to P relation.[19]

As widely discussed in the literature (see, e.g., the classic Rothschild and Stiglitz, 1970, and Marshall and Olkin, 1979), $X \succsim_{cx} Y$ intuitively means that the values of X are more dispersed than those of Y, while $X \succsim_{ssd} Y$ is the standard formalization of the statement "X is less risky than Y," provided P is the unanimously accepted model for uncertainty.

The convex order naturally induces a relation on $\mathcal{M}_1$ by

$$Q \succsim_{cx} Q' \text{ if and only if } \frac{dQ}{dP} \succsim_{cx} \frac{dQ'}{dP}.$$

The intuition is the same: the probability masses $dQ(\omega)$ are more scattered with respect to $dP(\omega)$ than the masses $dQ'(\omega)$.

[18] If $Z \in L^\infty_+$ then $X_1 + nZ \in \{\rho \le c\}$ for all $n \in \mathbb{N}$, and $Q(X_1) + nQ(Z) \ge b$ delivers $Q(Z) \ge 0$. Then Q is a non-zero positive linear functional, and if $Q \notin \mathcal{M}_{1,f}$ it is sufficient to normalize it.

[19] See Chong (1974) for this fact and for altenative characterizations of these orders.

An extended real valued function γ defined on a subset of L^1 is *law-invariant* (or *rearrangement invariant*) if and only if

$$X \sim Y \text{ implies } \gamma(X) = \gamma(Y);$$

while γ is *Schur concave* if and only if

$$X \succsim_{cx} Y \text{ implies } \gamma(X) \leq \gamma(Y).$$

Finally, γ *preserves second order stochastic dominance* if and only if

$$X \succsim_{ssd} Y \text{ implies } \gamma(X) \leq \gamma(Y).$$

Clearly, the latter property is desirable for a risk measure, under the assumption that all the agents agree on P. If X is considered to be less risky than Y, it is difficult for the supervising agency to require a higher reserve amount for X than for Y.

Theorem 10 *Let ρ be a quasiconvex and continuous from below risk measure. The following conditions are equivalent:*

(i) ρ preserves second order stochastic dominance;

(ii) $R(t,\cdot)$ is Schur concave on $\mathcal{M}_1$ for all $t \in \mathbb{R}$.

In this case,

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 q_{-X}(s)\, q_{\frac{dQ}{dP}}(s)\, ds, Q\right) \quad \forall X \in L^\infty \tag{13}$$

and

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 q_{\frac{dQ}{dP}}(s)\, q_Y(1-s)\, ds = -t\right\} \quad \forall (t,Q) \in \mathbb{R}\times\mathcal{M}_1. \tag{14}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then (i) and (ii) are equivalent to:

(iii) ρ is law-invariant;

(iv) $R(t,\cdot)$ is rearrangement invariant on $\mathcal{M}_1$ for all $t \in \mathbb{R}$.

Here q_Z denotes any *quantile* of $Z \in L^1$ (see, e.g., Föllmer and Schied, 2004),[20] and a probability space is *adequate* if and only if it is either finite and endowed with the uniform probability or non-atomic. We used the term "rearrangement invariant" rather than the equivalent "law-invariant" in (iv) since it gives a better intuition of what happens in the finite equidistributed case: $R(t,Q) = R(t, Q \circ \sigma)$ for all permutations σ of Ω and all $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$.

Proof. The proof heavily relies on the theory of rearrangement invariant Banach spaces developed by Luxemburg (1967) and Chong and Rice (1971). For convenience, the latter reference is denoted from now on by CR. Following its notation, if X is measurable, set

$$\begin{aligned}\delta_X(s) &= \inf\{x \in \mathbb{R} : P(\{\omega \in \Omega : X(\omega) > x\}) \leq s\} \\ &= \inf\{x \in \mathbb{R} : F_X(x) \geq 1-s\} = F_X^{-1}(1-s) = q_X^-(1-s)\end{aligned}$$

for all $s \in [0,1]$.

[20] Notice that we are not committing to any specific version of the quantile, e.g., the right continuous, or the left continuous, etc.

Step 1. If $Y \in L^\infty$ and $Q \in \mathcal{M}_1$, then

$$\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \precsim_{cx} Q\right\} = \left[\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds\right]. \tag{15}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds = \min\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\right\} \text{ and} \tag{16}$$

$$\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds = \max\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\right\}. \tag{17}$$

Proof of Step 1. [CR 13.4] and [CR 13.8] guarantee that, if Y and X belong to the set $M(\Omega, \mathcal{A}, P)$ of measurable functions and $\delta_{|Y|}\delta_{|X|} \in L^1([0,1])$, then

$$\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \precsim_{cx} X\right\} = \left[\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_X(s)\, ds\right].$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds = \min\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \sim X\right\} \text{ and}$$

$$\int_0^1 \delta_Y(s)\, \delta_X(s)\, ds = \max\left\{\int YX'dP : M(\Omega, \mathcal{A}, P) \ni X' \sim X\right\}.$$

The condition $\delta_{|Y|}\delta_{|X|} \in L^1([0,1])$ is implied by $\delta_{|Y|} \in L^p([0,1])$ and $\delta_{|X|} \in L^q([0,1])$, where either $p = \infty$ and $q = 1$ or $p = 1$ and $q = \infty$, which is equivalent to $Y \in L^p(\Omega)$ and $X \in L^q(\Omega)$ [CR 4.3]. In this case,

$$\left\{X' \in M(\Omega, \mathcal{A}, P) : X' \precsim_{cx} X\right\} = \left\{X' \in L^q : X' \precsim_{cx} X\right\}.$$

In fact, $X \in L^q$ and $X' \precsim_{cx} X$ imply $X' \in L^q$ [CR 10.2]. Therefore, if $Y \in L^p(\Omega)$ and $X \in L^q(\Omega)$, then

$$\left\{\int YX'dP : L^q \ni X' \precsim_{cx} X\right\} = \left[\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_X(s)\, ds\right]. \tag{18}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\int_0^1 \delta_Y(s)\, \delta_X(1-s)\, ds = \min\left\{\int YX'dP : L^q \ni X' \sim X\right\} \text{ and} \tag{19}$$

$$\int_0^1 \delta_Y(s)\, \delta_X(s)\, ds = \max\left\{\int YX'dP : L^q \ni X' \sim X\right\}. \tag{20}$$

If, in addition, X is a probability density (p.d.) and $X' \precsim_{cx} X$, then $X' \geq 0$ [CR 10.2] and $\mathbb{E}(X') = \mathbb{E}(X) = 1$, that is X' is a probability density. Finally, if $Y \in L^\infty$ and $Q \in \mathcal{M}_1$, then

$$\begin{aligned}
\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \precsim_{cx} Q\right\} &= \left\{\int YX'dP : X' \text{ is a p.d. and } X' \precsim_{cx} \frac{dQ}{dP}\right\} \\
&= \left\{\int YX'dP : L^1 \ni X' \precsim_{cx} \frac{dQ}{dP}\right\} \\
&= \left[\int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(1-s)\, ds, \int_0^1 \delta_Y(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds\right].
\end{aligned}$$

Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, then

$$\begin{aligned}\int_0^1 \delta_Y(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds &= \max\left\{\int YX'dP : L^1 \ni X' \sim \frac{dQ}{dP}\right\} \\ &= \max\left\{\int YX'dP : X' \text{ is a p.d. and } X' \sim \frac{dQ}{dP}\right\} \\ &= \max\left\{\mathbb{E}_{Q'}(Y) : \mathcal{M}_1 \ni Q' \sim Q\right\}.\end{aligned}$$

The formula for the minimum is proved in the same way. □

The next step is essentially due to Hardy, see, e.g., [CR 9.1]:

Step 2. Let $p = \infty$ and $q = 1$ or vice versa, $X, X' \in L^p$ and $Y \in L^q$.

(a) $X \precsim_{cx} X'$ implies $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$.

(b) $X \precsim_{cx} X'$ implies $\int_0^1 \delta_X(s)\,\delta_Y(1-s)\,ds \ge \int_0^1 \delta_{X'}(s)\,\delta_Y(1-s)\,ds$.

(c) $X \precsim_{icx} X'$ and $Y \ge 0$ implies $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$.

Proof of Step 2. $X, X' \in L^p$ and $Y \in L^q$ is equivalent to $\delta_X, \delta_{X'} \in L^p([0,1])$ and $\delta_Y \in L^q([0,1])$ [CR 4.3]. In particular, $\delta_X\delta_Y, \delta_{X'}\delta_Y \in L^1([0,1])$. Also notice that $f(s) \in L^q([0,1])$ if and only if $f(1-s) \in L^q([0,1])$, and

$$\int_0^1 f(s)\,ds = \int_0^1 f(1-s)\,ds.$$

(a) (resp., (b)) If $X \precsim_{cx} X'$, then $\int_0^w \delta_X(s)\,ds \le \int_0^w \delta_{X'}(s)\,ds$ for all $w \in [0,1]$ and $\int_0^1 \delta_X(s)\,ds = \int_0^1 \delta_{X'}(s)\,ds$, since $\delta_Y(s)$ is decreasing (resp., $\delta_Y(1-s)$ is increasing), then $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$ (resp., $\int_0^1 \delta_X(s)\,\delta_Y(1-s)\,ds \ge \int_0^1 \delta_{X'}(s)\,\delta_Y(1-s)\,ds$) [CR 9.1].

(c) If $X \precsim_{icx} X'$ and $Y \ge 0$, then $\int_0^w \delta_X(s)\,ds \le \int_0^w \delta_{X'}(s)\,ds$ for all $w \in [0,1]$ and δ_Y is decreasing and non-negative [CR 2.8], then $\int_0^1 \delta_X(s)\,\delta_Y(s)\,ds \le \int_0^1 \delta_{X'}(s)\,\delta_Y(s)\,ds$ [CR 9.1]. □

Step 3. If either $R(t,\cdot)$ is Schur concave on $\mathcal{M}_1$ for all $t \in \mathbb{R}$ or $(\Omega, \mathcal{A}, P)$ is adequate and $R(t,\cdot)$ is rearrangement invariant on $\mathcal{M}_1$ for all $t \in \mathbb{R}$, then

$$\rho(X) = \max_{Q\in\mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) \quad \forall X \in L^\infty. \tag{21}$$

Proof of Step 3. Let $X \in L^\infty$. Then, $\mathbb{E}_Q(-X) \le \int_0^1 \delta_{-X}(s)\,\delta_{dQ/dP}(s)\,ds$ for all $Q \in \mathcal{M}_1$, by (15), thus, monotonicity of R in the first component implies

$$\rho(X) = \max_{Q\in\mathcal{M}_1} R(\mathbb{E}_Q(-X), Q) \le \sup_{Q\in\mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right).$$

Conversely, for any $Q \in \mathcal{M}_1$, by (15) there exists $Q' \precsim_{cx} Q$ (resp., by (17) there exists $Q' \sim Q$) such that

$$\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds = \mathbb{E}_{Q'}(-X).$$

Thus,

$$R\left(\int_0^1 \delta_{-X}(s)\,\delta_{dQ/dP}(s)\,ds, Q\right) = R(\mathbb{E}_{Q'}(-X), Q) \le R(\mathbb{E}_{Q'}(-X), Q') \le \rho(X)$$

by Schur concavity (resp., rearrangement invariance). Therefore,

$$\sup_{Q\in\mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\,\delta_{\frac{dQ}{dP}}(s)\,ds, Q\right) \le \rho(X)$$

and the supremum is attained. $\square$

Step 4. (ii) implies (i) and (13), also (iv) implies (i) provided $(\Omega, \mathcal{A}, P)$ is adequate.

Proof of Step 4. By Step 3, (ii) guarantees that (21) holds, and the same is true for (iv) if $(\Omega, \mathcal{A}, P)$ is adequate. But (21) is equivalent to (13) since $\delta_Y(s) = q_Y^-(1-s)$ for $s \in [0,1]$.

Moreover, $X \succsim_{ssd} Y$ if and only if $-X \precsim_{icx} -Y$. Thus, Step 2.c implies $\int_0^1 \delta_{-X}(s)\, \delta_{dQ/dP}(s)\, ds \leq \int_0^1 \delta_{-Y}(s)\, \delta_{dQ/dP}(s)\, ds$ for all $Q \in \mathcal{M}_1$, and monotonicity of R allows to conclude that

$$\rho(X) = \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-X}(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds, Q\right) \leq \max_{Q \in \mathcal{M}_1} R\left(\int_0^1 \delta_{-Y}(s)\, \delta_{\frac{dQ}{dP}}(s)\, ds, Q\right) = \rho(Y).$$

Therefore, ρ preserves second order stochastic dominance and, in particular, it is law-invariant. $\square$

Step 5. If either ρ preserves second order stochastic dominance or $(\Omega, \mathcal{A}, P)$ is adequate and ρ is law-invariant, then, for all $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$,

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds \leq -t\right\} = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds = -t\right\}. \quad (22)$$

Proof of Step 5. Notice that if ρ preserves second order stochastic dominance, then it is Schur convex, that is, $X \precsim_{cx} Y$ implies $\rho(X) \leq \rho(Y)$. Let ρ be Schur convex (resp. law-invariant). First observe that

$$R(t,Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) \geq t\} = \inf\{\rho(X) : \mathbb{E}_Q(X) \leq -t\}$$

for all $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$.[21] Since ρ is Schur convex (resp., rearrangement invariant), then

$$\begin{aligned}\inf\{\rho(X) : \mathbb{E}_Q(X) \leq -t\} &= \inf\{\rho(Y) : \text{there exists } X \precsim_{cx} Y \text{ such that } \mathbb{E}_Q(X) \leq -t\} \\ (\text{resp. } &= \inf\{\rho(Y) : \text{there exists } X \sim_{cx} Y \text{ such that } \mathbb{E}_Q(X) \leq -t\}),\end{aligned}$$

but,

$$\inf\{\rho(Y) : \mathbb{E}_Q(X) \leq -t \text{ for some } X \precsim_{cx} Y\} = \inf\left\{\rho(Y) : \min\left\{\int \frac{dQ}{dP} X dP : L^\infty \ni X \precsim_{cx} Y\right\} \leq -t\right\}$$

$$(\text{resp., } \inf\{\rho(Y) : \mathbb{E}_Q(X) \leq -t \text{ for some } X \sim Y\} = \inf\left\{\rho(Y) : \min\left\{\int \frac{dQ}{dP} X dP : L^\infty \ni X \sim Y\right\} \leq -t\right\}).$$

By (18) and (19), for all $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$,

$$R(t,Q) = \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds \leq -t\right\} \leq \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds = -t\right\}.$$

Finally, assume per contra that $R(t,Q) < \inf\left\{\rho(Y) : \int_0^1 \delta_{dQ/dP}(s)\, \delta_Y(1-s)\, ds = -t\right\}$ for some $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$. This implies the existence of $Z \in L^\infty$ for which $\int_0^1 \delta_{dQ/dP}(s)\, \delta_Z(1-s)\, ds \leq -t$ and

$$\rho(Z) < \inf\left\{\rho(Y) : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Y(1-s)\, ds = -t\right\}.$$

Since $\delta_{Z+m} = \delta_Z + m$ for all $m \in \mathbb{R}$, then

$$\int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_{Z+m}(1-s)\, ds = \int_0^1 \delta_{\frac{dQ}{dP}}(s)\, \delta_Z(1-s)\, ds + m.$$

[21] Clearly, $R(t,Q) \geq \inf\{\rho(Y) : \mathbb{E}_Q(-Y) \geq t\}$. Conversely, assume per contra that $R(t,Q) > \inf\{\rho(Y) : \mathbb{E}_Q(-Y) \geq t\}$ for some $(t, Q) \in \mathbb{R} \times \mathcal{M}_1$. This implies the existence of $Z \in L^\infty$ for which $\mathbb{E}_Q(-Z) \geq t$ and $\rho(Z) < R(t,Q)$. Set $m = \mathbb{E}_Q(-Z) - t \geq 0$, then $Z + m \geq Z$, $\mathbb{E}_Q(-(Z+m)) = t$ and $R(t,Q) \leq \rho(Z+m) \leq \rho(Z) < R(t,Q)$, a contradiction. The second equality is trivial.

Choose $m \geq 0$ so that

$$\int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_{Z+m}(1-s)\,ds = -t,$$

then $Z+m \geq Z$, and $\rho(Z+m) \leq \rho(Z) < \inf\left\{\rho(Y) : \int_0^1 \delta_{dQ/dP}(s)\,\delta_Y(1-s)\,ds = -t\right\}$, a contradiction. □

Step 6. (i) implies (ii) and (14), also (iii) implies (ii) provided $(\Omega, \mathcal{A}, P)$ is adequate.

Proof of Step 6. By Step 5, (i) guarantees that (22) holds, and the same is true for (iii) if $(\Omega, \mathcal{A}, P)$ is adequate. But, the second part of (22) is equivalent to (14) since $\delta_Y(s) = q_Y^-(1-s)$ for $s \in [0,1]$. While the first part together with Step 2.b yields the following chain of implications

$$\begin{aligned}
Q \precsim_{cx} Q' &\Longrightarrow \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds \geq \int_0^1 \delta_{\frac{dQ'}{dP}}(s)\,\delta_Y(1-s)\,ds \text{ for all } Y \in L^\infty \\
&\Longrightarrow \left\{Y : \int_0^1 \delta_{\frac{dQ}{dP}}(s)\,\delta_Y(1-s)\,ds \leq -t\right\} \subseteq \left\{Y : \int_0^1 \delta_{\frac{dQ'}{dP}}(s)\,\delta_Y(1-s)\,ds \leq -t\right\} \quad \forall t \in \mathbb{R} \\
&\Longrightarrow R(t,Q) \geq R(t,Q') \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Hence, $R(t,\cdot)$ is Schur concave for all $t \in \mathbb{R}$. □

Finally, Steps 4 and 6 guarantee that (i)$\Longleftrightarrow$(ii), and in this case (13) and (14) hold. Moreover, if $(\Omega, \mathcal{A}, P)$ is adequate, the same steps deliver (iv)$\Longrightarrow$(i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii)$\Longrightarrow$(iv). ■

Theorem 10 considers law-invariant quasiconvex risk measures that are upper semicontinuous with respect to bounded pointwise convergence (see Proposition 8). Jouini, Schachermayer, and Touzi (2006) show that law-invariant convex monetary risk measures are automatically lower semicontinuous with respect to bounded pointwise convergence, provided $(\Omega, \mathcal{A}, P)$ is standard. Whether this remains true for quasiconvex risk measures is left for future research (but see Proposition 9).

5.1 Mean Value Premium Principle

Important examples of law-invariant quasiconvex risk measures that are continuous from below are those of the form

$$\rho(X) = \ell^{-1}\left(\mathbb{E}_P(\ell(-X))\right) \quad \forall X \in L^\infty, \tag{23}$$

where ℓ is a strictly increasing and convex loss function. The characterization of these measures is a version of the classic Nagumo-Kolmogorov-de Finetti Theorem and relies on two additional properties:

Constancy $\rho(m) = -m$ for all $m \in \mathbb{R}$.[22]

Conditional consistency Let $A \in \mathcal{A}$ and $X, Y, Z \in L^\infty$,

$$\rho(X\mathrm{I}_A) \geq \rho(Y\mathrm{I}_A) \Longleftrightarrow \rho(X\mathrm{I}_A + Z\mathrm{I}_{A^c}) \geq \rho(Y\mathrm{I}_A + Z\mathrm{I}_{A^c}).$$

The latter property is inspired by Savage (1954)'s "sure thing principle" and clearly hints at dynamic consistency (see, e.g., Ghirardato, 2002).[23] The seminal paper of Ellsberg (1961) shows how this assumption is non-controversial only if agents think that P is a reliable model of the uncertainty they face.[24]

[22]Throughout the paper, $\rho(m)$ is a little abuse for $\rho(m\mathrm{I}_\Omega)$.
[23]Indeed, in the Savagean perspective, these risk measures correspond to certainty equivalents of expected utility maximizers.
[24]See, e.g., Maccheroni, Marinacci, and Rustichini (2006) for a recent discussion of this issue.

Lemma 11 *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space. A law-invariant risk measure ρ satisfies constancy, conditional consistency, and continuity with respect to bounded pointwise convergence if and only if there exists a strictly increasing and continuous $\ell : \mathbb{R} \to \mathbb{R}$ such that*

$$\rho(X) = \ell^{-1}\left(\mathbb{E}_P\left(\ell(-X)\right)\right) \quad \forall X \in L^{\infty}.$$

The function ℓ is unique up to strictly increasing affine transformations, and it is convex if and only if ρ is quasiconvex.

Proof. Sufficiency is trivial. Necessity reduces to check that the function $\mathfrak{M} : \mathcal{D}^{\infty} \to \mathbb{R}$ defined for each distribution with bounded support $F = F_X$ by

$$\mathfrak{M}(F) = -\rho(X)$$

satisfies the assumptions of the Nagumo-Kolmogorov-de Finetti Theorem.

For the sake of completeness we include such check. Let $[a, b]$ be any closed interval in the real line and $\mathcal{D}(a, b)$ be the set of all simple probability distributions supported in $[a, b]$. The Dirac distribution concentrated in x is denoted by D_x.

Constancy guarantees that:

Step 1. For all $x \in [a, b]$, $\mathfrak{M}(D_x) = x$.

Step 2. If $F, G \in \mathcal{D}(a, b)$, $F \geq G$, and $F \neq G$, then $\mathfrak{M}(F) < \mathfrak{M}(G)$.

Proof of Step 2. Since $(\Omega, \mathcal{A}, P)$ is non-atomic there are two simple measurable functions $X \leq Y$ such that $F = F_X$ and $G = F_Y$. By monotonicity of ρ, $\mathfrak{M}(F) \leq \mathfrak{M}(G)$. Assume per contra that $\mathfrak{M}(F) = \mathfrak{M}(G)$, that is $\rho(X) = \rho(Y)$. If $X = Y$ then $F = G$, which is absurd. Thus (again by non-atomicity) there exist $n \in \mathbb{N}$, $A_1 \in \mathcal{A}$ with $P(A_1) = 1/n$, and $x, y \in \mathbb{R}$ such that

$$X(\omega) < x < y < Y(\omega) \quad \forall \omega \in A_1.$$

Therefore,

$$X \leq x\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c} \leq y\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c} \leq Y.$$

By monotonicity of ρ,

$$\rho\left(x\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c}\right) = \rho\left(y\mathrm{I}_{A_1} + X\mathrm{I}_{A_1^c}\right),$$

and so, by conditional consistency,

$$\rho(x\mathrm{I}_{A_1}) = \rho(y\mathrm{I}_{A_1}).$$

Let $A_2, ..., A_n$ be such that $\{A_i\}_{i=1}^n$ form a partition of Ω with $P(A_i) = 1/n$ for all i. By law-invariance

$$\rho(x\mathrm{I}_{A_i}) = \rho(y\mathrm{I}_{A_i}) \quad \forall i = 1, ..., n.$$

Repeated applications of conditional consistency then deliver

$$\begin{aligned}\rho(x\mathrm{I}_{\Omega}) &= \rho(x\mathrm{I}_{A_1} + x\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) = \rho(y\mathrm{I}_{A_1} + x\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) \\ &= \rho(y\mathrm{I}_{A_1} + y\mathrm{I}_{A_2} + x\mathrm{I}_{A_3} + ... + x\mathrm{I}_{A_n}) = ... = \rho(y\mathrm{I}_{\Omega}),\end{aligned}$$

which is absurd by constancy. □

Step 3. If $F, G, H \in \mathcal{D}(a, b)$, $\lambda \in (0, 1)$, and $\mathfrak{M}(F) = \mathfrak{M}(G)$, then $\mathfrak{M}(\lambda F + (1-\lambda) H) = \mathfrak{M}(\lambda G + (1-\lambda) H)$.

Proof of Step 3. For every $\lambda \in [0, 1]$, since $(\Omega, \mathcal{A}, P)$ is non-atomic, there are $X, Y, Z \in L^{\infty}$ and $A = A_{\lambda} \in \mathcal{A}$ that are independent and such that $F = F_X$, $G = F_Y$, $H = F_Z$, and $P(A) = \lambda$ (see, e.g., Billingsley, 1995, Theorem 5.3). Independence guarantees that $F_{W\mathrm{I}_A + W'\mathrm{I}_{A^c}} = \lambda F_W + (1-\lambda) F_{W'}$ if $W, W' \in \{X, Y, Z\}$.

If $\lambda = 1/2$, then $F_{W\mathrm{I}_A + W'\mathrm{I}_{A^c}} = 2^{-1}F_W + 2^{-1}F_{W'} = 2^{-1}F_{W'} + 2^{-1}F_W = F_{W'\mathrm{I}_A + W\mathrm{I}_{A^c}}$. Assume, per contra, $\mathfrak{M}\left(2^{-1}F + 2^{-1}H\right) \neq \mathfrak{M}\left(2^{-1}G + 2^{-1}H\right)$, then $\rho\left(X\mathrm{I}_A + Z\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + Z\mathrm{I}_{A^c}\right)$, by conditional consistency and law-invariance

$$\rho(X) = \rho\left(X\mathrm{I}_A + X\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + X\mathrm{I}_{A^c}\right) = \rho\left(X\mathrm{I}_A + Y\mathrm{I}_{A^c}\right) \gtrless \rho\left(Y\mathrm{I}_A + Y\mathrm{I}_{A^c}\right) = \rho(Y),$$

which contradicts $\mathfrak{M}(F) = \mathfrak{M}(G)$. Thus the statement is true for $\lambda = 2^{-1}$. Induction guarantees that it is true for any dyadic rational. Continuity with respect to bounded pointwise convergence of ρ and the Skorohod Theorem (see, e.g., Billingsley, 1995, Theorem 25.6) guarantee that the statement is true for any λ. □

For all $n \in \mathbb{N}$, $\mathfrak{M}$ satisfies the properties described in Steps 1-3 on $\mathcal{D}(-n, n)$. By the Nagumo-Kolmogorov-de Finetti Theorem,[25] for all $n \in \mathbb{N}$ there exists a unique strictly increasing continuous function $\phi_n : [-n, n] \to \mathbb{R}$ such that $\phi_n(0) = 0 = \phi_n(1) - 1$ and

$$\mathfrak{M}(F) = \phi_n^{-1}\left(\int_{\mathbb{R}} \phi_n(x)\, dF(x)\right) \quad \forall F \in \mathcal{D}(-n, n).$$

Define $\phi(x) = \phi_n(x)$ if $|x| \leq n$ to obtain $\mathfrak{M}(F) = \phi^{-1}\left(\int_{\mathbb{R}} \phi(x)\, dF(x)\right)$ for each simple probability distribution, then,

$$\rho(X) = -\phi^{-1}\left(\mathbb{E}_P(\phi(X))\right)$$

for all simple and measurable $X : \Omega \to \mathbb{R}$. Continuity with respect to bounded pointwise convergence yields the result for $\ell(\cdot) = -\phi(-\cdot)$.

Finally, if ρ is quasiconvex, Theorem 10 guarantees that ρ preserves second order stochastic dominance. Hence ℓ is convex. The converse is trivial. ■

Next result builds on Rockafellar (1971) and explicitly evaluates R for risk measures that admit an expected loss representation.

Proposition 12 *If $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly increasing convex function and $\rho(X) = \ell^{-1}\left(\mathbb{E}_P(\ell(-X))\right)$ for all $X \in L^\infty$, then*

$$R(t, Q) = \ell^{-1}\left(\max_{x \geq 0}\left[xt - \mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right) \quad \forall (t, Q) \in \mathbb{R}\times\mathcal{M}_1.$$

Observe that this amounts to say that $R(t, Q) = t - L(-t; Q, P)$, where $L(w; Q, P)$ is the generalized distance between probability measures considered by Bellini and Frittelli (2002) and corresponding to an initial endowment w and a utility $-\ell(-\cdot)$.

Proof. Observe that $\ell(\mathbb{R})$ is an open half line (l, ∞), with $l = \inf_{x\in\mathbb{R}} \ell(x)$. Then ℓ^{-1} can be extended to an extended-valued continuous and monotone function from $[-\infty, \infty]$ to $[-\infty, \infty]$ by setting $\ell^{-1}(x) = -\infty$ if $x \leq l$ and $\ell^{-1}(\infty) = \infty$. For all $(t, Q) \in \mathbb{R}\times\mathcal{M}_1$,

$$R(t, Q) = \inf\left\{\ell^{-1}\left(\mathbb{E}_P(\ell(-X))\right) : \mathbb{E}_Q(-X) = t\right\} = \ell^{-1}\left(\inf\left\{\mathbb{E}_P(\ell(-X)) : \mathbb{E}_Q(-X) = t\right\}\right).$$

Set $\phi(\cdot) = -\ell(-\cdot)$. Then

$$\begin{aligned}\inf\left\{\mathbb{E}_P(\ell(-X)) : \mathbb{E}_Q(-X) = t\right\} &= \inf\left\{-\mathbb{E}_P(-\ell(-X)) : \mathbb{E}_Q(-X) = t\right\} \\ &= -\sup\left\{\mathbb{E}_P(\phi(X)) : \mathbb{E}_Q(X) = -t\right\}.\end{aligned}$$

[25] See, e.g., Hardy, Littlewood, and Pólya (1934, Theorem 215).

But, the function $\Phi(X) = \mathbb{E}_P(\phi(X))$ for all $X \in L^\infty$ is concave, continuous, and monotone. Then, it follows immediately from Lemma 19 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b) and Corollary 2A of Rockafellar (1971) that

$$\sup\{\mathbb{E}_P(\phi(X)) : \mathbb{E}_Q(X) = -t\} = \min_{x\geq 0}[x(-t) - \Phi^*(xQ)] = \min_{x\geq 0}\left[x(-t) - \mathbb{E}_P\left(\phi^*\left(x\frac{dQ}{dP}\right)\right)\right].$$

Thus,

$$R(t,Q) = -\phi^{-1}\left(\min_{x\geq 0}\left[x(-t) - \mathbb{E}_P\left(\phi^*\left(x\frac{dQ}{dP}\right)\right)\right]\right) = \ell^{-1}\left(\max_{x\geq 0}\left[xt - \mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right),$$

as wanted. ■

5.2 Robust Mean Value Premium Principle

We conclude by introducing a new class of law-invariant quasiconvex risk measures. Consider a supervising agency whose decisions are taken by several supervisors $1, ..., k$, all of them using the mean value premium principle to rank risks. The cautious nature of the supervising task suggests that:

"a reserve for position X is deemed acceptable if and only if all supervisors agree."

Clearly, the minimum reserve that complies with this criterion is

$$\rho(X) = \min\left\{m \in \mathbb{R} : m \geq \ell_i^{-1}(\mathbb{E}_P(\ell_i(-X))) \quad \forall i = 1, ..., k\right\} = \max_{i=1,\ldots,k} \ell_i^{-1}(\mathbb{E}_P(\ell_i(-X))).$$

In light of this observation, in this section we study the quasiconvex risk measures defined by

$$\rho_{\mathcal{L}}(X) = \max_{\ell\in\mathcal{L}} \ell^{-1}(\mathbb{E}_P(\ell(-X))) \quad \forall X \in L^\infty$$

where $\mathcal{L}$ is a compact convex (or finite) set of strictly increasing and convex loss functions.[26]

Proposition 13 *Let $\mathcal{L}$ be a nonempty compact and convex set of strictly increasing and convex functions. Then $\rho_{\mathcal{L}} : L^\infty \to \mathbb{R}$ has the following properties:*

(a) it is a well defined, continuous with respect to bounded pointwise convergence, and quasiconvex risk measure that satisfies constancy and preserves second order stochastic dominance;

(b) for each $X \in L^\infty$, $\rho_{\mathcal{L}}(X) = \max_{\ell\in\operatorname{ext}(\mathcal{L})} \ell^{-1}(\mathbb{E}_P(\ell(-X)))$, where $\operatorname{ext}(\mathcal{L})$ is the set of extreme points of $\mathcal{L}$;

(c) $R_{\mathcal{L}}(t,Q) = \max_{\ell\in\mathcal{L}} \ell^{-1}\left(\max_{x\geq 0}\left[xt - \mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right)$ for all $(t,Q) \in \mathbb{R}\times\mathcal{M}_1$.

Proof. Given a closed and convex set C in L^∞. Consider the function $F : C \times \mathcal{L} \to \mathbb{R}$ defined by

$$F(X,\ell) = \ell^{-1}(\mathbb{E}_P(\ell(-X))) \quad \forall (X,\ell) \in C \times \mathcal{L}.$$

When $C = L^\infty$, then $\rho_{\mathcal{L}}(X) = \max_{\ell\in\mathcal{L}} F(X,\ell)$ for all $X \in L^\infty$.

[26] Here, $C(\mathbb{R})$ is endowed with the compact convergence topology, defined by the family of seminorms $\varphi_K(f) = \max_{x\in K}|f(x)|$ where K ranges over all compact subsets of $\mathbb{R}$. This locally convex topology is metrizable by $d(f,g) = \sum_{n=1}^{\infty} \frac{\max_{x\in[-n,n]}|f(x)-g(x)|}{2^n\left(1+\max_{x\in[-n,n]}|f(x)-g(x)|\right)}$. Subsets of $C(\mathbb{R})$ are endowed with the inherited topology.

Step 1. For each $X \in C$ the function $\digamma(X, \cdot) : \mathcal{L} \to \mathbb{R}$ is continuous and quasiaffine.[27]

Proof of Step 1. In order to prove the statement, it is enough to prove that, for each $c \in \mathbb{R}$, the sets $\{\ell \in \mathcal{L} : \digamma(X, \ell) \geq c\}$ and $\{\ell \in \mathcal{L} : \digamma(X, \ell) \leq c\}$ are closed and convex. Notice that, if $\ell \in \mathcal{L}$,

$$\digamma(X, \ell) \geq c \Leftrightarrow \ell^{-1}(\mathbb{E}_P(\ell(-X))) \geq c \Leftrightarrow \mathbb{E}_P(\ell(-X)) \geq \ell(c). \tag{24}$$

Assume that $\ell_n \in \{\digamma(X, \cdot) \geq c\}$ for all $n \in \mathbb{N}$ and $\ell_n \to \ell$. Then $\ell_n(c) \to \ell(c)$ and $\mathbb{E}_P(\ell_n(-X)) \to \mathbb{E}_P(\ell(-X))$.[28] This, matched with (24), implies that $\mathbb{E}_P(\ell(-X)) \geq \ell(c)$ and $\ell \in \{\digamma(X, \cdot) \geq c\}$. Thus $\{\digamma(X, \cdot) \geq c\}$ is closed. Next, consider $\ell_1, \ell_2 \in \{\digamma(X, \cdot) \geq c\}$ and $\lambda \in (0, 1)$. By (24), $\mathbb{E}_P(\ell_1(-X)) \geq \ell_1(c)$ and $\mathbb{E}_P(\ell_2(-X)) \geq \ell_2(c)$, then

$$\begin{aligned}\mathbb{E}_P((\lambda \ell_1 + (1-\lambda)\ell_2)(-X)) &= \lambda \mathbb{E}_P(\ell_1(-X)) + (1-\lambda)\mathbb{E}_P(\ell_2(-X)) \\ &\geq \lambda \ell_1(c) + (1-\lambda)\ell_2(c) = (\lambda \ell_1 + (1-\lambda)\ell_2)(c)\end{aligned}$$

that is, $\lambda \ell_1 + (1-\lambda)\ell_2 \in \{\digamma(X, \cdot) \geq c\}$. Thus, $\{\digamma(X, \cdot) \geq c\}$ is convex. The same arguments yield closure and convexity of $\{\digamma(X, \cdot) \leq c\}$. □

Step 2. $\rho_{\mathcal{L}}$ is a well defined risk measure that satisfies constancy and that preserves second order stochastic dominance.

Proof of Step 2. Let $C = L^\infty$, and arbitrarily choose $X \in L^\infty$. Since $\mathcal{L}$ is compact and $\digamma(X, \cdot)$ is continuous, then $\rho_{\mathcal{L}}(X) = \max_{\ell \in \mathcal{L}} \digamma(X, \ell)$ is well defined. If $m \in \mathbb{R}$ then $\ell^{-1}(\mathbb{E}_P(\ell(-m))) = -m$ for all $\ell \in \mathcal{L}$ and hence $\rho_{\mathcal{L}}(m) = -m$. Finally, if $X \succsim_{ssd} Y$, then $-X \precsim_{icx} -Y$, and $\ell^{-1}(\mathbb{E}_P(\ell(-X))) \leq \ell^{-1}(\mathbb{E}_P(\ell(-Y)))$ for all $\ell \in \mathcal{L}$ so that $\rho_{\mathcal{L}}(X) \leq \rho_{\mathcal{L}}(Y)$. This implies that $\rho_{\mathcal{L}}$ preserves second order stochastic dominance (and *a fortiori* $\rho_{\mathcal{L}}$ is decreasing and hence a risk measure). □

Step 3. For each $\ell \in \mathcal{L}$ the function $\digamma(\cdot, \ell) : C \to \mathbb{R}$ is continuous and quasiconvex.

The proof is trivial. Moreover, taking $C = L^\infty$, since suprema of lower semicontinuous and quasiconvex functions are lower semicontinuous and quasiconvex, thus Step 3 implies the following.

Step 4. $\rho_{\mathcal{L}}$ is lower semicontinuous and quasiconvex.

Step 5. $\rho_{\mathcal{L}}$ is continuous with respect to bounded pointwise convergence.

Proof of Step 5. By Proposition 9, Step 2, and Step 4, it is enough to prove that $\rho_{\mathcal{L}}$ is continuous from below. If $X_n \nearrow X$, the Dominated Convergence Theorem guarantees that $\digamma(X_n, \ell) \searrow \digamma(X, \ell)$ for all $\ell \in \mathcal{L}$. This is equivalent to say that $\digamma(X_n, \cdot) \searrow \digamma(X, \cdot)$ pointwise on $\mathcal{L}$. By Step 1 and Dini's Theorem [2, Theorem 2.66], $\digamma(X_n, \cdot) \to \digamma(X, \cdot)$ uniformly on $\mathcal{L}$ and, in particular, $\max_{\ell \in \mathcal{L}} \digamma(X_n, \ell) \to \max_{\ell \in \mathcal{L}} \digamma(X, \ell)$, that is $\rho_{\mathcal{L}}(X_n) \to \rho_{\mathcal{L}}(X)$. □

Step 6. For each $X \in L^\infty$, $\rho_{\mathcal{L}}(X) = \max_{\ell \in \mathrm{ext}(\mathcal{L})} \ell^{-1}(\mathbb{E}_P(\ell(-X)))$, where $\mathrm{ext}(\mathcal{L})$ is the set of extreme points of $\mathcal{L}$.

Proof of Step 6. The statement follows from the Bauer Maximum Principle [2, Corollary 7.75], after observing that $\digamma(X, \cdot)$ is explicitly quasiconvex. In fact, $\digamma(X, \ell_1) < \digamma(X, \ell_2)$ implies $\digamma(X, \lambda \ell_1 + (1-\lambda)\ell_2) < \digamma(X, \ell_2)$ for all $\ell_1, \ell_2 \in \mathcal{L}$ and $\lambda \in (0, 1)$. Specifically, $\ell_1^{-1}(\mathbb{E}_P(\ell_1(-X))) < \ell_2^{-1}(\mathbb{E}_P(\ell_2(-X)))$ implies

$$\mathbb{E}_P(\ell_1(-X)) < \ell_1\left(\ell_2^{-1}(\mathbb{E}_P(\ell_2(-X)))\right)$$

[27] That is both quasiconvex and quasiconcave.

[28] Take a version of X such that $-X(\Omega) \subseteq [a, b]$, since $\ell_n \to \ell$ uniformly on $[a, b]$, then $0 \leq \int |\ell_n(-X) - \ell(-X)| \, dP \leq \sup_{\omega \in \Omega} |\ell_n(-X(\omega)) - \ell(-X(\omega))| \leq \sup_{x \in [a,b]} |\ell_n(x) - \ell(x)| \to 0$.

thus

$$\begin{aligned}\mathbb{E}_P\left(\left(\lambda\ell_1+(1-\lambda)\,\ell_2\right)(-X)\right) &= \lambda\mathbb{E}_P\left(\ell_1(-X)\right)+(1-\lambda)\,\mathbb{E}_P\left(\ell_2(-X)\right)\\ &< \lambda\ell_1\left(\ell_2^{-1}\left(\mathbb{E}_P\left(\ell_2(-X)\right)\right)\right)+(1-\lambda)\,\ell_2\left(\ell_2^{-1}\left(\mathbb{E}_P\left(\ell_2(-X)\right)\right)\right)\\ &= \left(\lambda\ell_1+(1-\lambda)\,\ell_2\right)\left(\ell_2^{-1}\left(\mathbb{E}_P\left(\ell_2(-X)\right)\right)\right)\end{aligned}$$

hence $\left(\lambda\ell_1+(1-\lambda)\,\ell_2\right)^{-1}\mathbb{E}_P\left(\left(\lambda\ell_1+(1-\lambda)\,\ell_2\right)(-X)\right)<\ell_2^{-1}\left(\mathbb{E}_P\left(\ell_2(-X)\right)\right)$, as wanted. □

Step 7. $R_{\mathcal{L}}(t,Q)=\max_{\ell\in\mathcal{L}}\ell^{-1}\left(\max_{x\geq 0}\left[xt-\mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right)$ for all $(t,Q)\in\mathbb{R}\times\mathcal{M}_1$.

Proof of Step 7. By Step 2, Step 4, Step 5, and Lemma 4, it follows that

$$R_{\mathcal{L}}(t,Q)=\inf\left\{\rho_{\mathcal{L}}(Y):\mathbb{E}_Q(-Y)=t\right\}\quad\forall(t,Q)\in\mathbb{R}\times\mathcal{M}_1.$$

By Step 1, Step 3, the Sion Minimax Theorem [47, Corollary 3.3], and Proposition 12, it follows that

$$\begin{aligned}R_{\mathcal{L}}(t,Q) &= \inf_{X\in\{Y\in L^\infty:\mathbb{E}_Q(-Y)=t\}}\max_{\ell\in\mathcal{L}}F(X,\ell)=\max_{\ell\in\mathcal{L}}\inf_{X\in\{Y\in L^\infty:\mathbb{E}_Q(-Y)=t\}}F(X,\ell)\\ &= \max_{\ell\in\mathcal{L}}\inf_{X\in\{Y\in L^\infty:\mathbb{E}_Q(-Y)=t\}}\ell^{-1}\left(\mathbb{E}_P\left(\ell(-X)\right)\right)=\max_{\ell\in\mathcal{L}}\ell^{-1}\left(\max_{x\geq 0}\left[xt-\mathbb{E}_P\left(\ell^*\left(x\frac{dQ}{dP}\right)\right)\right]\right).\end{aligned}$$

□

Finally, (a) follows from Steps 2, 4, and 5, (b) is Step 6, (c) is Step 7. ■

In particular, observe that the case where $\mathcal{L}$ finite is encompassed by point (b).

6 Final Remarks

Though for mathematical convenience we considered quasiconvex risk measures defined on $L^\infty(\Omega,\mathcal{A},P)$, a parallel analysis can be carried out in any function space with unit,[29] like for example the space $B(\Omega,\mathcal{A})$ of bounded and measurable functions and the space $C_b(\Omega)$ of bounded and continuous functions (provided Ω is a topological space).

A more delicate case is the one of quasiconvex risk measures defined on $L^p(\Omega,\mathcal{A},P)$ for $p\in[1,\infty)$; here the results of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b) no longer apply directly. In view of the importance of these spaces in mathematical finance, in Appendix A we show that a more general version of the key Lemma 4 can be proved for these spaces. This is another novel technical contribution of this paper.

A Unbounded Random Variables

In this appendix we generalize Lemma 4 to any L^p space with $p\geq 1$. The generalization goes in two directions: unbounded random variables are allowed and the requirement of upper semicontinuity is relaxed.

We adhere to the notation and terminology previously used. For $p\in[1,\infty)$, the topological dual $L^p(\Omega,\mathcal{A},P)^*$ of $L^p(\Omega,\mathcal{A},P)$ is isometrically isomorphic to $L^q(\Omega,\mathcal{A},P)$ with $q^{-1}+p^{-1}=1$, which in turn can be identified with the subspace of all countably additive set functions on $\mathcal{A}$ that are absolutely continuous with respect to P and whose Radon-Nykodim derivative is q-integrable. The subset of countably additive probabilities with q-integrable density is denoted by

$$\mathcal{M}_{1,q}=\left\{Q\in\mathcal{M}_1:\frac{dQ}{dP}\in L^q\right\}\quad\forall p\in[1,\infty)$$

[29] That is, in any Riesz space of functions with order unit and endowed with the supnorm. This is the general setup of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b).

while $\mathcal{M}_{1,q} = \mathcal{M}_{1,f}$ if $p = \infty$. Like in the $p = \infty$ case, given $X \in L^p(\Omega, \mathcal{A}, P)$ and $\mu \in L^p(\Omega, \mathcal{A}, P)^*$, we indifferently write: $\mu(X)$, $\int X d\mu \left(= \int X \frac{d\mu}{dP} dP \right)$, or even $\mathbb{E}_\mu(X)$ if μ can be identified with an element of $\mathcal{M}_{1,q}(\Omega, \mathcal{A}, P)$.

A subset C of L^p, is *evenly convex* if and only if for each $\bar{X} \notin C$ there exists a linear and continuous functional μ on L^p such that

$$\mu(\bar{X}) < \mu(X) \quad \forall X \in C.$$

Clearly, evenly convex sets are convex and the intersections of evenly convex sets are evenly convex. Moreover, by standard separation results, it follows that if a set is open (or closed) and convex, then it is evenly convex.

A *risk measure* now is a decreasing function $\rho : L^p \to [-\infty, \infty]$ and we consider the following quasiconvexity property:

Even Quasiconvexity The set $\{X \in L^p : \rho(X) \le \alpha\}$ is evenly convex for all $\alpha \in \mathbb{R}$.

Clearly, evenly quasiconvex risk measures are quasiconvex and it is easy to show that quasiconvex upper semicontinuous risk measures (the ones considered in Lemma 4) are evenly quasiconvex.

In order to provide a generalization of Lemma 4, we have to introduce a new class of functions: $\mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$ for all $q \in [1, \infty]$. Set $\mathbb{R}^\diamond = \mathbb{R} \setminus \{0\}$. A subset C of $\mathbb{R} \times \mathcal{M}_{1,q}$ is $\diamond$*-evenly convex* if and only if for each $(\bar{t}, \bar{Q}) \in (\mathbb{R} \times \mathcal{M}_{1,q}) \setminus C$ there exists $(s, X) \in \mathbb{R}^\diamond \times L^p$ such that

$$\bar{t}s + \mathbb{E}_{\bar{Q}}(X) < ts + \mathbb{E}_Q(X) \quad \forall (t, Q) \in C.$$

Analogously, a function $R : \mathbb{R} \times \mathcal{M}_{1,q} \to [-\infty, \infty]$ is $\diamond$*-evenly quasiconcave* if and only if the set $\{(t, Q) \in \mathbb{R} \times \mathcal{M}_{1,q} : R(t, Q) \ge \alpha\}$ is $\diamond$-evenly convex for all $\alpha \in \mathbb{R}$. Finally, $\mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$ denotes the class of functions $R : \mathbb{R} \times \mathcal{M}_{1,q} \to [-\infty, \infty]$ that are $\diamond$-evenly quasiconcave, increasing in the first component, and such that $\inf_{t \in \mathbb{R}} R(t, Q) = \inf_{t \in \mathbb{R}} R(t, Q')$ for all $Q, Q' \in \mathcal{M}_{1,q}$.[30]

We are now ready to state the anticipated generalization of Lemma 4. Let us remark that while the case $p = \infty$ follows from the results of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b), this is not the case if $p \in [1, \infty)$.

Theorem 14 *A function $\rho : L^p \to [-\infty, \infty]$ is an evenly quasiconvex risk measure if and only if there exists $R \in \mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$ such that*

$$\rho(X) = \sup_{Q \in \mathcal{M}_{1,q}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^p. \tag{25}$$

The function $R \in \mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$ for which (25) holds is unique and satisfies

$$R(t, Q) = \inf\{\rho(X) : \mathbb{E}_Q(-X) = t\} \quad \forall (t, Q) \in \mathbb{R} \times \mathcal{M}_{1,q}. \tag{26}$$

Proof. If $p = \infty$, notice that L^∞ is a normed Riesz space with order unit I_Ω, $\mathcal{M}_{1,f}$ is the positive unit ball of its topological dual, and $-\rho$ is an evenly quasiconcave, and monotone increasing function. The statement then follows from Lemma 8 and Theorem 2 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008b).

Else, if $p \in [1, \infty)$, then L^p is a normed Riesz space, but it does not admit an order unit and $\mathcal{M}_{1,q}$ is not the positive unit ball of its topological dual. Therefore, we cannot invoke the previous results. Next we provide a direct proof.

[30] For $p = \infty$, $\mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,f}) \supseteq \mathcal{R}_0(\mathbb{R} \times \mathcal{M}_{1,f}) \supseteq \mathcal{R}_1(\mathbb{R} \times \mathcal{M}_{1,f})$.

"Only if." Suppose ρ is evenly quasiconvex and define R by (26). By the argument we used in footnote 21,

$$R(t,Q) = \inf\{\rho(Y) : \mathbb{E}_Q(-Y) \geq t\} \quad \forall (t,Q) \in \mathbb{R} \times \mathcal{M}_{1,q}.$$

In particular, for each $X \in L^p$

$$\rho(X) \geq R(\mathbb{E}_Q(-X), Q) \quad \forall Q \in \mathcal{M}_{1,q} \tag{27}$$

thus

$$\rho(X) \geq \sup_{Q \in \mathcal{M}_{1,q}} R(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^p. \tag{28}$$

Fix $\bar{X} \in L^p$. If $\bar{X}$ is not a global minimum, then there is a sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ that tends to $\rho(\bar{X})$ and such that $\bar{X} \notin \{\rho \leq r_n\} \neq \emptyset$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since $\{\rho \leq r_k\}$ is evenly convex, there exists μ_k in the topological dual of L^p such that $\mu_k(\bar{X}) < \mu_k(X)$ for all $X \in \{\rho \leq r_k\}$. Without loss of generality, μ_k can be identified with an element Q_k of $\mathcal{M}_{1,q}$.[31] It follows that, $\{\rho \leq r_k\} \subseteq \{X \in L^p : \mathbb{E}_{Q_k}(-X) < \mathbb{E}_{Q_k}(-\bar{X})\}$, that is, $\{\rho > r_k\} \supseteq \{X \in L^p : \mathbb{E}_{Q_k}(-X) \geq \mathbb{E}_{Q_k}(-\bar{X})\}$. Thus, $R(\mathbb{E}_{Q_k}(-\bar{X}), Q_k) \geq r_k$ and

$$\rho(\bar{X}) \geq \sup_{Q\in\mathcal{M}_{1,q}} R(\mathbb{E}_Q(-\bar{X}), Q) \geq R(\mathbb{E}_{Q_k}(-\bar{X}), Q_k) \geq r_k$$

for all $k \in \mathbb{N}$. Passing to the limits, this implies equality in (28).

It only remains to show that $R \in \mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$. Let $Q \in \mathcal{M}_{1,q}$. If $t \geq t'$ then $\{X \in L^p : \mathbb{E}_Q(-X) \geq t\} \subseteq \{X \in L^p : \mathbb{E}_Q(-X) \geq t'\}$. This implies that $R(t,Q) \geq R(t',Q)$. Next, observe that $R(t,Q) \geq \inf_{X\in L^p} \rho(X)$ for all $(t,Q) \in \mathbb{R} \times \mathcal{M}_{1,q}$. Hence, $\inf_{t\in\mathbb{R}} R(t,Q) \geq \inf_{X\in L^p} \rho(X)$ for all $Q \in \mathcal{M}_{1,q}$. Conversely, consider $\{X_n\}_{n\in\mathbb{N}} \subseteq L^p$ such that $\rho(X_n) \downarrow \inf_{X\in L^p} \rho(X)$. For all $Q \in \mathcal{M}_{1,q}$, $\rho(X_n) \geq R(\mathbb{E}_Q(-X_n), Q) \geq \inf_{t\in\mathbb{R}} R(t,Q)$ for all $n \in \mathbb{N}$. This implies that $\inf_{t\in\mathbb{R}} R(t,Q) \leq \inf_{X\in L^p} \rho(X)$ for all $Q \in \mathcal{M}_{1,q}$. It follows that

$$\inf_{t\in\mathbb{R}} R(t,Q) = \inf_{X\in L^p} \rho(X) = \inf_{t\in\mathbb{R}} R(t,Q') \quad \forall Q, Q' \in \mathcal{M}_{1,q}.$$

Finally, we show that $(t,Q) \mapsto R(t,Q)$ is $\diamond$-evenly quasiconcave. Fix $\alpha \in \mathbb{R}$ and define

$$U_\alpha = \{(t,Q) \in \mathbb{R} \times \mathcal{M}_{1,q} : R(t,Q) \geq \alpha\}.$$

Suppose that U_α is neither empty nor $\mathbb{R} \times \mathcal{M}_{1,q}$. Pick $(\bar{t}, \bar{Q}) \in \mathbb{R} \times \mathcal{M}_{1,q}$ such that $(\bar{t}, \bar{Q}) \notin U_\alpha$. Then, it follows that $R(\bar{t}, \bar{Q}) < \alpha$. This implies that there exists $\bar{X} \in L^p$ such that $\mathbb{E}_{\bar{Q}}(-\bar{X}) \geq \bar{t}$ and $\rho(\bar{X}) < \alpha$. But $R(t,Q) \geq \alpha$ for all $(t,Q) \in U_\alpha$, which implies that $\mathbb{E}_Q(-\bar{X}) < t$ for all $(t,Q) \in U_\alpha$.[32] This, in turn, implies that

$$\bar{t} + \mathbb{E}_{\bar{Q}}(\bar{X}) \leq 0 < t + \mathbb{E}_Q(\bar{X}) \quad \forall (t,Q) \in U_\alpha$$

as wanted.

"If." Suppose (25) holds. We first prove that ρ is evenly quasiconvex. Pick $\alpha \in \mathbb{R}$. We prove that $\{\rho \leq \alpha\}$ is evenly convex. If $\{\rho \leq \alpha\} = L^p$ or $\{\rho \leq \alpha\} = \emptyset$ then there is nothing to prove. Otherwise, let $\bar{X} \notin \{\rho \leq \alpha\}$. By (25), there exists $\bar{Q} \in \mathcal{M}_{1,q}$ for which $R(\mathbb{E}_{\bar{Q}}(-\bar{X}), \bar{Q}) > \alpha$. Let $X \in \{\rho \leq \alpha\}$. Suppose, by contradiction, that $\mathbb{E}_{\bar{Q}}(X) \leq \mathbb{E}_{\bar{Q}}(\bar{X})$. Then, since R is increasing in the first component, $\rho(X) \geq R(\mathbb{E}_{\bar{Q}}(-X), \bar{Q}) \geq R(\mathbb{E}_{\bar{Q}}(-\bar{X}), \bar{Q}) > \alpha$, a contradiction. In other words, there exists $\bar{Q} \in \mathcal{M}_{1,q}$ such that $\mathbb{E}_{\bar{Q}}(\bar{X}) < \mathbb{E}_{\bar{Q}}(X)$ for all $X \in \{\rho \leq \alpha\}$. Next, suppose that $X, Y \in L^p$ and $X \geq Y$. Then,

[31] First, observe that μ_k is a positive linear functional. Let $X \in \{\rho \leq r_k\}$, for each $Y \in L^p_+$ and all $n \in \mathbb{N}$, $\rho(X + nY) \leq \rho(X)$ and $X + nY \in \{\rho \leq r_k\}$. Therefore, $\mu_k(\bar{X}) < \mu_k(X + nY) = \mu_k(X) + n\mu_k(Y)$ for all $n \in \mathbb{N}$. This implies $\mu_k(Y) \geq 0$. Moreover, denoting by $d\mu_k/dP$ the L^q representative of μ_k, $\mu_k \neq 0$ implies $\mu_k(\mathrm{I}_\Omega) = \int_\Omega (d\mu_k/dP)\, dP = \|d\mu_k/dP\|_1 > 0$. Finally, set $Q_k = \mu_k/\mu_k(\mathrm{I}_\Omega)$.

[32] $\mathbb{E}_{Q'}(-\bar{X}) \geq t'$ for some $(t', Q') \in U_\alpha$ would imply $R(t', Q') \leq \rho(\bar{X}) < \alpha$, a contradiction.

$\mathbb{E}_Q(X) \geq \mathbb{E}_Q(Y)$ for all $Q \in \mathcal{M}_{1,q}$ and $R(\mathbb{E}_Q(-X), Q) \leq R(\mathbb{E}_Q(-Y), Q)$ for all $Q \in \mathcal{M}_{1,q}$. By (25), $\rho(X) \leq \rho(Y)$, proving that ρ is a risk measure.

Finally, assume that ρ admits representation (25) for some $R \in \mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$. In order to prove uniqueness it is sufficient to show that R satisfies (26).

Claim. For each $(\bar{t}, \bar{Q}) \in \mathbb{R} \times \mathcal{M}_{1,q}$,

$$R(\bar{t}, \bar{Q}) = \sup_{Q \in \mathcal{M}_{1,q}} \left(\inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} R(\mathbb{E}_Q(-X), Q) \right). \tag{29}$$

Proof of the Claim. Consider the program

$$\pi(Q, \bar{Q}, \bar{t}) = \inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} R(\mathbb{E}_Q(-X), Q)$$

with $Q \in \mathcal{M}_{1,q}$. It is sufficient to show that $\pi(Q, \bar{Q}, \bar{t}) \leq \pi(\bar{Q}, \bar{Q}, \bar{t}) = R(\bar{t}, \bar{Q})$ for all $Q \in \mathcal{M}_{1,q}$. For the second equality just notice that, since R is increasing in the first component,

$$\pi(\bar{Q}, \bar{Q}, \bar{t}) = \inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} R(\mathbb{E}_{\bar{Q}}(-X), \bar{Q}) \geq R(\bar{t}, \bar{Q}).$$

Furthermore, since $\bar{Q} \in \mathcal{M}_{1,q}$, there exists $\bar{Y} \in L^p$ such that $\mathbb{E}_{\bar{Q}}(-\bar{Y}) = \bar{t}$ implying the inverse inequality. Next, fix $Q \in \mathcal{M}_{1,q}$. We have two cases:

- Suppose $Q \in \operatorname{span}(\bar{Q})$. Then, $Q = \alpha \bar{Q}$ for some $\alpha \in \mathbb{R}$. Since $Q, \bar{Q} \in \mathcal{M}_{1,q}$, we have that $\alpha = 1$. It follows that $Q = \bar{Q}$ and $\pi(Q, \bar{Q}, \bar{t}) = \pi(\bar{Q}, \bar{Q}, \bar{t}) = R(\bar{t}, \bar{Q})$.
- Suppose $Q \notin \operatorname{span}(\bar{Q})$. By the Fundamental Theorem of Duality (see, e.g., Aliprantis and Border, 2006), $\ker(\bar{Q}) \nsubseteq \ker(Q)$. That is, there exists $Z \in L^p$ such that $\mathbb{E}_{\bar{Q}}(Z) = 0$ and $\mathbb{E}_Q(Z) \neq 0$. Choose $\bar{Y} \in L^p$ such that $\mathbb{E}_{\bar{Q}}(\bar{Y}) = -\bar{t}$, then the straight line $\bar{Y} + \alpha Z$ is thus included into the hyperplane $\{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) = \bar{t}\}$. Hence, since R belongs to $\mathcal{R}(\mathbb{R} \times \mathcal{M}_{1,q})$,

$$\pi(Q, \bar{Q}, \bar{t}) \leq \inf_{\alpha \in \mathbb{R}} R(\mathbb{E}_Q(-\bar{Y} - \alpha Z), Q) = \inf_{t \in \mathbb{R}} R(t, Q) = \inf_{t \in \mathbb{R}} R(t, \bar{Q}) \leq R(\bar{t}, \bar{Q}).$$

In sum, $\pi(Q, \bar{Q}, \bar{t}) \leq R(\bar{t}, \bar{Q})$ for all $Q \in \mathcal{M}_{1,q}$ and $\pi(\bar{Q}, \bar{Q}, \bar{t}) = R(\bar{t}, \bar{Q})$. □

Let $(\bar{t}, \bar{Q}) \in \mathbb{R} \times \mathcal{M}_{1,q}$. Observe that

$$\inf\{\rho(X) : \mathbb{E}_{\bar{Q}}(-X) = \bar{t}\} = \inf\{\rho(X) : \mathbb{E}_{\bar{Q}}(-X) \geq \bar{t}\} = \inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} \sup_{Q \in \mathcal{M}_{1,q}} R(\mathbb{E}_Q(-X), Q).$$

By the Claim, $R(\bar{t}, \bar{Q}) = \sup_{Q \in \mathcal{M}_{1,q}} \left(\inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} R(\mathbb{E}_Q(-X), Q) \right)$. The general maxmin inequality implies

$$R(\bar{t}, \bar{Q}) = \sup_{Q \in \mathcal{M}_{1,q}} \inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} R(\mathbb{E}_Q(-X), Q) \leq \inf_{X \in \{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\}} \sup_{Q \in \mathcal{M}_{1,q}} R(\mathbb{E}_Q(-X), Q), \tag{30}$$

it remains to prove the converse inequality. If $R(\bar{t}, \bar{Q}) = \sup_{(t,Q) \in \mathbb{R} \times \mathcal{M}_{1,q}} R(t, Q)$, the equality in (30) is easily checked. Otherwise, set $\alpha = R(\bar{t}, \bar{Q})$. We have $\alpha < \infty$. Moreover, for each scalar $\beta > \alpha$, $U_\beta = \{(t, Q) \in \mathbb{R} \times \mathcal{M}_{1,q} : R(t, Q) \geq \beta\}$ is $\diamond$-evenly convex and $(\bar{t}, \bar{Q}) \notin U_\beta$. If β is small enough then U_β is neither empty nor $\mathbb{R} \times \mathcal{M}_{1,q}$.[33] Therefore, there are $\bar{X} \in L^p$ and $s \neq 0$ such that

$$s\bar{t} + \mathbb{E}_{\bar{Q}}(\bar{X}) < st + \mathbb{E}_Q(\bar{X}) \qquad \forall (t, Q) \in U_\beta. \tag{31}$$

[33] Take (t', Q') such that $R(t', Q') > R(\bar{t}, \bar{Q})$ and set $\gamma = R(t', Q')$. For all $\beta \in (\alpha, \gamma)$, $(t', Q') \in U_\beta$ and $(\bar{t}, \bar{Q}) \notin U_\beta$.

Since R is increasing in the first component, it is easy to see that $s > 0$.[34] Set $\lambda = -\bar{t} - \mathbb{E}_{\bar{Q}}\left(s^{-1}\bar{X}\right)$ and $\hat{X} = s^{-1}\bar{X} + \lambda$. It follows that $\mathbb{E}_{\bar{Q}}\left(\hat{X}\right) = -\bar{t}$, and for each $(t, Q) \in U_\beta$

$$\begin{aligned} st + \mathbb{E}_Q\left(\bar{X}\right) > s\bar{t} + \mathbb{E}_{\bar{Q}}\left(\bar{X}\right) &\Longrightarrow \mathbb{E}_Q\left(s^{-1}\bar{X} + \lambda\right) + t > \mathbb{E}_{\bar{Q}}\left(s^{-1}\bar{X} + \lambda\right) + \bar{t} \\ &\Longrightarrow \mathbb{E}_Q\left(\hat{X}\right) + t > \mathbb{E}_{\bar{Q}}\left(\hat{X}\right) + \bar{t} \\ &\Longrightarrow \mathbb{E}_Q\left(\hat{X}\right) + t > 0. \end{aligned}$$

Therefore, it follows that if $0 \geq \mathbb{E}_Q\left(\hat{X}\right) + t$ then $(t, Q) \notin U_\beta$. For each $Q \in \mathcal{M}_{1,q}$, set $t_Q = \mathbb{E}_Q\left(-\hat{X}\right)$, then $\mathbb{E}_Q\left(\hat{X}\right) + t_Q = 0$. Therefore, $(t_Q, Q) = \left(\mathbb{E}_Q\left(-\hat{X}\right), Q\right) \notin U_\beta$ for all $Q \in \mathcal{M}_{1,q}$. This implies $R\left(\mathbb{E}_Q\left(-\hat{X}\right), Q\right) < \beta$ for all $Q \in \mathcal{M}_{1,q}$. Since $\hat{X} \in \left\{Y \in L^p : \mathbb{E}_{\bar{Q}}(-Y) \geq \bar{t}\right\}$, we have that

$$\alpha \leq \inf_{X \in \left\{Y \in L^p : E_{\bar{Q}}(-Y) \geq \bar{t}\right\}} \sup_{Q \in \mathcal{M}_{1,q}} R\left(\mathbb{E}_Q(-X), Q\right) \leq \sup_{Q \in \mathcal{M}_{1,q}} R\left(\mathbb{E}_Q\left(-\hat{X}\right), Q\right) \leq \beta.$$

This is true for each β in a right neighborhood of α, thus $\inf_{X \in \left\{Y \in L^p : E_{\bar{Q}}(-Y) \geq \bar{t}\right\}} \sup_{Q \in \mathcal{M}_{1,q}} R\left(\mathbb{E}_Q(-X), Q\right) = \alpha$, as desired. ■

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[34] If per contra $s < 0$, take $(t', Q') \in U_\beta$, then by monotonicity $(t' + n, Q') \in U_\beta$ for all $n \in \mathbb{N}$. Therefore, it would follow that $st' + sn + \mathbb{E}_{Q'}\left(\bar{X}\right) > s\bar{t} + \mathbb{E}_{\bar{Q}}\left(\bar{X}\right)$ for all $n \in \mathbb{N}$, which is absurd.

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