Macroeconomics Sequence, Block I

The Solow Model

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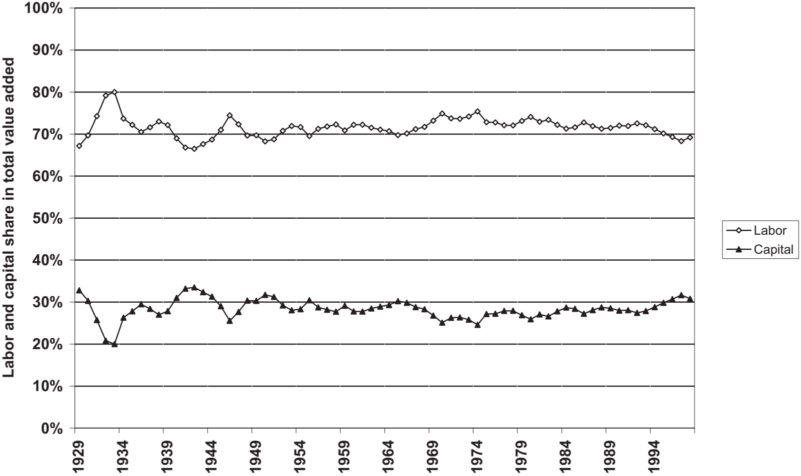
The Kaldor Facts

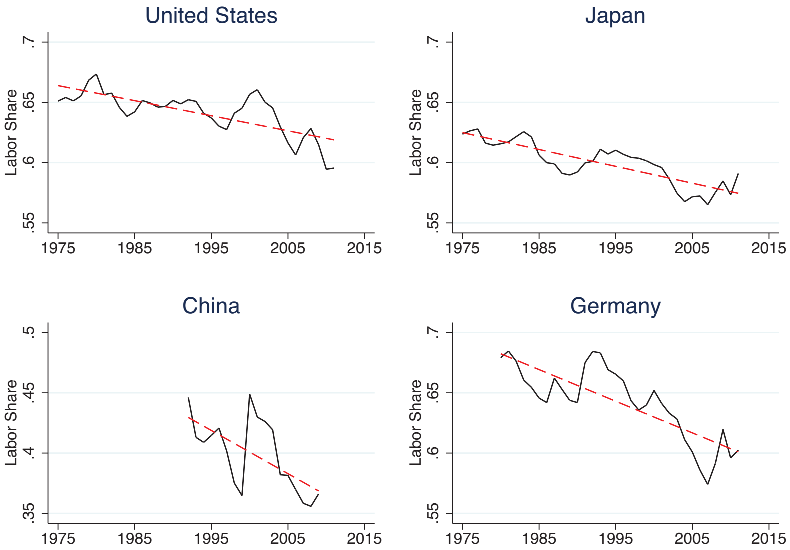
- Nicolas Kaldor '50 found some regularities in long-term US economic data (also Mitchell and Kuznets) which are roughly true today for several contries
    1. Constant GDP (Y) growth rate (US since 1870, approx. 1.8%)
    2. The 'great ratios' K/Y, C/Y, I/Y are approx. constant, i.e., K,Y and C growth roughly at the same rate (for the US: C/Y=.7, I/Y=.2)
    3. N and r do not change much ($r_{US} \in [0.02, 0.06]$)
    4. Y/N (GDP per capita) growth rate changes across countries (now, Y/N in top 5% is 20 times that in bottom 5%, constant income disparity: both rich and poor growth.)
    5. Income shares rK/Y and wN/Y are roughly constant (1/3 and 2/3 respectively), and high capital share ⇒ High investment
- These facts guided researchers in the development of the Neoclassical Growth Theory (Solow '56 and Cass-Koopmans '65). Key ingredient the (Neoclassical) production function ⇒

U.S. Capital and Labor Shares



Accuracy of Kaldor Facts?

Labor shares from Karabarbounis and Neiman (2014)



The Neoclassical Production Function ...

Output in period t depends positively on Labour input N_t, existing Capital K_t and the state of the technology A_t, and it satisfies two key conditions: CRS and DR

$$Y_t = F(K_t, N_t, A_t) \qquad \text{with} \qquad F_K, F_N > 0$$

A1. Constant Returns to Scale (CRS)

$F(\lambda K, \lambda N, A) = \lambda F(K, N, A)$ (you can replicate it elsewhere).

A2. Decreasing Returns (DR)

$$\text{Both} \qquad F_{KK} < 0 \qquad \text{and} \qquad F_{NN} < 0$$

F is also assumed to satisfy few other limit conditions:

$$\lim_{K\to 0} F_K = \lim_{N\to 0} F_N = +\infty \quad \text{(Inada)} \quad \text{and} \quad \lim_{K\to\infty} F_K = \lim_{N\to\infty} F_N = 0$$

... The Neoclassical Production Function

Example: Cobb-Douglas

$$Y = K^{\alpha}(AN)^{1-\alpha} = (A)^{1-\alpha}K^{\alpha}(N)^{1-\alpha} \quad \text{(check it satisfies A1, A2, etc..)}$$

Output per capita as a function of capital per capita

$$\begin{aligned} y &= \frac{Y}{N} = \frac{(A)^{1-\alpha}K^{\alpha}N^{1-\alpha}}{N} = \frac{\hat{A}K^{\alpha}N^{1-\alpha}}{N} \\ &= \frac{\hat{A}K^{\alpha}N^{1-\alpha}}{N^{\alpha}N^{1-\alpha}} \\ &= \frac{\hat{A}K^{\alpha}}{N^{\alpha}} = \hat{A}\left(\frac{K}{N}\right)^{\alpha} = \hat{A}k^{\alpha} = \hat{A}f(k). \end{aligned}$$

where $k = K/N$ is capital per worker, or capital per capita.

- Notice that we only have used CRS to derive the expression.
- DR will imply that f is concave (see Figure)
- The limit conditions imply f is as in the figure

A Simple Dynamic Economy

1. Saving function à la Keynes (behavioral hp): $S_t = sY_t$
2. Capital accumulation rule (standard): $K_{t+1} = (1-\delta)K_t + I_t$
3. Capital market eq.: $I_t = S_t$ (no public sector, closed economy)

$$\text{(1-2-3)} \quad \Rightarrow K_{t+1} = (1-\delta)K_t + S_t = (1-\delta)K_t + sY_t$$

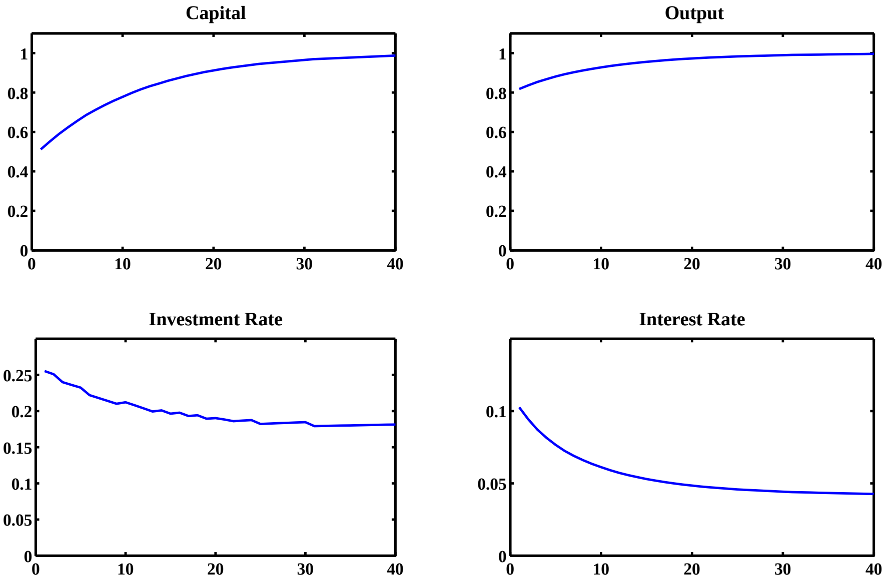
Now let's get all this in per-capita terms:

- If we assume that N and A are constant, we get

$$\begin{aligned} \frac{K_{t+1}}{N} &= (1-\delta)\frac{K_t}{N} + \frac{sY_t}{N} \\ k_{t+1} &= (1-\delta)k_t + sy_t \qquad \text{or} \\ k_{t+1} - k_t &= sy_t - \delta k_t \end{aligned}$$

where $\frac{K_t}{N} = k_t$ and $\frac{Y_t}{N} = y_t$ are capital per-capita and income per-capita respectively

Neoclassical Transitions



The Steady State

- We are in the steady state when $k_{t+1} = k_t$.
- Recalling that $k_{t+1} - k_t = sy_t - \delta k_t$, the steady state level of capital per capita solves:

$$\begin{aligned} sy_t &= \delta k_t \\ s\hat{A}f(k_t) &= \delta k_t \\ s\hat{A}f(k^*) &= \delta k^* \end{aligned}$$

- Cobb-Douglas Example: Assume $f(k) = k^\alpha$

$$s\hat{A}k^\alpha = \delta k \Rightarrow \frac{\hat{A}s}{\delta} = k^{1-\alpha} \Rightarrow k^* = \left(\frac{s\hat{A}}{\delta}\right)^{1/(1-\alpha)}$$

k^* increases with s, α and $\hat{A}$; decreases with δ.

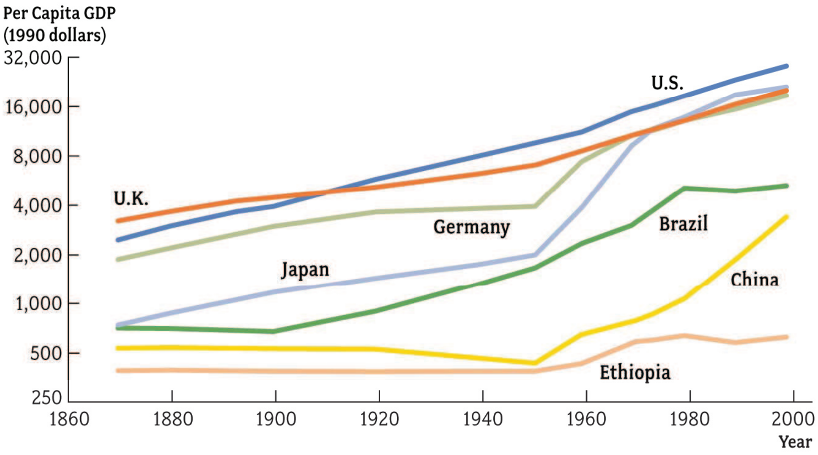


FIGURE 1.1 Per Capita GDP in Seven Countries, 1870–2000

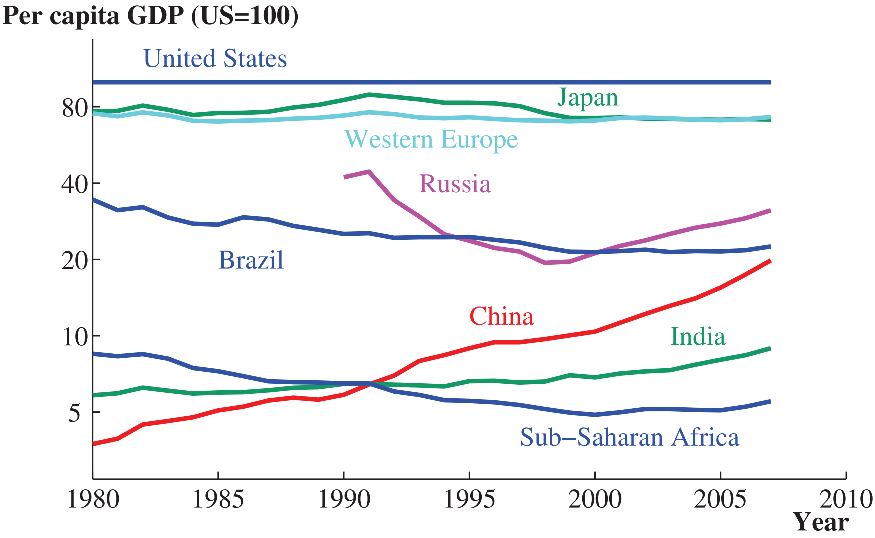
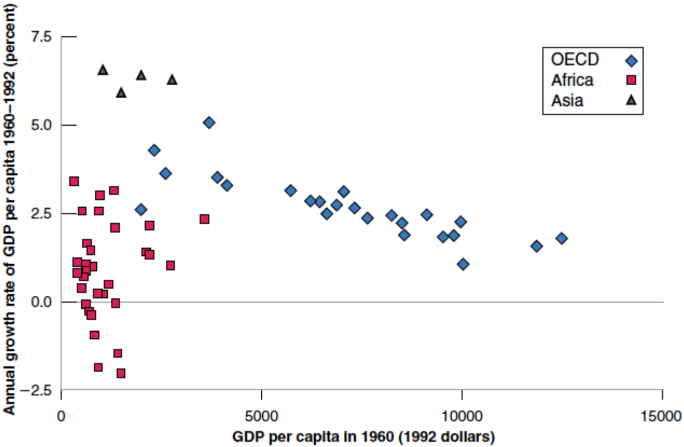

Per capita GDP (US=100)
United States
Japan
Western Europe
Russia
Brazil
China
India
Sub–Saharan Africa
80
40
20
10
5
1980
1985
1990
1995
2000
2005
2010
Year

Figure: Is it there any 'absolute' convergence?

Growth Rate of GDP per Capita, 1960–1992, Versus GDP per Capita in 1960; OECD, Africa, and Asia.
Asian countries are converging to OECD levels. There is no evidence of convergence for African countries.
Source: See Figure 10-2.



'Conditional' Convergence

Consider the following 'growth regression':

$$g_{y,i}^{1960/1990} = \alpha + \beta y_{0,i} + \gamma edu_{0,i} + \varepsilon_i \qquad i = 1, 2, \ldots n$$

Within a group classified by 1960 human capital measures (such as schooling), 1960 saving rates, and other indicators,
a higher income in 1960 is positively correlated with lower growth rates g_y, that is $\beta < 0$.

NB: If we do not add the controls for education, we get $\beta = 0$ (no 'absolute' convergence)

Normative: The Golden Rule

- The good market equilibrium implies $C + I = Y$. Since $I = S$, using (ii) we have

$$\frac{C}{N} = \frac{(1-s)Y}{N} \Rightarrow c = (1-s)y = (1-s)\hat{A}f(k)$$

- $s = 0 \Rightarrow k^* = 0 \Rightarrow f(0) = 0 \Rightarrow c^* = 0.$
- and $s = 1 \Rightarrow c^* = 0$ as well.
- The golden rule saving rate s_G solves:

$$\max_s \; c^*(s) := (1-s)\hat{A}f(k^*(s))$$

- Cobb-Douglas example: $y^*(s) = \hat{A}(k^*(s))^{\alpha} = \hat{A}\left(\frac{s\hat{A}}{\delta}\right)^{\alpha/(1-\alpha)}$

$$\max_s \, (1-s)\hat{A}\left(\frac{s\hat{A}}{\delta}\right)^{\alpha/(1-\alpha)} \Rightarrow s_G = \alpha. \qquad \text{(Check at home)}$$

Positive: Predictions of the Solow Model

1. The model does not predict long-run or sustained growth
   - Diminishing returns to reproducible factors (K)
   - .. and limiting condition: $\lim_{K\to\infty} F_K = 0$
2. Conditional Convergence:
   - Define the growth rate of capital as $\gamma_{k_t} := \frac{k_{t+1}-k_t}{k_t}$, we have:

$$\gamma_{k_t} = \frac{sy_t - \delta k_t}{k_t} = \frac{s\hat{A}k_t^{\alpha} - \delta k_t}{k_t} = \frac{s\hat{A}}{k_t^{1-\alpha}} - \delta$$

     (Check that at $k_t = k^*$ we have $\gamma_{k_t} = 0$)
   - The above condition is key in several empirical studies (growth regressions)
   - 'After controlling for observable differences across countries (which affect the steady state level) the lower is the initial capital, the faster they growth' (Conditional Convergence)

Sources of Sustained Growth

1. Physical Capital Accumulation $F(K_t, \bar{A}N)$
2. Human Capital Accumulation $F(K_t, \bar{A}H_t)$, H_t (education)
3. Technological Progress $F(K_t, A_tN)$ A_t (innovation)

- 3a. Exogenous technological progress: $A_{t+1} = (1+g)A_t$
- If $F(K_t, A_tN)$, the GDP per effective worker is

$$\hat{y}_t = Y_t/NA_t = \hat{f}(K_t/A_tN) = \hat{f}(\hat{k}_t)$$

where $\hat{k}_t$ is now capital per effective worker.

$$\begin{aligned} \frac{K_{t+1}}{A_tN} &= (1-\delta)\frac{K_t}{A_tN} + \frac{sY_t}{A_tN} \\ \Rightarrow \frac{A_{t+1}}{A_t}\frac{K_{t+1}}{A_{t+1}N} &= (1-\delta)\frac{K_t}{A_tN} + \frac{sY_t}{A_tN} \\ \Rightarrow (1+g)\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + s\hat{y}_t \end{aligned}$$

Solow with exogenous technological progress

If we rearrange the law of motion, we get:

$$\begin{aligned}(1+g)\hat{k}_{t+1} &= s\hat{y}_t + (1-\delta)\hat{k}_t \\ \Rightarrow (1+g)\left(\hat{k}_{t+1} - \hat{k}_t\right) &= s\hat{y}_t - (\delta+g)\hat{k}_t\end{aligned}$$

- NB1: g decreases the capital per effective worker. Why?
- NB2: We can use the same diagram as before in $\hat{k}$ $\Rightarrow \hat{k}^* = \left(\frac{s}{\delta+g}\right)^{1/(1-\alpha)}$
- NB3: If $\hat{y} = Y/AN$ is constant then income per capita $y = Y/N$ grows as long as A grows (we get sustained growth). Recall $\hat{k}$:

$$k^*(t) = \hat{k}^* A(t) = \left(\frac{s}{\delta+g}\right)^{1/(1-\alpha)} A(t) \qquad \Rightarrow \gamma_k = g$$

In a sustained (balanced) growth we must have $\gamma_y = \gamma_k$ (prove it!)

A Couple of Revision Exercises

1. Consider the standard Solow model with no technological progress (i.e., $A_t = \bar{A}$), and assume there is population growth, that is, $N_{t+1} = (1+n)N_t$. Show that the model implies a sustained growth path where total GDP Y_t and aggregate capital K_t both growth at the same rate n.

2. Now consider the model in the previous exercise and suppose the economy is on its balanced growth path and the saving rate is s_0. Now, suppose the saving rate increases to $s_1 > s_0$. Describe what will happen to the path of Y_t and K_t. Do it by drawing two well commented graphs where in the horizontal axis you have time t, while in the vertical axis you have $\ln Y_t$ and $\ln K_t$ respectively.

References

Any advanced book in macroeconomics analyzes the Solow-Swan model.
Here below you find (alternative) references that include the Ramsey-Cass-Koopmans moodel we will see next.
In parenthesis optional chapters on endogenous growth models.

1. Romer: *Advanced Macroeconomics*, McGraw-Hill, Chapters: 1, 2, (& 3).
2. Barrow and Sala-i-Martin: *E*conomic Growth, McGraw-Hill, Chapters: Intro, 1, 2 (& 4).