Macroeconomics Sequence, Block I

The Contraction Mapping Theorem and The Properties of the Value Function

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Useful Mathematics: Complete Metric Spaces

Definition 1 A metric space (X, d) is a set X, together with a metric (or distance function) $d : X \times X \to \mathbb{R}$, such that for all $x, y, z \in X$ we have:
(i) $d(x, y) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$; and
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Example: $X = \mathbb{R}$ and $d(x, y) = |x - y|$

Definition 9: Convergence Let (X, d) a metric space . The sequence $\{x_n\}_{n=0}^{\infty}$ is convergent to $y \in X$ if for each real number $\varepsilon > 0$ there exists a natural number N such that for any $n \geq N$ we have $d(x_n, y) < \varepsilon$. And we write $x_n \to y$, or $\lim_{n\to\infty} x_n = y$.

Definition 10: Cauchy We say that a sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy if for each $\varepsilon > 0$ there exists a natural number N such that for any $n, m \geq N$ we have $d(x_n, x_m) < \varepsilon$.

Definition 20: Complete Metric Spaces A metric space (X, d) is said to be Complete if any Cauchy Sequence is convergent in X.

Useful Mathematics: Example

Here is an example of a non-complete metric space.

- Let again $X = \mathbb{R}$, with the following metric

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ \max\left\{\frac{1}{1+|x|}, \frac{1}{1+|y|}\right\} & \text{otherwise.} \end{cases}$$

- We can check that d is actually a metric!
- Now consider the sequence $\{x_n\}_{n=0}^{\infty}$ of integers $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ \dots\ x_n = n,\ \dots$.
- It is easy to see that as m and n increase, the distance $d(x_n, x_m) = d(n, m)$ goes to zero. Indeed, if $n, m \geq N$, then $d(n, m) \leq \frac{1}{1+N}$. Hence the sequence is Cauchy.
- However, it is easy to see that the sequence $\{x_n\}_{n=0}^{\infty} = \{n\}_{n=0}^{\infty}$ does not converge to any real number x, since for any $x < \infty$ we have $d(x, n) \geq \frac{1}{1+|x|} > 0$ for all n.

The Contraction Mapping Theorem

Definition 22: Contraction Mapping Let (X, d) a metric space and $T : X \to X$ a function mapping X into itself. T is a Contraction (with modulus β) if for some $\beta < 1$ we have

$$d(Tx, Ty) \leq \beta d(x, y), \text{ for all } x, y \in X.$$

Theorem 7: If (X, d) is a complete metric space and $T : X \to X$ is a contraction with modulus β, then
(i) T has exactly one fixed point x^* in X, i.e. $x^* = Tx^*$, and
(ii) for any $x_0 \in X$ we have $d(T^n x_0, T^n x^*) \leq \beta^n d(x_0, x^*)$, $n = 0, 1, 2, \ldots$

Proof: Start with any $x_0 \in X$ and construct the sequence: $x_n = T^n x_0$.
Notice that $d(x_{n+1}, x_n) \leq \beta^n d(x_1, x_0)$. Now, let $m = n + p + 1$:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq \beta^p d(x_{n+1}, x_n) + \beta^{p-1} d(x_{n+1}, x_n) + ... + d(x_{n+1}, x_n) \\ &\leq \frac{1}{1-\beta} d(x_{n+1}, x_n) \leq \frac{\beta^n}{1-\beta} d(x_1, x_0) \end{aligned}$$

Proof of the Contraction Mapping Theorem

The sequence is hence Cauchy. Since (X, d) is complete this sequence must converge, i.e. there exists a x^* such that

$$\lim_{n\to\infty} x_n = x^*.$$

By the continuity of T (in fact T is uniformly continuous) we have

$$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

It remains to show that $x^* = Tx^*$ is unique. Call $x^{**} = Tx^{**}$ the second fixed point. Note that

$$d\left(x^{**}, x^*\right) = d\left(Tx^{**}, Tx^*\right) \leq \beta d\left(x^{**}, x^*\right),$$

a contradiction as long as $d\left(x^{**}, x^*\right) > 0$. Hence, $d\left(x^{**}, x^*\right) = 0$, that is x^* and x^{**} must in fact be the same point.

- For part (ii) see Theorem 3.2 in Stokey-Lucas-Prescott (1991).

Sufficient Conditions for a Contraction

Theorem 6: Let $X \subset \mathbb{R}^l$. Both the set $\mathbf{B}(X)$ of bounded functions and the set $\mathbf{C}(X)$ of bounded and continuous functions $f : X \to \mathbb{R}$ together with the "sup" metric $d_\infty(f, g) \equiv \sup_t |f(t) - g(t)|$ are Complete Metric Space (They are linear hence Banach Space).

Theorem 8: (Blackwell)
Let $T : \mathbf{B}(X) \to \mathbf{B}(X)$ be an operator satisfying:
(i) $f, g \in \mathbf{B}(X)$ and $f(x) \leq g(x)$ $\forall x \in X$, implies $(Tf)(x) \leq (Tg)(x)$ $\forall x \in X$, and
(ii) there exists some $0 \leq \beta < 1$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \; \forall f \in \mathbf{B}(X), \; a \geq 0, \; x \in X.$$

Then T is a contraction with modulus β.

The general Framework

$$
\begin{aligned}
V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad (1) \\
s.t.\ x_0 &\in X \\
x_{t+1} &\in \Gamma(x_t) \quad \text{for all } t.
\end{aligned}
$$

Time invariant function F, and correspondence Γ; $\beta \in [0,1)$. We assume Γ to be non empty for all $x \in X$.

Recall that the BPO is equivalent to the possibility of writing the value function V^* as Bellman Functional Equation:

$$
V(x_0) = \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta V(x_1), \qquad (2)
$$

Basic Result: Existence and Uniqueness

Assumption 4.1 $\Gamma(x)$ is non-empty for all $x \in X$.

Assumption 4.2 F is bounded and $\beta \in [0, 1)$.

Theorem Assume 4.1 and 4.2 and consider the metric space $(\mathcal{B}(X), d_\infty)$ of bounded functions with the sup norm. Then the Bellman operator T defined by

$$(TW)(x) = \sup_{x' \in \Gamma(x)} F(x, x') + \beta W(x') \tag{3}$$

(i) maps $\mathcal{B}(X)$ into itself; (ii) has a unique fixed point $V \in \mathcal{B}(X)$.

Properties I: Continuity

Assumption 4.3 $\Gamma(x)$ is continuous.

Assumption 4.4 F is continuous and $\beta \in [0, 1)$.

Theorem 15 Assume 4.1, 4.2, 4.3 and 4.4 then
(i) the fixed point V is continuous (and bounded);
(ii) the policy correspondence

$$G(x) = \{y \in \Gamma(x) : V(x) = F(x, y) + \beta V(y)\}$$

is non empty, compact valued, and upper semi-continuous.

Properties II: Concavity and Differentiability

Assumption 4.7 Γ has a convex graph

Assumption 4.8 F is concave

Theorem 16 Assume 4.1, 4.2, 4.7 and 4.8. Then
(i) The fixed point V is concave.
(ii) If F is differentiable then V is continuously differentiable and

$$V'(x) = \frac{\partial F(x, g(x))}{\partial x} = F_1(x, g(x))$$

for any $x \in intX$ such that the policy is interior, i.e. $g(x) \in int\Gamma(x)$.

Proofs

They all work in the same way.

- Assumption 4.1 and 4.2, (and 4.3 and 4.4) [and 4.7 and 4.8] guarantee that the operator T maps bounded (and continuous) [and concave] functions into bounded (and continuous) [and concave] functions.
- The space of bounded (and continuous) [and concave] functions is a complete metric space and T is a contraction.
- We can hence apply the contraction mapping theorem and show that the fixed point $V = TV$ has these properties.
- Differentiability uses a bit more advanced stuff. See SLP.
- The same can be done for <u>Monotonicity</u>.
  When $F(x, x')$ is monotone increasing in x and the feasibility set $\Gamma(x)$ widens with x, i.e. if $x' \geq x$ then $\Gamma(x) \subset \Gamma(x')$;
  Under 4.1, 4.2 and Monotonicity, $V(\cdot)$ <u>is</u> bounded <u>increasing</u>.

The Maximum Theorem

Theorem of the Maximum. For all $x \in X$ let

$$v(x) := \sup_{x' \in \Gamma(x)} h(x, x').$$

If $h(\cdot, \cdot)$ is continuous and $\Gamma(\cdot)$ is continuous and non-empty, $v(\cdot)$ is continuous and the policy correspondence

$$g(x) := \{y \in \Gamma(x) | v(x) = h(x, y)\}$$

is non-empty and compact valued.

Figure

This theorem allows us to show that under 4.3 and 4.4 the Bellman operator maps continuous function into continuous functions.