Macroeconomics Sequence, Block I

The Optimal Model of Growth: Euler Equations vs Dynamic Programming

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The Neoclassical Growth Model I

$$\begin{aligned}
V^*(k_0) &= \max_{\{k_{t+1},\, i_t,\, c_t,\, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& \quad s.t. \\
k_{t+1} &= (1-\delta)k_t + i_t \qquad (1) \\
c_t + i_t &\leq F(k_t, n_t) \\
c_t,\ k_{t+1} &\geq 0,\ n_t \in [0,1]\,;\ k_0 \text{ given.}
\end{aligned}$$

Constraint (1) is the law of motion of the state variable k_t. $V^*(k_0)$ is the value function of the problem.

- Notice that $n_t = 1$ and the resource constraint is satisfied with equality.
- We can hence simplify the constraints by defining

$$f(k_t) = F(k_t, 1) + (1-\delta)k_t.$$

The Neoclassical Growth Model II

- The above specified simplifications deliver

$$\begin{aligned} V^*(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\ & \quad s.t. \\ 0 &\leq k_{t+1} \leq f(k_t);\ k_0 \text{ given.} \end{aligned} \tag{2}$$

- (2) describes the problem in Sequential form: It is a 'static' problem with objective function

$$W(k_0, k_1, k_2, ...) = \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

- If $f(k) = k$ we have the Cake eating problem. If $f(k) = (1 + r)k + y$ we have a simple saving (storage) problem.

Euler Equation

Taking the usual first order conditions we get the Euler Equations

$$u'\left(f(k^*_{t-1}) - k^*_t\right) = \beta f'(k^*_t) u'\left(f(k^*_t) - k^*_{t+1}\right) \qquad t = 1, 2, ...$$

Economic Intuition: Intertemporal efficiency.

$$\begin{aligned} \frac{u'(c^*_{t-1})}{\beta u'(c^*_t)} &= f'(k^*_t) = \frac{\partial F(k^*_t, 1)}{\partial k_t} + (1-\delta), \\ MRS &= MRT(c_{t-1}, c_t) \end{aligned}$$

Since consumption and investment are the same good, a marginal reduction of period $t-1$ consumption implies a one-to-one increase on i_{t-1}, which in turn increases by $\frac{\partial F(k_t,1)}{\partial k_t} + (1-\delta)$ the amount of goods available for period t consumption.

Euler Equation and Steady state

- Under standard assumptions (neoclassical production function, with $\lim_{k\to\infty} F_k = 0$) the economy allows for a steady state for capital and consumption.
- Graph of $c^0(k) = f(k) - k$, (which implies $k_{t+1} = k_t$), and the locus of points where $c_{t+1} = c_t$.
- In the steady state, the Euler equation implies $\frac{1}{\beta} = f'(k^{ss})$.
- Assuming a Cobb-Douglas production function:

$$F(k,n) = Ak^{\alpha}n^{1-\alpha} \text{ with } \alpha \in (0,1)$$

the Euler equation becomes

$$\frac{1}{\beta} = \alpha A k^{\alpha-1} + (1-\delta) \implies k^{ss} = \left(\frac{A\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$$

where $\rho = \beta^{-1} - 1$ is the agent's discount rate.

The Euler Variational Approach I

- The approach considers perturbations on the optimal contract (which is supposed to exist)
- And finds Necessary conditions for the conjectured contract to be optimal
- We are looking for a feasible deviation from the optimal *interior* program $\{k_{t+1}^*\}_{t=0}^{\infty}$, where interiority simple requires

$$\text{both } k_{t+1}^* > 0 \text{ and } k_{t+1}^* < f(k_t^*) \text{ for all } t.$$

- The perturbation is aimed at changing k_{t+1}^* (and i_t^*, i_{t+1}^*), while keeping unchanged all k_s^* for $s \neq t+1$, in particular both k_t^* and k_{t+2}^*.
- The perturbation must be small enough to keep feasibility

Euler variational approach II

- Let ε any real number in an open neighborhood O of zero
- For each ε, the perturbed plan $\{\hat{\imath}_t^{\varepsilon}, \hat{k}_{t+1}^{\varepsilon}\}_{t=0}^{\infty}$ is constructed from $\{i_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ as follows: $\hat{k}_{t+1}^{\varepsilon} = k_{t+1}^* + \varepsilon$, and $\hat{k}_s^{\varepsilon} = k_s^*$ for $s \neq t+1$.
- Such perturbation implies: $\hat{\imath}_t^{\varepsilon} = i_t^* + \varepsilon$ and $\hat{\imath}_{t+1}^{\varepsilon} = i_{t+1}^* - (1-\delta)\,\varepsilon$ and $\hat{\imath}_s^{\varepsilon} = i_s^*$ for $s \neq t, t+1$.
- If we denote by $\hat{V}_0(\varepsilon)$ the value associated to the perturbed plan for each $\varepsilon \in O$, the optimality of the original plan implies $\hat{V}_0(\varepsilon) \leq V_0^*$ for all $\varepsilon \in O$, and $\hat{V}_0(0) = V_0^*$.
- Stated in other terms, $\varepsilon = 0$ is the optimal solution to

$$\max_{\varepsilon \in O} \ \hat{V}_0(\varepsilon).$$

The first order condition is $\hat{V}_0'(0) = 0$.

Obtaining the Euler Equations

- Since k_s^* are untouched, both for $s \leq t$ and $s \geq t+2$ the derivative with respect to ε of all terms are zero but period t and $t+1$ returns. We hence have:

$$\hat{V}_0'(\varepsilon) = \frac{d}{d\varepsilon}\left\{u(f(k_t^*) - k_{t+1}^* - \varepsilon) + \beta u(f(k_{t+1}^* + \varepsilon) - k_{t+2}^*)\right\}$$

- The condition $\hat{V}_0'(0) = 0$ hence delivers the Euler equation

$$u'(f(k_t^*) - k_{t+1}^*) = \beta f'(k_{t+1}^*) u'(f(k_{t+1}^*) - k_{t+2}^*).$$

The Euler Approach: Another Concavity Requirement

The Euler conditions check for 'unilateral' variations on the optimal plan.

To get the idea of the Euler Approach, consider the simple problem

$$V^* = \max_{(x,y) \in X \times Y} h(x, y)$$

The optimal pair $(x^*, y^*) : h(x^*, y^*) \geq h(x, y)$ for all $(x, y) \in X \times Y$, satisfies (among other things)

$$\begin{aligned} h(x^*, y^*) &\geq h(x, y^*) \text{ for all } x \in X, \text{ and} \\ h(x^*, y^*) &\geq h(x^*, y) \text{ for all } y \in Y. \end{aligned}$$

Now, for example, specify

$$h(x, y) = \sqrt{x}\sqrt{y}, \quad \text{with} \quad x, y \in [0, 3].$$

What is the optimal pair? What about $(0, 0)$?

The Euler Approach: The Transversality Condition

Since with the first order conditions we only check for one period deviations, one checks in addition the Transversality condition.

Proposition 3 Assume u and f to be bounded, increasing, continuous, concave, and differentiable. If the (interior) sequence $\{k_t^*\}_{t=1}^{\infty}$ satisfies the Euler equations (given k_0) and

$$\lim_{T\to\infty} \beta^T u'(f(k_T^*) - k_{T+1}^*)f'(k_T^*)k_T^* \leq 0$$

then $\{k_t^*\}_{t=0}^{\infty}$ is an optimal sequence.

Intuition: First order conditions are sufficient for global 'finite period optimality' (concavity). What about infinite deviations? Since $\beta^T u' f'$ is the $t = 0$ price of capital k_T, if

$$\lim_{T\to\infty} \beta^T u'(f(k_T^*) - k_{T+1}^*)f'(k_T^*)k_T^* > 0$$

The agent is holding valuable capital, i.e. the value of capital has not been exhausted, and perhaps U_0 can be increased further.

The Bellman Principle of Optimality (B.P.O.) I

The optimal growth problem (2) can be studied by solving the following functional equation

$$V(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta V(k'), \tag{3}$$

where the 'unknown' in the equation is the value function V. (3) is the Recursive formulation (R.F.) of the problem.

1. The R.F. simplifies the problem making it easier to understand.
2. The R.F. has several computational advantages. See numerical methods.
3. Instead of an optimal sequence $(k_0^*, k_1^*, k_2^*, \ldots)$ (Euler), the solution to the recursive problem delivers a policy $k' = g(k)$, i.e. an optimal rule for any k.

The Bellman Principle of Optimality II

In order to obtain the recursive formulation one need to identify the state variables.

> *"In some problems, the state variables and the transformations are forced upon us; in others there is a choice in these matters and the analytic solution stands or fall upon this choice; in still others, the state variables and sometimes the transformations must be artificially constructed. Experience alone, combines with often laborious trial and error, will yield suitable formulations of involved processes." Bellman (1957).*

The Principle of Optimality is

> *"An optimal [plan] has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal [plan] with regard to the state resulting from the first decision." Bellman (1957).*

The Bellman Principle of Optimality III

The B.P.O. is based on two key facts (recall our simple case):

(i) typically we can 'split' the max:

$$V^* = \max_{(x,y)\in X\times Y} h(x,y) = \max_{x\in X}\left\{\max_{y\in Y} h(x,y)\right\}.$$

(ii) quite often we can also 'pass' the max over. Example: Profit maximization

$$\begin{aligned} \pi^*(p,w) &= \max_{z,y}\ py - wz \\ s.t. &\qquad y \le f(z), \end{aligned}$$

where y is output, z are inputs; p and w are prices. It is well known:

$$\pi^*(p,w) = \max_{y}\ py - C(y;w),$$

where, for each y, C is the cost function

$$C(y;w) = \min_{z}\ wz \qquad s.t. \quad y \le f(z). \tag{4}$$

The Bellman Principle of Optimality III: (continued)

Proposition 1 *The 'true' value function V^* Solves the Bellman FE.*

Proof :(assume existence) Just use the B.P.O. to our model

$$V^*(k_0) = \max_{0 \le k_1 \le f(k_0)} \left\{ \max_{\{k_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \right\}$$

$$= \max_{0 \le k_1 \le f(k_0)} \left\{ \max_{\{k_{t+2}\}_{t=0}^{\infty}} u(f(k_0) - k_1) + \beta \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1}) - k_{t+2}) \right\}$$

$$= \max_{0 \le k_1 \le f(k_0)} u(f(k_0) - k_1) + \beta \max_{\{k_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1}) - k_{t+2})$$

$$= \max_{0 \le k_1 \le f(k_0)} u(f(k_0) - k_1) + \beta V^*(k_1).$$

That is, we obtain the functional equation (3) with $V = V^*$!

The B.P.O. III: (continued)

The Bellman Principle of Optimality is a much tighter then the Euler approach.

- With the Principle of Optimality we guarantee

$$u(f(k_{t-1}^*) - k_t^*) + \beta V^*(k_t^*) \geq u(f(k_{t-1}^*) - k_t) + \beta V^*(k_t)$$

- Recall that with the Euler approach we were checking instead

$$u(f(k_{t-1}^*) - k_t^*) + \beta u(f(k_t^*) - k_{t+1}^*) \geq u(f(k_{t-1}^*) - k_t) + \beta u(f(k_t) - k_{t+1}^*)$$

When the time horizon is finite, the B.P.O. is equivalent to global optimality.
When the horizon is infinite we need a form of 'Transversality', since again we only check for one-stage deviations.

The Verification Theorem

Proposition 2 *If V solves the Bellman functional equation (3), and for all feasible sequences $\{k_t\}_{t=0}^{\infty}$ with $0 \leq k_{t+1} \leq f(k_t)$ we have*

$$\lim_{T\to\infty} \beta^T V(k_T) = 0$$

then V is the true value function (i.e. $V = V^*$) and any sequence generated by the optimal policy starting from k_0 is optimal.
Proof : Bellman equation implies global 'finite period optimality':

$$\begin{aligned}
V(k_0) &= \max_{k_1} u(f(k_0) - k_1) + \beta V(k_1) \\
&= \max_{k_1,k_2} u(f(k_0) - k_1) + \beta u(f(k_1) - k_2) + \beta^2 V(k_2) \\
&\qquad \cdots \\
V(k_0) &= \max_{k_1,k_2,..} \sum_{t=0}^{T-1} \beta^t u(f(k_t) - k_{t+1}) + \beta^T V(k_T).
\end{aligned}$$

Since k_T belongs to a feasible sequence and $\beta^T V(k_T) \to 0$, done!

The Cake Eating Problem I: Sequential Approach

If we set $f(k) = k$ the problem is that of eating a cake of size k_0. In terms of the controls alone it can be written as

$$\begin{aligned} V^*(k_0) &= \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \quad s.t. \\ \sum_{t=0}^{\infty} c_t &\leq k_0;\ c_t \geq 0;\ k_0 \text{ given.} \end{aligned}$$

$\Rightarrow$ Do it at home!

The Cake Eating Problem II: Recursive Approach

The associated functional equation is

$$V(k) = \max_{0 \leq k' \leq k} \ln(k - k') + \beta V(k').$$

We can adopt a guess and verify procedure.

(i) Guess that $V(k) = A + B\ln(k)$ and

(ii) take FOC to verify that we can find constants (A and B) such that V solves the functional equation.

$$\frac{1}{k - k'} = \beta \frac{B}{k'} \quad \Rightarrow \quad k' = g(k) = \frac{\beta B}{1 + \beta B} k.$$

Plugging it into our initial problem we have

$$V(k) = A + \frac{1}{1 - \beta} \ln k,$$

hence $B = \frac{1}{1-\beta}$, optimal policy $g(k) = \beta k$, or $c(k) = (1 - \beta)k$.

Finite Horizons and guided guesses

Consider the one-period Gale's cake eating problem

$$V_1(k) = \max_{0 \le k' \le k} \ln\left(k - k^{'}\right).$$

$$V_1(k) = \max_{0 \le k' \le k} \ln\left(k - k^{'}\right) + \beta V_0(k') = \max_{0 \le k' \le k} \ln\left(k - k^{'}\right).$$

where $V_0(k) \equiv 0$.
Since $k' = g_0(k) \equiv 0$, we have $V_1(k) = \ln\left(k - g_0(k)\right) = \ln(k)$.

$$\begin{aligned}
V_2(k) &= \max_{0 \le k' \le k} \ln\left(k - k^{'}\right) + \beta V_1(k') \\
&= A_2 + (1+\beta)\ln(k), \\
&\quad \cdots \\
V_n(k) &= A_n + (1 + \beta + \beta^2 + \cdots + \beta^{n-1}) ln(k) \\
&\quad \cdots \\
V(k) &= V_\infty(k) = A_\infty + \frac{1}{1-\beta}\ln(k).
\end{aligned}$$

Bellman Operator and Contraction Mapping

Recall the previous slide:

$$V_1(k) = \max_{0 \le k' \le k} u\left(k - k^{'}\right) + \beta V_0(k') = (TV_0)(k).$$

T is called the Bellman operator. T maps functions into functions.

$$\begin{aligned} V_2(k) &= T^2 V_0 = (TV_1)(k) = \max_{0 \le k' \le k} u\left(k - k^{'}\right) + \beta V_1(k'), \\ &\dots \\ V_n(k) &= T^n V_0 = (TV_{n-1})(k) = \max_{0 \le k' \le k} u\left(k - k^{'}\right) + \beta V_{n-1}(k'). \end{aligned}$$

The function V can be seen as a fixed point of the $T-$ operator:

$$\lim_{n \to \infty} T^n V_0 = V_\infty = TV_\infty.$$

Contraction Mapping Theorem $\Rightarrow$ the fixed point V exists, is unique, is obtainable as a limit 'point' V_∞ from an arbitrary V_0. (algorithm)