# Chapter 3:The Constant Expected Returns Model

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## 1. The Constant Expected Returns Model

The CER model assumes that an asset's return over time is normally distributed with a constant mean and constant variance. The model allows for the returns on different assets to be contemporaneously correlated but that the correlations are constant over time. Returns are independent over time both across assets and within the same asset. The CER model constitutes the simplest specification of our general statistical model for asset returns.

$$\begin{aligned} r_{i,t} &= \mu_i + \sigma_i \epsilon_{it} \\ \epsilon_{it} &\sim NID\,(0,1) \\ cov\,(\epsilon_{it}, \epsilon_{js}) &= \begin{cases} \sigma_{ij} & t = s \\ 0 & t \neq s \end{cases} \end{aligned}$$

As we have already discussed in the introduction, the CER model is consistent with the view on the behaviour of asset prices andfinancial returns that can be summarized in the following points:

- The CAPM provides a good measure of risk and thus a good explanation for why some stocks earn higher average returns than others according to the simple model

$$\boldsymbol{\mu} - r^f \mathbf{e} = \boldsymbol{\beta} \odot [(\mu_M - r^f)\mathbf{e}],$$

where $\odot$ denotes the element-by-element multiplication operator, $\boldsymbol{\beta}$ is the $N \times 1$ vector of CAPM (unconditional) betas for each of the $N$ risky assets and $\mathbf{e}$ an $(N \times 1)$ vector of ones;[1]

- Excess returns are close to unpredictable; any predictability is a statistical artifact or cannot be exploited after transaction costs are imputed to actual trades based on such alleged predictability, i.e., whatever is our information set $\mathcal{I}_t$, $E[\mathbf{r}_{t+1} - r^f \mathbf{e} | \mathcal{I}_t] = E[\mathbf{r}_{t+1} - r^f \mathbf{e}] = \boldsymbol{\mu} - r^f \mathbf{e}$, there is nothing to be learnt from $\mathcal{I}_t$ for practical purposes;

- Volatility and covariances are approximately constant over time, i.e., $\boldsymbol{\Sigma}_t \equiv Var[\mathbf{r}_{t+1} - r^f \mathbf{e} | \mathcal{I}_t] = Var[\mathbf{r}_{t+1} - r^f \mathbf{e}] = \boldsymbol{\Sigma}$.

- asset prices behave as a (log) random walk with drift

[1] As an example, consider the triangular matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and the column vector $[-4\ 4]'$. We know that normally, the product would give

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}.$$

In this case the product is not even emphasized in the notation, i.e., $Ax = A \cdot x$. Using the dot product, we obtain instead

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ 0 & 12 \end{bmatrix},$$

i.e., each colum of the matrix is multiplied by the column vector.

1.1. *Regression Model Representation*

Consider the following simultaneous equations linear regression model for a sample of size $T$ concerning observations on a vector of $N$ returns:[2]

$$\mathbf{y}^{+} = \mathbf{x}^{+}\boldsymbol{\delta}^{+} + \mathbf{u}^{+}, \tag{1}$$

where $\mathbf{y}^{+}$ is a $(NT \times 1)$ vector, $\mathbf{x}^{+}$ is a $NT \times \sum_{i=1}^{N} K_i$ matrix ($K_i$ is the number of regressors available at each point in time), $\boldsymbol{\delta}^{+}$ is a $\sum_{i=1}^{N} K_i \times 1$ vector of unknown parameters, and $\mathbf{u}^{+}$ is a $(NT \times 1)$ vector of residuals:

$$\mathbf{y}^{+} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \vdots \\ \mathbf{y}_N \end{pmatrix}, \quad \mathbf{x}^{+} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2 & \cdots & \vdots \\ \mathbf{0} & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{x}_N \end{pmatrix},$$

$$\boldsymbol{\delta}^{+} = \begin{pmatrix} \boldsymbol{\delta}_1 \\ \vdots \\ \vdots \\ \boldsymbol{\delta}_N \end{pmatrix}, \quad \mathbf{u}^{+} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \vdots \\ \mathbf{u}_N \end{pmatrix}.$$

Moreover, given that we consider predictive models in which $\mathbf{X}^{+}$ contains variables observed in periods preceeding that of those included in $\mathbf{y}^{+}$, we can safely that they are orthogonal to the residuals and therefore

$$p \ \lim \frac{1}{T}\mathbf{x}^{+\prime}\mathbf{u}^{+} = 0.$$

Clearly, all vectors and matrices are simply stacking $N$ times the $T$-long histories for each of the return series, each of the explanatory variables, as well as the residuals. Also notice the special diagonal structure of $\mathbf{X}^{+}$ (the + indeed stands for "augmented" to emphasize that the stacking operation has allowed us to capture in one single system of simultaneous regression models, $N$ such models, each with structure:

$$\mathbf{y}_i = \mathbf{x}_i\boldsymbol{\delta}_i + \mathbf{u}_i,$$

[2] In the lecture notes, it is possible that the number of assets $G$ may have been called $N$. To impose some uniformity in notations, we simply set $N = G$ here and the two carry no special meaning or differentiation.

$i = 1, 2, ..., N$. (**??**) is then easy to use to derive inferences on means, variance and covariances:

$$\begin{aligned} E[\mathbf{y}^{+}] &= \mathbf{x}^{+}\boldsymbol{\delta}^{+} \\ Var[\mathbf{y}^{+}] &= Var[\mathbf{u}^{+}] \end{aligned}$$

assuming that the regressors collected in $\mathbf{X}^{+}$ are predetermined (e.g., this will be the case when $\mathbf{X}^{+}$ simply collects past values of asset return themselves). Because in this model $\boldsymbol{\Sigma}_t \equiv Var[\mathbf{y}^{+}|\mathcal{I}_t] = Var[\mathbf{u}^{+}]$, the third view is automatically enforced.

If $\mathbf{u}_i$ is assumed to have standard white noise properties, i.e., $E[\mathbf{u}_i] = \mathbf{0}$ and $E[\mathbf{u}_i\mathbf{u}_i'] = \sigma_{ii}\mathbf{I}_T$ (i.e., all residuals are not serially correlated although they can be contemporaneously correlated) where $\sigma_{ii} = \sigma_i^2$, then the following properties hold for $\mathbf{u}^{+}$:

$$\begin{aligned} E\left(\mathbf{u}^{+}\right) &= \mathbf{0}_{NT} \\ E[\mathbf{u}^{+}(\mathbf{u}^{+})'] &= \begin{pmatrix} E\left(\mathbf{u}_1\mathbf{u}_1'\right) & E\left(\mathbf{u}_1\mathbf{u}_2'\right) & \cdots & E\left(\mathbf{u}_1\mathbf{u}_N'\right) \\ E\left(\mathbf{u}_2\mathbf{u}_1'\right) & E\left(\mathbf{u}_2\mathbf{u}_2'\right) & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ E\left(\mathbf{u}_N\mathbf{u}_1'\right) & \cdots & \cdots & E\left(\mathbf{u}_N\mathbf{u}_N'\right) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{11}\mathbf{I}_T & \sigma_{12}\mathbf{I}_T & \cdots & \sigma_{1N}\mathbf{I}_T \\ \sigma_{21}\mathbf{I}_T & \sigma_{22}\mathbf{I}_T & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \sigma_{N1}\mathbf{I}_T & \cdots & \cdots & \sigma_{NN}\mathbf{I}_T \end{pmatrix} = \boldsymbol{\Sigma} \otimes \mathbf{I}_T. \end{aligned}$$

where each block of the covariance matrix $E[\mathbf{u}^{+}(\mathbf{u}^{+})']$ is $T \times T$ by construction. Here $\otimes$ denotes a standard Kronecker product (For any two matrices $\mathbf{A}\,(m \times n)$ and $\mathbf{B}\,(p \times q)$, define as the Kronecker product, $\mathbf{A} \otimes \mathbf{B}$**,** the matrix $(mp \times nq)$ obtained by multiplying each element of $\mathbf{A}$ by $\mathbf{B}$).[3] $\boldsymbol{\Sigma}$ is non-singular covariance matrix. Notice that $\boldsymbol{\Sigma}$ will be a full matrix (i.e., it will not be simply a diagonal matrix) when the shocks hitting the returns on different risky assets are potentially simultaneously correlated, so that the

[3]For intance

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} h & i & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} ah & ai & al & bh & bi & bl \\ am & an & ao & bm & bn & bo \\ ap & aq & ar & bp & bq & br \\ ch & ci & cl & dh & di & dl \\ cm & cn & co & dm & dn & do \\ cp & cq & cr & dp & dq & dr \end{bmatrix}.$$

If you contemplate the result for a while you understand the meaning of the diffusive operation. Notice that the Kronecker product of a $N_c \times N_r$ matrix by a $M_c \times M_r$ matrix, gives a new $N_r M_r \times N_c M_c$ matrix.

off-diagonal elements of $\mathbf{\Sigma}$ are non-zero. In this case OLS will be consistent but not efficient. efficiency requires and estiamtion method that takes into account the existence of non-zero off diagonal elements in $\mathbf{\Sigma}$.

### 1.2. *Seemingly Unrelated Regressions Estimators (SURE)*

SURE is a a full-information estimators that expoit the information generated by the whole system of equations. To analyse full-information estimators, we need to introduce some new properties, related to the Kronecker product:

- $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$, whenever the matrices $\mathbf{AC}$ and $\mathbf{BD}$ are defined;
- $(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$;
- $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$, whenever the matrices $\mathbf{A}^{-1}$and $\mathbf{B}^{-1}$ are defined.

To see how SURE is derived consider the following decomposition for $\Sigma^{-1}$:

$$\Sigma^{-1} = \mathbf{HH}', \tag{2}$$

which always exists. From (2) we have:

$$\mathbf{H\Sigma H}' = I_G.$$

By pre-multiplying (**??**) by $\mathbf{H}' \otimes I_T$, we obtain:

$$(\mathbf{H}' \otimes I_T)\,\mathbf{y}^+ = (\mathbf{H}' \otimes I_T)\,\mathbf{x}^+\boldsymbol{\delta}^+ + (\mathbf{H}' \otimes I_T)\,\mathbf{u}^+, \tag{3}$$

where residuals of (3) feature a diagonal variance-covariance matrix

$$\begin{aligned} E\left((\mathbf{H}' \otimes I_T)\,\mathbf{u}^+\mathbf{u}^{+\prime}\,(\mathbf{H}' \otimes I_T)'\right) &= (\mathbf{H}' \otimes I_T)\,(\Sigma \otimes I_T)\,(\mathbf{H}' \otimes I_T)' \\ &= (\mathbf{H}'\Sigma \otimes I_T)\,(\mathbf{H} \otimes I_T) \\ &= I_G \otimes I_T = I_{GT}. \end{aligned}$$

We are now left with the following transformed model:

$$(\mathbf{H}' \otimes I_T)\,\mathbf{y}^+ = (\mathbf{H}' \otimes I_T)\,\mathbf{x}^+\boldsymbol{\delta}^+ + (\mathbf{H}' \otimes I_T)\,\mathbf{u}^+, \tag{4}$$

$$\mathbf{y}^* = \mathbf{x}^*\boldsymbol{\delta}^* + \mathbf{u}^*, \tag{5}$$

$$E\,(\mathbf{u}^*) = 0,\ E\,(\mathbf{u}^*\mathbf{u}^{*\prime}) = I_{GT}, \tag{6}$$

in which the variance-covariance matrix is diagonal, however, moreover :

$$p\ \lim \frac{1}{T}\mathbf{x}^{*\prime}\mathbf{u}^{*} = 0.$$

Using these properties we can derive the following estimator:

$$\begin{aligned}
\widehat{\boldsymbol{\delta}}^{+} &= \left(\mathbf{x}^{*\prime}\mathbf{x}^{*}\right)^{-1}\mathbf{x}^{*\prime}\mathbf{y}^{*} \\
&\left(\mathbf{x}^{+\prime}\left(\Sigma^{-1}\otimes\mathbf{I}_{T}\right)\mathbf{x}^{+}\right)^{-1}\mathbf{x}^{+\prime}\left(\Sigma^{-1}\otimes\mathbf{I}_{T}\right)\mathbf{y}^{+},
\end{aligned}$$

which is known as the seemingly unrelated regression equations (SURE) or Zellner's estimator.

An interesting specific case of the SURE estimator is obtained when each equation of the system contains the same set of regressors:

$$\begin{aligned}
\mathbf{x}^{+} &= \begin{pmatrix} \mathbf{x}_1 & 0 & 0. & 0 \\ 0 & \mathbf{x}_2 & . & . \\ 0 & . & . & . \\ 0 & . & . & \mathbf{x}_G \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{x} & 0 & 0. & 0 \\ 0 & \mathbf{x} & . & . \\ 0 & . & . & . \\ 0 & . & . & \mathbf{x} \end{pmatrix} = I\otimes\mathbf{x}.
\end{aligned}$$

By substituting for $\mathbf{x}^{+}$ in the expression for the Zellner estimator, we obtain:

$$\begin{aligned}
\widehat{\boldsymbol{\delta}}_1^{+} &= \left((I\otimes\mathbf{x})'\left(\Sigma^{-1}\otimes\mathbf{I}_{T}\right)(I\otimes\mathbf{x})\right)^{-1}(I\otimes\mathbf{x})'\left(\Sigma^{-1}\otimes\mathbf{I}_{T}\right)\mathbf{y}^{+} \\
&= \left(I\Sigma^{-1}I\otimes\mathbf{x}'I_{T}\mathbf{x}\right)^{-1}\left(I\Sigma^{-1}\otimes\mathbf{x}'\mathbf{I}_{T}\right)\mathbf{y}^{+} \\
&= \left(\Sigma\otimes(\mathbf{x}'\mathbf{x})^{-1}\right)\left(\Sigma^{-1}\otimes\mathbf{x}'\right)\mathbf{y}^{+} \\
&= \left(I_G\otimes(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\right)\mathbf{y}^{+},
\end{aligned}$$

which gives a compact representation of the OLS estimators applied equation by equation.

### 1.3. *CER model representation*

The CEr model in which $E[\mathbf{r}_{t+1}|\mathcal{I}_t] = E[\mathbf{r}_{t+1}] = \boldsymbol{\mu}$ implies that each equation contains only the one common regressors: a vector of ones. In this case we have for the i-th return:

:

$$\mathbf{y}_i = \mathbf{e}_T\delta_i + \mathbf{u}_i,$$

where

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{bmatrix}, \quad \mathbf{X}_i = \mathbf{e}_T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The OLS/SURE estimates of the relevant parameters are then simply

$$\hat{\delta}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it} = \bar{r}_i \quad \hat{\sigma}_{11} = \hat{\sigma}_1^2 = \frac{1}{T}\sum_{t=1}^{T} \left(r_{it} - \bar{r}_i\right)^2,$$

which are sample mean and sample variance.

## 2. A Static Asset Allocation Problem with Constant Expected Returns

Consider the case of an investor who believes returns have constant first and second moments. adopts a buy and hold portfolio strategy for a single period of any fixed length (the length is not a decision variable in the asset allocation) from time $t$ to time $T$. Let's denote with $\mathbf{r}$ the random vector of linear total returns from time $t$ to time $T$ from a given menu of $N$ risky assets for interval $[t, T]$, $\mathbf{r} \sim \mathcal{D}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.[4] The investor can also invest at time $t$ in a security the price of which at $T$ is known at $t$ (typically, a non-defaultable Aaa bond), called risk-free security. Let $r^f$ be the discretely compounded non-random return from this investment over every single period. Short sales are admitted without any constraints. For simplicity, we ignore transaction costs and any other frictions.

The investor's strategy is to invest in the riskfree bond and in the $N$ risky assets (stocks) at time $t$ and then liquidate the investment at time $T$. The relative weights invested in each of the risky assets are in collected in the column vector $\mathbf{w}$, while $(1 - \mathbf{w}'\mathbf{e}_N)$ is the relative amount invested in the riskfree security ($\mathbf{e}_N$ is a $N \times 1$ column vector of ones). Given a degree of risk aversion $\lambda$, a standard *mean-variance* description of this allocation problem is the following:

$$\max_{\mathbf{w}} \left(1 - \mathbf{w}'\mathbf{e}\right) r^f + \mathbf{w}'\boldsymbol{\mu} - \frac{1}{2}\lambda(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w})$$

where $E[\mathbf{r}] = \left(1 - \mathbf{w}'\mathbf{e}\right) r^f + \mathbf{w}'\boldsymbol{\mu} = r^f + \mathbf{w}'(\boldsymbol{\mu} - r^f\mathbf{e})$ and $Var[\mathbf{r}] = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$.[5] In this mean-variance setup the distribution of returns is fully described by the first two moments and the minimization of the variance of the portfolio implies that by ruling out big losses also

[4]Notice that in $\mathcal{D}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathcal{D}$ is not necessarily multivariate normal.

[5]The presence of the $1/2$ coefficient in the objective function simplies the problem and has no material effects on the qualitative findings.

big gains are ruled out. The solution of this problems determines the portfolio weights in terms of the preferences of the investor, as capture by the parameter $\lambda$, and the (known) mean and the covariance matrix describing the joint distribution of returns. Because this is an unconstrained convex problem, the first-order conditions (FOCs) are necessary and sufficient and define the following system of $N$ linear equations in $N$ unknowns, the portfolio weights $\mathbf{w} \in \mathcal{R}^N$:

$$(\boldsymbol{\mu} - r^f \mathbf{e}) - \lambda \boldsymbol{\Sigma} \mathbf{w} = \mathbf{0}.$$

Solving the FOCs yields:

$$\hat{\mathbf{w}} = \frac{1}{\lambda} \boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - r^f \mathbf{e} \right), \tag{7}$$

where $\left( \boldsymbol{\mu} - r^f \mathbf{e} \right)$ defines the vector of risk premia for the $N$ risky assets. (7) defines the solution to a standard mean-variance portfolio program and it is one of the most crucial and commonly used results in all of financial economics. Of course, in order to make this approach to portfolio allocation operational, knowledge of $\lambda$ needs to be paired with estimates (better, forecasts of future values) of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ (or $\boldsymbol{\mu} - r^f \mathbf{e}$, when more convenient or appropriate).

Consider now the special case in which $\hat{\mathbf{w}}' \mathbf{e} = 1$, that is no investment in the riskfree bond is allowed. The optimal portfolio in this case is the famous *tangency portfolio*:

$$\mathbf{e}' \hat{\mathbf{w}} = \frac{1}{\lambda} \mathbf{e}' \boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - r^f \mathbf{e} \right) = 1 \Longrightarrow \lambda = \mathbf{e}' \boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - r^f \mathbf{e} \right)$$

so that (7) becomes in this case

$$\hat{\mathbf{w}}^T = \frac{\boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - r^f \mathbf{e} \right)}{\mathbf{e}' \boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - r^f \mathbf{e} \right)}, \tag{8}$$

where the $T$ in $\hat{\mathbf{w}}^T$ stands for tangency. Several comments are in order

(1) The weights in the tangency portfolio do not depend on the risk aversion parameter $\lambda$.Individual preferences only influence the allocation between the risk free security and the risky portfolio but do not influence the allocation among different risky assets .

(2) Given that the optimal risky portfolio is uniquely determined, the tangency portfolio must then coincide with the market portfolio. Agents maximize their utility by taking a linear combination of the market portfolio and and the risk-free securities. Note that in this case we can express the return on any portfolio in the following way:

$$\begin{aligned} r^p &= (1 - \beta) r^f + \beta r^M \\ r^p - r^f &= \beta \left( r^p - r^M \right) \end{aligned}$$

and the CAPM holds.

(3) Efficient portfolios are those with the highest expected return for a given level of risk.If we summarize the expected return-risk (mean-variance) properties of the feasible portfolios in a plot with portfolio expected return, $\boldsymbol{\mu}^P$, on the vertical axis and portfolio standard-deviation, $\sigma^P$, on the horizontal axis, then all efficient portfolios can be represented as points in the space $\left(\sigma^P, \boldsymbol{\mu}^P\right)$ and the efficient frontier is the line that connects all these points. given the properties of the tangency portfolios, weights for all portfolios on the efficient frontier are obtained by inputing different values for the risk-free rate in (8)

(4) If the CER model is adopted, then the optimal asset allocation problem and the computation of the efficient frontier are easily implemented by using sample mean and sample variance covaraince matrix of returns as inputs for deriving the relevant frontiers and optimal portfolios. Notes that in this context the horizon a which returns are considered does not matter as the relation between one-period retruns and multi-period returns in the CER implies that the tangecy portfolios and the efficient frontier do not depend on the horizon at which returns are defined.

## 3. **What Happens in Practice ?**

What happens when asset allocation based on the CER model is implemented on the data?

First, estimates/forecasts of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are not constant over time: think of an investor who as avalaible a sample from time $\mathrm{t}_0$ to time $\mathrm{t}_T$ to decide optimal asset allcation over period the $T+1...T+k$. The estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ based on the sample $\mathrm{t}_0$ ...$\mathrm{t}_T$ are usually very different from those based on the sample $T+1...T+k$ and optimal asset allocation ex-ante does not coincide with the optimal asset allocation ex-post. Moreover, the CER approach to portfolio allocation can lead to dramatic swings in optimal portfolio weights for small changes in investment views and conditions, as given by the estimates/forecasts of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. There is a simple reason for these common findings: too much sampling error in the estimation of the vector of expected returns and, due to this, an asset allocation which is idiosyncratic to the specific estimation sample. This result is easily understood by using regression analysis to obtain a confidence interval on the estimates of the mean returns obtained within the simple econometric model underlying the CER: the standard error associated to the OLS estimate $\hat{\delta}_i = \bar{r}_i$ is typically large. The presence of estimation error not only leads to high sensibility of optimal weights on input parameters, but also can generate so called *financial irrelevant* portfolios, namely, financial portfolios which are mostly concentrated on few assets. The estimation error increases as either the sample size decreases for a given portfolio size or the size of the portfolio increases for a given amount of historical information. There are several ways to mititgate the negative effects of the estimation error in a mean-variance framework. We shall consider two well-known alternatives widely adopted in the industry: the first is to use methods that keep the simplest possible estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ but fully recognize that the resulting estimates are simply realizations of sample estimators that may imply considerable *parameter* (also called estimation) *uncertainty*; the second approach consists of allowing the investor to have different views from those that will lead to hold the market portfolio and provide a method to derive weights in the optimal portfolio determined by an optimal combination of weights in the market portfolio and weights reflecting the the views specific to the individual investor.

Turning to the cross section of assets, remember that the solution of the static asset allocation problem implies that for each portfolio (including those portfolio made of a single asset) we have:

$$E\left(r^i - r^f\right) = \beta_i E\left(r^M - r^f\right)$$

So the heterogeneity in excess returns to different assets can be explained only the the different exposure to a single risk factor, the market excess returns. Given a sample of observations on $r_t^i$, $r_t^f$ and $r_t^M$, then $\hat{\beta}_i$can be estimated by OLS regression over the time series of returns, then the following second-pass equations can be estimated over the cross-section of returns:

$$\overline{r_i} = \gamma_0 + \gamma_1 \hat{\beta}_i + \epsilon_i \quad i = 1, \ldots, m \tag{9}$$

(9) asserts that $\overline{r_i}$ is a linear function of $\hat{\beta}_i$ plus an error term (we need to insert an error term as the relevant betas have been estimated in the first stage).

If CAPM is valid, then $\gamma_0$ and $\gamma_1$ should satisfy

$$\gamma_0 = 0 \text{ and } \gamma_1 = \overline{r_M},$$

where $\overline{r_M}$ is the mean market excess return. When the model is estimated with appropriate methods[6], the restriction $\gamma_0 = 0$ is typically rejected (Fama-French(1992)). The different

[6]Appropriate methods must take consider that , as returns are afffected by common shocks, the variance-covariance matrix of the residuals in (9) is not diagonal (see Fama-MacBeth(1973) ).

exposure to a single factor model cannot explain the observed cross-sectional behaviour of returns. This evidence paved the way to the estimation of multi-factor models of returns. Fama-French(1993) introduced a three-factor model based on the integration ofthe CAPM with a "small-minus-big" market value (SMB) and "high-minus-low" book-to-market ratio (HML), are based on portfolios of stocks sorted according to the two characteristics of interest. Each factor is equivalent to a zero-cost arbitrage portfolio that takes a long position in high book-to-market (small-size) stocks and finances this with a short position in low book-to-market (large-size) stocks. Jegadeesh and Titman(1993) discovered the importance of a further additional factor in explaining excess returns: momentum. An investment strategy that buys stocks that have performed well and sells stocks that have performed poorly over the past 3- to 12-month period generates significant excess returns over the following year. It is interesting to note that augmenting the CAPM with SMB and HML does not challenge per se the CER model, which still hold as valid if the constant expected return model can be applied to the two aditional factors. However, momentum provides direct evidence against the CER model as it indicates that the conditional expections of future returns is not constant. We shall immediately discuss the resampled optimal-mean variance portfolio and the Black-Litterman approach, while we shall defer the discussion of momentum related strategies to the next chapters.

### 3.1. *The resampled optimal mean-variance portfolio*

A first possibility to deal with estimation uncertainty is to implement *bootstrap methods* to derive the optimal portfolio allocation. Consider the estimation of a simple multivariate model, in which the only regressor is a constant for the returns $r_t^i$ on $N$ assets, $i = 1, 2,$ ..., $N$:

$$\begin{aligned}
r_{1t} &= \hat{\mu}_1 + \hat{u}_{1t} \\
r_{2t} &= \hat{\mu}_2 + \hat{u}_{2t} \\
&\dots \\
r_{Nt} &= \hat{\mu}_N + \hat{u}_{Nt}
\end{aligned}$$

$$\begin{bmatrix} \hat{u}_{1t} & \hat{u}_{2t} & \dots & \hat{u}_{Nt} \end{bmatrix}' \sim \mathcal{N}\left(\mathbf{0}, \hat{\mathbf{\Sigma}}\right).$$

Notice that the fact that each observed return can always be decomposed as $r_{it} = \hat{\mu}_i + \hat{u}_{it}$ obtains by definition. Moreover, standard least squares algebra shows that in this case $E[\hat{\mathbf{u}}_t] = \mathbf{0}$ as claimed. The idea of resampling the mean-variance portfolio solution is not to stop at replacing $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ with $\hat{\boldsymbol{\mu}} \equiv [\hat{\mu}_1\ \hat{\mu}_2\ \dots\ \hat{\mu}_N]'$ and $\hat{\mathbf{\Sigma}}$ in the classical formula

$\hat{\mathbf{w}} = \lambda^{-1}\hat{\boldsymbol{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}} - r^f\mathbf{e}\right)$ but to implement the following algorithm. Collect of the residuals from estimation in the following $T \times N$ matrix:

$$\hat{\mathbf{U}} \equiv \begin{bmatrix} \hat{u}_{11} & \hat{u}_{21} & \dots & \hat{u}_{N1} \\ \hat{u}_{12} & \hat{u}_{22} & \dots & \hat{u}_{N2} \\ \vdots & \dots & \ddots & \vdots \\ \hat{u}_{1T} & \hat{u}_{2T} & \dots & \hat{u}_{NT} \end{bmatrix}.$$

At this point, draw a new sample of size $T$ of residuals by extracting randomly $T$ rows from $\hat{\mathbf{U}}$. Extracting the rows at random and with replacement is important because it ensures that the covariance structure of the residuals is preserved. Given these new, re-sampled residuals collected in a vector $\hat{\mathbf{u}}_t^1$ $(t = 1, 2, \dots, T)$ and the estimates $\hat{\boldsymbol{\mu}}$, we proceed to generate a new artificial sample of returns using

$$\mathbf{r}_t^1 = \hat{\boldsymbol{\mu}} + \hat{\mathbf{u}}_t^1,$$

where the subscript "1" alludes to the fact that this represents the first iteration of the algorithm. At this point, a new OLS estimation of the model is performed on this artificial data, obtaining as an outcome a pair of new, bootstrapped estimates, $\hat{\boldsymbol{\mu}}^1$ and $\hat{\boldsymbol{\Sigma}}^1$ and, using the classical formula, $\hat{\mathbf{w}}^1$. At this point the algorithm is iterated a second time, re-sampling a new set of residuals $\hat{\mathbf{u}}_t^2$ $(t = 1, 2, \dots, T)$ to obtain $\hat{\boldsymbol{\mu}}^2$, $\hat{\boldsymbol{\Sigma}}^2$, and $\hat{\mathbf{w}}^2$. This algorithm is then replicated $B$ times, where $B$ is in general a large number (let's say 5,000 or 10,000 times), using the fact that at the $b$th iteration one simply draws a new sample of size $T$ of residuals by extracting randomly $T$ rows from $\hat{\mathbf{U}}$, generate a new artificial sample of returns using

$$\mathbf{r}_t^b = \hat{\boldsymbol{\mu}} + \hat{\mathbf{u}}_t^b,$$

perform OLS estimation of $\hat{\boldsymbol{\mu}}^b$ and $\hat{\boldsymbol{\Sigma}}^b$ to obtain $\hat{\mathbf{w}}^b$, for $b = 1, 2, \dots, B$.

This total number $B$ of replications of this procedure will generate $B$ optimal portfolio allocations $\{\hat{\mathbf{w}}^b\}_{b=1}^B$. At this point, the desidered vector of re-sampled, optimized portfolio weights may be represented by the average, across the $B$ bootstraps, of the weights in $\{\hat{\mathbf{w}}^b\}_{b=1}^B$:

$$\breve{\mathbf{w}}^{boot} = \frac{1}{B}\sum_{b=1}^{B}\hat{\mathbf{w}}^b.$$

This method and the resulting average portfolio allocation across bootstraps acknowledges the effects of estimation uncertainty and is generally more stable across different sample whenever the instability in the portfolio allocation is generated by estimation uncertainty rather than by a true structural break (or other forms of statistical instability such as regimes) in the distribution of the vector of risky asset returns. Using the same procedure resampled the efficient frontier can be computed together with indication of the uncertainty associated to its estimation.

### 3.2. *Black and Litterman's approach*

In the early 90s the quantitative research group at Goldman Sachs proposed a model for portfolio selection which is based on the mixed estimation approach developed in Theil and Goldberger (1961). This model, popularly known as the Black-Littermann model (BL henceforth), has become one of the most prominent portfolio allocation method to avoid the standard drawbacks of the mean-variance framework, i.e Corner solutions, portfolio instability, high sensitivity to the inputs $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (see Black and Littermann, 1990, and Black and Litterman, 1991). The basic idea is Bayesian updating of prior information. The investor updates the market view (implied by the equilibrium CAPM model) by using her own views via the Bayes rule. The economic intuition is that the investor start by holding a simple rescaled version of the market portfolio, then deviates from the market portfolio according to views on specific returns or linear combination of them. The weight attributed to each asset depends on views and their associated uncertainty. In other words, the BL optimal portfolio deviates from the market value-weighted weights if the investor's views substantially deviates from those implied by the market. The main contribution of the method it to discipline the asset manager action. A numerical specification of views and of their associated confidence is an input of the method and not an output, the output is the optimal combination. The Bayesian method proposed by Black and Littermann ensures then the most efficient implementation of the expressed views into a vector of portfolio weights.

Given the knowledge of the market capitalization and therefore of the market portfolio weights $\mathbf{w}_{mkt}$ and some estimates of the variance-covariance matrix of returns, we can use the optimal portfolio allocation condition to derive the expected returns consistent with the market capitalization:

$$\mathbf{w}_{mkt} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu}{-}r^f\mathbf{e}\right) \Longrightarrow \boldsymbol{\mu}_{mkt} = \lambda\boldsymbol{\Sigma}\mathbf{w}_{mkt} + r^f\mathbf{e}.$$

Assume now that the portfolio manager holds some (normally distributed, for simplicity) views on a subset of size $Q \leq N$ of the $N$ expected returns included in the market portfolio:

$$\mathbf{P}\boldsymbol{\mu}_r \sim \mathcal{N}_Q\left(\mathbf{v}, \boldsymbol{\Gamma}\right),$$

where $\mathcal{N}_Q$ denotes a $q$-variate multivariate normal distribution, $\boldsymbol{\mu}_r$ is a vector of $N$ expected returns, and $\mathbf{P}$ is an appropriate $Q \times N$ selection matrix that selects the subset of returns on which there are subjective views expressed by the investor. The views are expressed as a vector of mean expected returns $\mathbf{V}$ and a diagonal variance-covariance matrix $\boldsymbol{\Gamma}$, expressing the confidence on the views. Such subjective views have to be balanced against the distribution of returns implied by the market portfolio:

$$\boldsymbol{\mu}_r \sim \mathcal{N}\left(\boldsymbol{\mu}_{mkt}, \tau\boldsymbol{\Sigma}\right),$$

where $\tau$ is a scalar smaller than one (and conventionally set to 1/3 by Black and Littermann and most of the subsequent literature) to filter out of the estimated covariance matrix of returns the impact of their random variation (i.e., to take into account the effect of noise in small samples).

Black and Littermann's approach aims then at generating a value for the expected return vector $\boldsymbol{\mu}_{BL}$ by optimally combining the distribution of returns implied in the market capitalization and the subjective views of the portfolio manager. This is obtained by solving the following optimization problem:

$$\boldsymbol{\mu}_{BL} = \arg\min_{\boldsymbol{\mu}} \left(\boldsymbol{\mu} - \boldsymbol{\mu}_{mkt}\right)' \left(\tau\boldsymbol{\Sigma}\right)^{-1} \left(\boldsymbol{\mu} - \boldsymbol{\mu}_{mkt}\right) + \left(\mathbf{P}\boldsymbol{\mu} - \mathbf{v}\right)' \boldsymbol{\Gamma}^{-1} \left(\mathbf{P}\boldsymbol{\mu} - \mathbf{v}\right).$$

This is a weighted least squares problems, where the weights depend on covariance matrix. When the diagonal elements of $\Gamma$ all approach zero, that is, when there is infinite confidence in the subjective views by the investor, the problem becomes a constrained least squares problem where the relevant constraint is $\mathbf{P}\boldsymbol{\mu}_{BL} = \mathbf{v}$. On the other hand, when $\boldsymbol{\Gamma}$ has diagonal elements diverging to infinity (no confidence in the views), the solution to the problem is simply $\boldsymbol{\mu}_{BL} = \boldsymbol{\mu}_{mkt}$.

The first order conditions for the solution of the problem can be written as follows:

$$2\left(\tau\boldsymbol{\Sigma}\right)^{-1}\left(\boldsymbol{\mu}_{BL} - \boldsymbol{\mu}_{mkt}\right) + 2\mathbf{P}'\boldsymbol{\Gamma}^{-1}\left(\mathbf{P}\hat{\boldsymbol{\mu}}_{BL} - \mathbf{v}\right) = \mathbf{0}$$

from which we can derive:

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_{BL} &= \left(\left(\tau\boldsymbol{\Sigma}\right)^{-1} + \mathbf{P}'\boldsymbol{\Gamma}^{-1}\mathbf{P}\right)^{-1}\left(\left(\tau\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\mu}_{mkt} + \mathbf{P}'\boldsymbol{\Gamma}^{-1}\mathbf{v}\right) \\
&= \left[\underbrace{\left(\left(\tau\boldsymbol{\Sigma}\right)^{-1} + \mathbf{P}'\boldsymbol{\Gamma}^{-1}\mathbf{P}\right)^{-1}\left(\tau\boldsymbol{\Sigma}\right)^{-1}}_{=\boldsymbol{\Psi}}\right]\boldsymbol{\mu}_{mkt} + \left[\underbrace{\left(\left(\tau\boldsymbol{\Sigma}\right)^{-1} + \mathbf{P}'\boldsymbol{\Gamma}^{-1}\mathbf{P}\right)^{-1}\mathbf{P}'\boldsymbol{\Gamma}^{-1}}_{=\mathbf{I}_N - \boldsymbol{\Psi}}\right]\mathbf{v}
\end{aligned}$$

This expression emphasizes that $\boldsymbol{\mu}_{BL}$ is obtained by optimally combining market views ($\boldsymbol{\mu}_{mkt}$) with the investor's views ($\mathbf{v}$), through a rather complex weighting matrices given by $\mathbf{\Psi}$ and $\mathbf{I}_N - \mathbf{\Psi}$, respectively. Also note that $\boldsymbol{\mu}_{BL}$ can be equivalently written as:

$$\hat{\boldsymbol{\mu}}_{BL} = \boldsymbol{\mu}_{mkt} + \mathbf{K}\left(\mathbf{v} - \mathbf{P}\boldsymbol{\mu}_{mkt}\right) \qquad \mathbf{K} = \left(\tau\mathbf{\Sigma}\right)\mathbf{P}'\left(\mathbf{P}\tau\mathbf{\Sigma}\mathbf{P}' + \mathbf{\Gamma}\right)^{-1}.$$

At this point, given $\hat{\boldsymbol{\mu}}_{BL}$ the optimal BL portfolio weights are obtained by the usual formula:

$$\hat{\mathbf{w}}_{BL} = \frac{1}{\lambda}\hat{\mathbf{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}}_{BL} - r^f\mathbf{e}\right) \text{ or } \hat{\mathbf{w}}_{BL}^T = \frac{\mathbf{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}}_{BL} - r^f\mathbf{e}\right)}{\mathbf{e}'\mathbf{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}}_{BL} - r^f\mathbf{e}\right)}.$$

Similarly to how optimal portfolio weights in the tangency portfolio are computed, the BL efficient frontier can also be computed using $\hat{\boldsymbol{\mu}}_{BL}$ and $\hat{\mathbf{\Sigma}}$ as inputs.

## 4. Going to the Data: Asset Allocation and the CER model with MATLAB

To illustrate what happens in practice consider the case of an investor who consumes in euro and sees the German 3-month rate as the risk free. The risky assets available for portfolio allocation are German US and UK shares and the German 10-Year government bond. Data at monthly frequency over the sample 1978-2013, on aggregate stock price indexes price index, dividends, bond yields to maturity, the exchange rates and the return on the safe assets are available in the file STOCKINT2013.XLS. The following time series are saved in the successive columns of the EXCEL file

The time-series in the STOCKINT2013.XLS files

| identifier | description |
|---|---|
| BDBRYLD | GERMANY BENCHMARK BOND 10 YR (DS) - RED. YIELD |
| BDINTER3 | BD FIBOR - 3 MONTH (MTH.AVG.) |
| TOTMKBD(PI) | GERMANY-DS Market - PRICE INDEX |
| TOTMKBD(DY) | GERMANY-DS Market - DIVIDEND YIELD |
| TOTMKUS(PI) | US-DS Market - PRICE INDEX |
| TOTMKUS(DY) | US-DS Market - DIVIDEND YIELD |
| TOTMKUK(PI) | UK-DS Market - PRICE INDEX |
| TOTMKUK(DY) | UK-DS Market - DIVIDEND YIELD |
| USDOLLR | US $ TO UK £ - EXCHANGE RATE |
| EUUSBOE | US $ TO EUR - EXCHANGE RATE |

Files for running the empirical exerise should be organized as follows: a directory will contain the main programme named `main_chap4.m`, the main directory will have three sudirectories label: INPUT, OUTPUT and UTILITIES. The INPUT directory contains the data, the UTILITIES directory contains subroutines to be called within `main_chap4.m`,while the OUTPUT directory will be used to store tables and figures generated within the programme.

The organization of the directories is set-up, the data are imported and prepared to be used for different samples with the following commands:

```
close all; clear all; clc; pause(0.01), randn('seed',sum(clock)), rand('seed',sum(
warning off
TSTART = tic;
% ------------- Load folders ----------------------------------------
addpath([pwd '\Input\']);
addpath([pwd '\Utilities\']);
addpath([pwd '\Output\']);
% ------------- Import data -----------------------------------------
[data,textdata,raw] = xlsread('STOCKINT2013.xls','Monthly');
date=datenum(textdata(4:end,1),'dd/mm/yyyy');
```

Next all the relevant data transformation are implemented

```
% BUILDING RETURNS
% ------------- German risk-free please note dating ---------
GER.RiskFree = log(1+(lag(data(:,2))/(100*12)));
GER.RiskFree(1,1)=NaN;
% German Long term bond YTM and period holding returns
GER.LTBond.Yield = (log(1+(data(:,1))/(100*12)));
GER.LTBond.Duration = 12*((1-(1+(data(:,1)/(100))).^(-10)))./(1-(1+(data(:,1)/(100
GER.LTBond.Ret = (lag(GER.LTBond.Duration,1,NaN).*lag(GER.LTBond.Yield,1,NaN)-(lag
GER.LTBond.ExRet = GER.LTBond.Ret - GER.RiskFree;
% ------------- German Stocks Monthly excess returns ------------------
GER.Stock.DY = data(:,4)/(100*12); % Monthly dividend yield
GER.Stock.dy = log(GER.Stock.DY); % Monthly log dividend yield
GER.Stock.Index = data(:,3); % Monthly stock index
GER.Stock.Ret = log(GER.Stock.Index./lag(GER.Stock.Index,1,NaN)+GER.Stock.DY);
GER.Stock.ExRet = GER.Stock.Ret - GER.RiskFree;
% ------------- Forex Exchanges ----------------------------------------
FX.EUvsUS=data(:,10);
FX.USvsUK=data(:,9);
FX.EUvsUK=FX.EUvsUS.*FX.USvsUK;
FX.r_EUvsUS = log(data(:,10)./lag(data(:,10),1,NaN));
FX.r_USvsUK = log((data(:,9))./lag(data(:,9),1,NaN));
```

```
FX.r_EUvsUK = FX.r_EUvsUS + FX.r_USvsUK;
% -------------- US Stocks Monthly excess Returns -------------------------
US.Stock.DY = data(:,6)/(100*12); % US dividend yield
US.Stock.dy = log(data(:,6)); % US log dividend yield
US.Stock.Index = data(:,5); % US stock index
US.Stock.Ret = log(US.Stock.Index./lag(US.Stock.Index,1,NaN)+US.Stock.DY);

% Returns in local currency
US.Stock.Ret = US.Stock.Ret + FX.r_EUvsUS; % Returns in euros (german perspective)

US.Stock.ExRet = US.Stock.Ret - GER.RiskFree; % Excess returns in euros
% ------------- UK Stocks Monthly Excess Returns --------------------------
UK.Stock.DY = data(:,8)/(100*12); % UK dividend yield
UK.Stock.dy = log(data(:,8)); % UK log dividend yield
UK.Stock.Index = data(:,7); % UK stock index
UK.Stock.Ret = log(UK.Stock.Index./lag(UK.Stock.Index,1,NaN)+UK.Stock.DY);

UK.Stock.Ret = UK.Stock.Ret + FX.r_EUvsUK; % Returns in euros
UK.Stock.ExRet = UK.Stock.Ret - GER.RiskFree; % Excess returns in euros
```

Let us now imagine a situation in which, given data available over the period 1978-2003 the investor has to decide an asset allocation between the four risky assets (the tangency portfolio) to be maintained over the period 2004-2007.

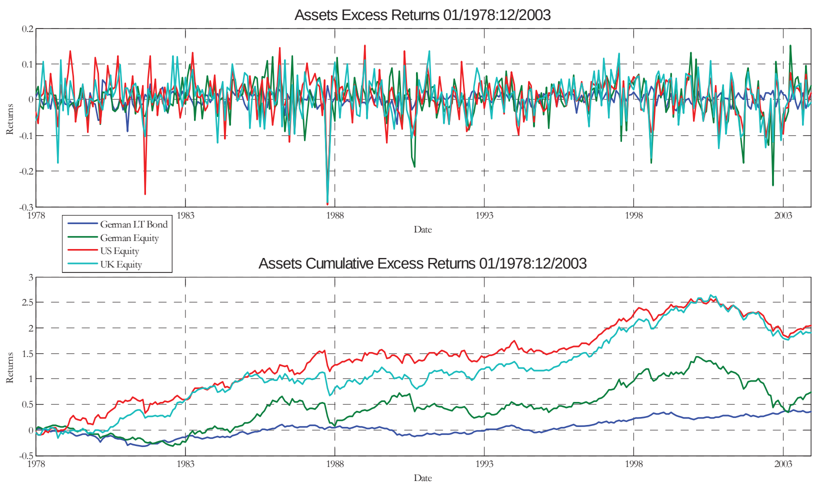
### 4.1. *Exploratory Data Analysis*

As a first step an exploratory data analysis is conducted by visualizing excess returns and cumulative excess returns over the period 1978-2003.

```
% % ------------------------------------------------------------------------
% ------------------- Exploratory Data Analysis -----------------
% -------------------------------------------------------------------------
s_start = '31/01/1978';
s_end = '31/12/2003';
date_find = datenum([s_start; s_end],'dd/mm/yyyy');
```

```
ss = datefind(date_find(1,1),date);
se = datefind(date_find(2,1),date);
T=ss-se+1;
%---------- Historical Excess Returns Matrix ---------------------------
R = [GER.LTBond.ExRet(ss:se) GER.Stock.ExRet(ss:se) UK.Stock.ExRet(ss:se)
US.Stock.ExRet(ss:se)];
Perf = cumsum(R);
% Plot returns and cumulative returns performances in figure(1)
run Figure001
```

Note that the programme calls a routine named `Figure001.m` and saved in UTILITIES to produce the relevant graphs. The following figure is generated:




### 4.2. *Optimal Static Asset Allocation*

Next optimal static asset allocation is implemented using the data available. In particular the Tangency portfolio is constructed alongwith the efficient frontier and a graph illustrating the solution is obtained.

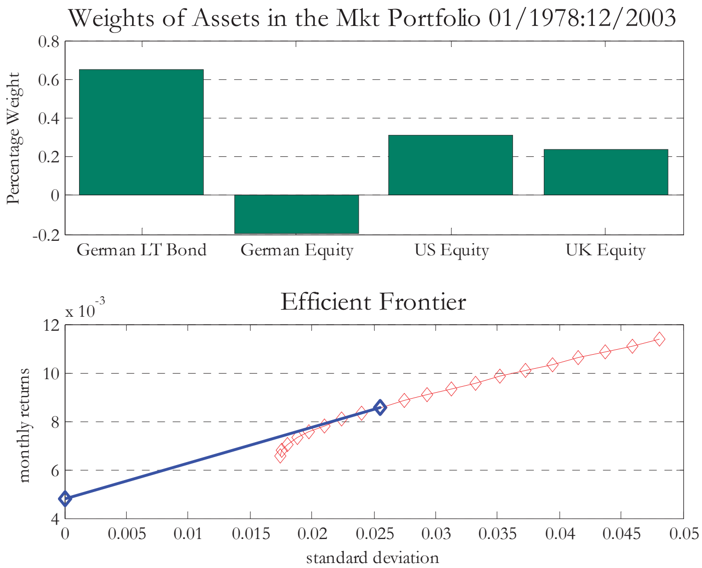
```
% % ------------------------------------------------------------------
% ------------- Compute the weights and Efficient Frontier using information
upto 2003
% ------------------------------------------------------------------
% Compute unconditional means and VarCov
```

```
muR = mean(R)';
SigmaR = (T/(T-1))*cov(R); %compute robust variance covariance matrix
% Optimal weights on the tangency portfolio
WeightsTP = ((SigmaR^(-1))*muR)./(ones(size(R,2),1)'*(SigmaR^(-1)*muR));
%Compute the efficient frontier and plot it
NumPortf = 20; % number of efficient portfolios to be computed
 % we do not compute all efficient portfolios in the
 % efficient frontier (there is a continuum of them)
 % we compute only a subset of them, equally spaced between
 % the minimum variance portfolio and the portfolio with the
 % highest return.
mu=mean(R)+ mean(GER.RiskFree(ss:se)); %generate mean returns
% generate mean variance efficient frontier
[WeightsMV, MuMeanVariance, StdMeanVariance] = effront(mu, SigmaR, NumPortf);

%mean variance for risk fre and tangency portfolio
Mu_bm = zeros(2, 1);
SD_bm = zeros(2, 1);
Mu_bm(1,1) = mean(GER.RiskFree(ss:se));
SD_bm(1,1)=0;
Mu_bm(2,1)=mu*WeightsTP;
SD_bm(2,1)=sqrt(WeightsTP'*SigmaR*WeightsTP);
% Plot weights in the tangency portfolio and the efficient frontier
run Figure002
```

Note that two external routine are called: `effront.m` generates the efficient frontier and `figure2.m` plots the efficient frontier and the tangency portfolio.

Running figure2.m at the end of this section of the programme, the following figure is obtained:




### 4.3. *Testing the model*

Given the availability ex-post of the data over the period 2004-2007 the model can be testd by assessing its ex-post performance, and by comparing the asset allocation optimal ex-ante with that optimal ex-post.

The following section of the programme implemets all these steps:

```
 % % -----------------------------------------------------------------------
% Compute the cumulative performances out-of-sample
% ---------------------------------------------------------------------
p_start = '30/01/2004';
p_end = '31/12/2007';
date_find=datenum([p_start; p_end],'dd/mm/yyyy');
ps=datefind(date_find(1,1),date);
pe=datefind(date_find(2,1),date);
n = pe-ps+1;
R2 = [GER.LTBond.ExRet(ps:pe) GER.Stock.ExRet(ps:pe) UK.Stock.ExRet(ps:pe)
US.Stock.ExRet(ps:pe)];
PerfR2 = cumsum(R2);
Rport = R2*WeightsTP;
PerfPort1 = cumsum(Rport);
% Plot the cumulative performances out-of-sample
```