PV Models for Consumption and asset prices

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1 Consumption and Asset Pricing puzzles

Consider the case of a representative consumer, who allocates his wealth among a number of i risky assets, with a period return of R^i and a riskfree asset with a return of R^f.

$$\sum_{j=0}^{\infty} E_t \left[U\left(C_{t+j}\right) \left(\frac{1}{1+\delta}\right)^j \right] \tag{1}$$

$$\begin{aligned}
\sum P_{t+j} A_{i,t+j} &= \left(1 + R^f_{t+j-1,t+j}\right) A_{1,t+j-1} P_{t+j-1} + & (2) \\
& \sum \left(1 + R^i_{t+j-1,t+j}\right) A_{i+1,t+j-1} P_{t+j-1} + & (3) \\
& + P_{t+j} Y_{t+j} - P_{t+j} C_{t+j} & (4)
\end{aligned}$$

The optimization problem is :

$$\underset{C_{t+j} A_{i,t+j}}{max} \sum_{j=0}^{\infty} E_t L_{t+j}$$

$$L_{t+j} = \left(\frac{1}{1+\delta}\right)^j \left[U\left(C_{t+j}\right) - \lambda_{t+j} V_{t+j} \right]$$

$$\begin{aligned}
V_{t+j} &= \sum A_{i,t+j} - \left(1 + R^f_{t+j-1,t+j}\right) A_{1,t+j-1} \frac{P_{t+j-1}}{P_{t+j}} - & (5) \\
& - \sum \sum \left(1 + R^i_{t+j-1,t+j}\right) A_{i+1,t+j-1} \frac{P_{t+j-1}}{P_{t+j}} - Y_{t+j} + C_{t+j} & (6)
\end{aligned}$$

FOC are

$$E_t \frac{\partial L_{t+j}}{\partial C_{t+j}} = E_t \left[\left(\frac{1}{1+\delta}\right)^j U'\left(C_{t+j}\right) - \left(\frac{1}{1+\delta}\right)^j \lambda_{t+j} \right] = 0 \tag{7}$$

$$E_t\frac{\partial L_{t+j}}{\partial A_{1,t+j}}=E_t\left[\left(\frac{1}{1+\delta}\right)^j\lambda_{t+j}-\left(\frac{1}{1+\delta}\right)^{j+1}\lambda_{t+j+1}\left(1+R^f_{t+j,t+j+1}\right)\frac{P_{t+j}}{P_{t+j+1}}\right]=0 \tag{8}$$

$$E_t\frac{\partial L_{t+j}}{\partial A_{i,t+j}}=E_t\left[\left(\frac{1}{1+\delta}\right)^j\lambda_{t+j}-\left(\frac{1}{1+\delta}\right)^{j+1}\lambda_{t+j+1}\left(1+R^i_{t+j,t+j+1}\right)\frac{P_{t+j}}{P_{t+j+1}}\right]=0 \tag{9}$$

Substituting the multipliers out :

$$E_tU'\left(C_{t+j}\right)=E_t\left[\frac{1}{1+\delta}U'\left(C_{t+j+1}\right)\left(1+R^F_{t+j,t+j+1}\right)\frac{P_{t+j}}{P_{t+j+1}}\right]$$
$$E_tU'\left(C_{t+j}\right)=E_t\left[\frac{1}{1+\delta}U'\left(C_{t+j+1}\right)\left(1+R^i_{t+j,t+j+1}\right)\frac{P_{t+j}}{P_{t+j+1}}\right]$$

And therefore :

$$E_t\left[\frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)}\left(\frac{P_t}{P_{t+1}}R^f_{t,t+1}\right)\right]=E_t\left[\frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)}\left(\frac{P_t}{P_{t+1}}R^i_{t,t+1}\right)\right]$$

Given that Cov(X,Y)=E(XY)-E(X)E(Y), we have in terms of real rates

$$E_t\left(r^i_{t,t+1}-r^f_{t,t+1}\right)=-\frac{COV\left[\left(\frac{U'(C_{t+1})}{U'(C_t)}\right),r^i_{t,t+1}-r^f_{t,t+1}\right]}{E_t\left[\frac{U'(C_{t+1})}{U'(C_t)}\right]} \tag{10}$$

given that when x and y are distribute as a multivariate normal we have:

$$Cov\left(X,Y\right)=Cov\left(X,\log Y\right)E\left(Y\right)$$

we have:

$$E_t\left(r^i_{t,t+1}-r^f_{t,t+1}\right)=-COV\left(\log\left(\frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)}\right),\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right)$$

which, by using a CRRA $\left(U(C_t)=\frac{C_t^{1-\gamma}}{1-\gamma}\right)$specification, we can finally write as:

$$E_t\left(\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right)=\gamma COV\left(\Delta\ln\left(C_{t+1}\right),\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right)$$

Bringing this relation to the data generates a number of puzzles:

1) **The Equity Premium Puzzle:**

Using ex-post data as a measure of expectations generates a γ of about 38, with a lower bound of 33

$$\begin{aligned} E_t\left(R_{t+1}^e\right) &= \gamma COV\left(\Delta \ln\left(C_{t+1}\right), \left(r_{t,t+1}^i - r_{t,t+1}^f\right)\right) \\ &= \gamma\rho(\Delta c, R_{t+1}^e)\sigma\left(\Delta c\right)\sigma\left(R_{t+1}^e\right) \\ E_t\left(R_{t+1}^e\right) &= 0.08, \sigma\left(R_{t+1}^e\right) = 0.16, \sigma\left(\Delta c\right) = 0.015 \end{aligned}$$

2) **the risk-free rate puzzle**

Even **if we accept high risk aversion** using the CRRA Utility function and the fact that when a random variable X is conditionally lognormally distributed we have:

$$\log E_t\left(X\right) = E_t\left(\log\left(X\right)\right) + \frac{1}{2}Var_t\left(\log\left(X\right)\right)$$

from the first order condition of the optimization problem we have :

$$0 = -\log\left(1+\delta\right) - \gamma E_t\left(\Delta c_{t+1}\right) + E_t \log\left(1 + r_{t,t+1}^i\right) + \frac{1}{2}\left(\sigma_i^2 + \gamma^2\sigma^2\left(\Delta c_{t+1}\right) - 2\gamma\sigma_{ic}\right)$$

from which we can write for the risk-free asset :

$$r_{t,t+1}^f \simeq \delta + \gamma E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\gamma^2\sigma^2\left(\Delta c_{t+1}\right)$$

which predict that consumption will respond very little to the risk free rate and it is very hard to reconcile with the data, given that if $\gamma = 33, E_t\left(\Delta c_{t+1}\right) = 0.01, \sigma^2\left(\Delta c_{t+1}\right) = 0.015^2$, we have

$$\begin{aligned} r_{t,t+1}^i &= \delta + 33 * 0.01 - \frac{1}{2}33^2 0.015^2 \\ r_{t,t+1}^i &= \delta + 0.33 - 0.13 \end{aligned}$$

and **we need a negative discount rate to generate positive and plausible values for the policy rate**.

2 Hansen Jagannathan Bounds

The FOC can be rewritten as:

$$\begin{aligned} 1 &= E_t\left[\frac{1}{1+\delta}\frac{U'(C_{t+1})}{U'(C_t)}\left(1+R^i_{t,t+1}\right)\frac{P_t}{P_{t+1}}\right] \\ 1 &= E_t\left[M_{t+1}r^i_{t,t+1}\right] \end{aligned}$$

In general given a candidate M_{t+1}, its validity can be tested by looking at the sample counterpart of the moment restrictions

$$E(M\mathbf{x}-\mathbf{q})=0$$

where $\mathbf{x}$ is a vector of gross-real returns and $\mathbf{q}$ is a vector of ones.

Consider now the regression of a unobservable discount factor y onto a constant and the vector of $\mathbf{x}$ of returns observed by the econometrician

$$y=\mathbf{a}+\mathbf{x}'\mathbf{b}+\mathbf{e}$$

The standard OLS formula gives

$$\mathbf{b}=\left[cov\left(\mathbf{x},\mathbf{x}'\right)^{-1}\right]cov\left(\mathbf{x},y\right)$$

of course this regression cannot be run in practice, givent that y is not observed.

However, the moment restrictions imply that

$$cov\left(\mathbf{x},y\right)=\mathbf{q}-\mathbf{E}\left(\mathbf{x}\right)\mathbf{E}\left(y\right)$$

and therefore we have

$$\mathbf{b}=\left[cov\left(\mathbf{x},\mathbf{x}'\right)^{-1}\right]\left[\mathbf{q}-\mathbf{E}\left(\mathbf{x}\right)\mathbf{E}\left(y\right)\right]$$

given the properties of the regression model we have then

$$var\left(y\right)=var\left(\mathbf{x}'\mathbf{b}\right)+var\left(\mathbf{e}\right)$$

And $\left(var\left(\mathbf{x}'\mathbf{b}\right)\right)^{0.5}$ gives a lower bound on the standard deviation of y.

Consider for example the case in which the boundary is computed using the excess returns of stocks over Treasury Bills. In this case we have:

$$\begin{aligned} E_t\left[M_{t+1}\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right] &= 0 \\ COV\left[M_{t+1}\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right] &= -E_t\left(r^i_{t,t+1}-r^f_{t,t+1}\right)E_t\left[M_{t+1}\right] \\ \mathbf{b} &= \frac{-E_t\left(r^i_{t,t+1}-r^f_{t,t+1}\right)E_t\left[M_{t+1}\right]}{\sigma^2_t\left[\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right]} \end{aligned}$$

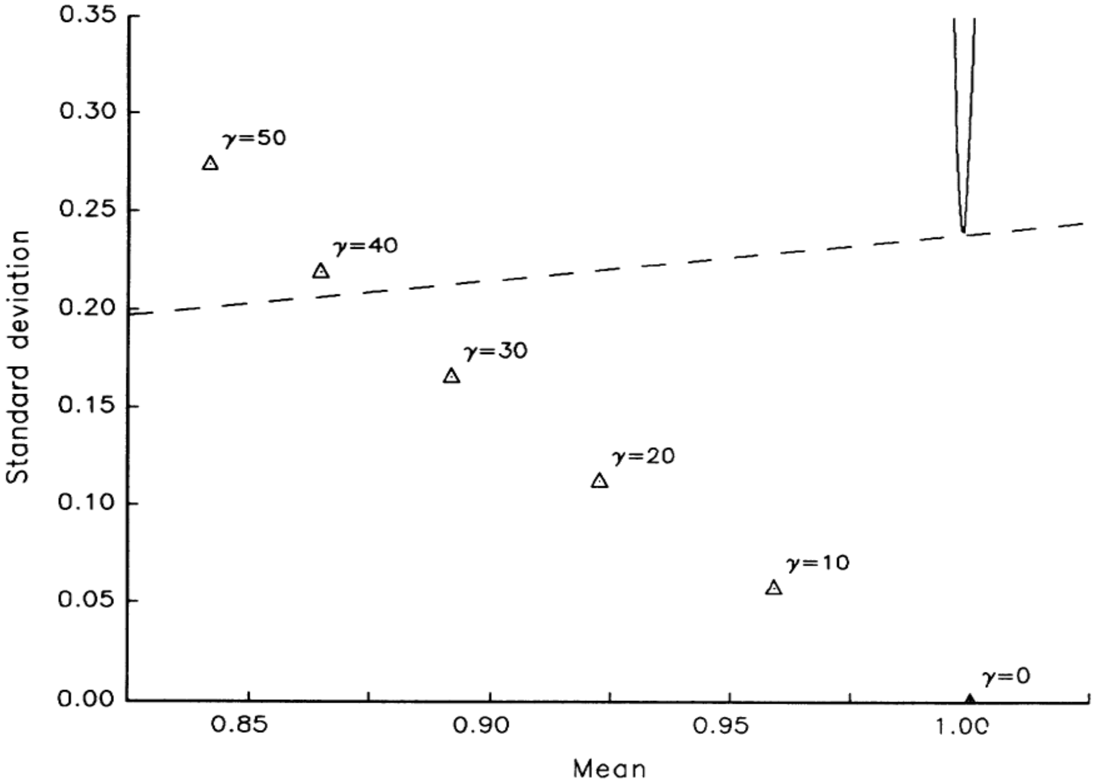
$$
\begin{aligned}
\left(var\left(\mathbf{x}'\mathbf{b}\right)\right)^{0.5} &= \frac{E_t\left[\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right]}{\sigma_t\left[\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right]}E_t\left[M_{t+1}\right] \quad if\ E_t\left(r^i_{t,t+1}-r^f_{t,t+1}\right)E_t\left[M_{t+1}\right]>0 \\
\sigma_t\left(M_{t+1}\right) &\geq \frac{E_t\left[\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right]}{\sigma_t\left[\left(r^i_{t,t+1}-r^f_{t,t+1}\right)\right]}E_t\left[M_{t+1}\right]
\end{aligned}
$$

the Sharpe-ratio for the US stock market is about one-half implying a minimum annualized standard deviation of 50 per cent for the SDF, which is a rather high number with a variable that should fluctuate close to one with a lower bound of zero.

Alternative the volatility in the stochastic discount factor implied by two returns(stocks and three-month bills) can be computed as follows:

$$
\begin{aligned}
cov\left(\mathbf{x},y\right) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} E_t\left[\left(r^s_{t,t+1}\right)\right]\mathbf{E}\left(y\right) \\ E_t\left[\left(r^{tb}_{t,t+1}\right)\right]\mathbf{E}\left(y\right) \end{bmatrix} \\
\mathbf{b} &= \begin{bmatrix} \sigma^2_s & \sigma_{s,tb} \\ \sigma_{s,tb} & \sigma^2_{tb} \end{bmatrix}^{-1} cov\left(\mathbf{x},y\right) \\
\left(var\left(\mathbf{x}'\mathbf{b}\right)\right)^{0.5} &= cov\left(\mathbf{x},y\right)' \begin{bmatrix} \sigma^2_s & \sigma_{s,tb} \\ \sigma_{s,tb} & \sigma^2_{tb} \end{bmatrix}^{-1} cov\left(\mathbf{x},y\right)
\end{aligned}
$$

Which gives rise to the cup shaped region in Figure 1 (Conchrane Hansen(1992)). Note that the Figure reports also the bound computed using excess returns of Stocks over TBills and triangles that represent the mean variance-space for of the stochastic discount factor generated by power utility $\left(M_{t+1}=\left(c_{t+1}/c_t\right)^{-\gamma}\right)$ with different coefficient of risk aversions. The figure illustrates both the equity premium and the riskfree rate puzzles.



0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Standard deviation
0.85
0.90
0.95
1.00
Mean
γ=50
γ=40
γ=30
γ=20
γ=10
γ=0

2.1 Regression and HJ bounds

Assume now that a predictive regression is run

$$
\begin{aligned}
z_{t,t+1} &= \beta_o + \beta_1' P_t + u_{t+1} \\
\sigma_z^2 &= \sigma_{E(z)}^2 + \sigma_u^2 \\
R^2 &= \frac{\sigma_{E(z)}^2}{\sigma_z^2}
\end{aligned}
$$

The HJ bounds can be written as:

$$
\begin{aligned}
\sigma_{E(z)}^2 &\leq E_t\left[\left(\beta_o + \beta_1' P_t\right)^2\right] \leq \frac{\sigma_z^2 \sigma_M^2}{\left(E_t\left(M_{t+1}\right)\right)^2} \\
R^2 &\leq \sigma_M^2 \left(1 + r_{t,t+1}^f\right)^2
\end{aligned}
$$

3 Consumption strikes back

3.1 Relaxing additivity over states: separating Risk Aversion and Intertemporal Substitution

In standard CRRA model there is an important restriction: the elasticity of intertemporal substitution is the reciprocal of the coefficient of risk aversion. Risk aversion has to do with substituting consumption across different states and it is meaningful even in an atemporal setting, intertemporal subsitution has to do with substituion of consumption over time and it is meaningful also in a deterministic setting.

To relax such a restriction, consider the Epstein-Zin-Weil objective function, defined recursively by:

$$
\begin{aligned}
U_t &= \left\{(1-\delta) C_t^{\frac{1-\gamma}{\delta}} + \delta \left(E_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}} \\
\theta &= \frac{1-\gamma}{1-\frac{1}{\psi}}
\end{aligned}
$$

When $\theta = 1$ we have the usual recursion, ψ is the elasticity of intertemporal substitution, which can be different from the reciprocal of the coefficient of relative risk aversion γ.Epstein and Zin(1989).

The intertemporal budget constraint for a representative agent can be written as follows:

$$
W_{t+1} = \left(1 + R_{a,t+1}\right)\left(W_t - C_t\right)
$$

where $R_{a,t+1}$ is the return on the aggregate portfolio that includes human wealth. $R_{a,t+1}$ correspond to the return on an asset that delivers aggregate consumption as a dividend in each period

The utility function and the budget constraint imply an Euler equation of the form:

$$1 = E_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left\{ \frac{1}{(1 + R_{a,t+1})} \right\}^{1-\theta} ((1 + R_{i,t+1})) \right]$$

If we assume that asset returns and consumption are homoscedastic and jointly lognormal, then this implies that the riskless real rate is :

$$r_{f,t+1} = -\log \delta + \frac{\theta - 1}{2} \sigma_m^2 - \frac{\theta}{2\psi^2} \sigma_c^2 + \frac{1}{\psi} E_t (\Delta c_{+1}) \tag{11}$$

and, for a generic asset including the market portfolio, we have:

$$E_t (r_{i,t+1}) - r_{f,t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1 - \theta) \sigma_{im} \tag{12}$$

which is interesting in that we have a weigthed average of two famous models.

It is interesting to combine the Euler equation with the intertemporal budget constraint and derive a consumption function. To this end log-linearize the budget constraint around the mean log consumption-wealth ratio to obtain:

$$\begin{aligned} \Delta w_{t+1} &\simeq r_{m,t+1} + k + \left(1 - \frac{1}{\rho} \right) (c_t - w_t) \\ \rho &= 1 - \exp \left(\overline{c - w} \right) \end{aligned}$$

By solving forward:

$$c_t - w_t = E_t \left[\sum_{j=1}^{\infty} \rho^j (r_{m,t+j} - \Delta c_{t+j}) \right] + \frac{\rho k}{1 - \rho}$$

The following consumption function is obtained by combining the Euler equation and the intertemporal budget constraint:

$$c_t - w_t = (1 - \psi) E_t \left[\sum_{j=1}^{\infty} \rho^j r_{m,t+j} \right] + \frac{\rho (k - \mu_m)}{1 - \rho} \tag{13}$$

The solved-out consumption function (**??**) shows that the log consumption-wealth ratio is a constant plus $(1 - \psi)$ times the discounted value of expected future returns on invested wealth.

The consumption function implies that

$$
\begin{aligned}
c_{t+1}-E_t\left(c_{t+1}\right) &= \left(r_{m,t+1}-E_t r_{m,t+1}\right)+ \\
&\quad +\left(1-\psi\right)\left(E_{t+1}\left[\sum_{j=1}^{\infty}\rho^j r_{m,t+j+1}\right]-E_t\left[\sum_{j=1}^{\infty}\rho^j r_{m,t+j+1}\right]\right)
\end{aligned}
$$

So there is a direct relation between unexpected return on invested wealth and unexpected return in consumption independently from the parameters of the utility function.

This relation also implies that:

$$
\begin{aligned}
Cov_t\left(r_{i,t+1},\Delta c_{t+1}\right) &= \sigma_{ic}=\sigma_{im}+\left(1-\psi\right)\sigma_{ih} \\
\sigma_{ih} &= Cov_t\left(r_{i,t+1},E_{t+1}\left[\sum_{j=1}^{\infty}\rho^j r_{m,t+j+1}\right]-E_t\left[\sum_{j=1}^{\infty}\rho^j r_{m,t+j+1}\right]\right)
\end{aligned}
$$

Substituting this relation back in the Euler equation determining the risk premium for any risky asset we have:

$$
\begin{aligned}
E_t\left(r_{i,t+1}\right)-r_{f,t+1}+\frac{\sigma_i^2}{2} &= \theta\frac{\sigma_{ic}}{\psi}+\left(1-\theta\right)\sigma_{im} && (14) \\
&= \frac{\theta}{\psi}\left(\sigma_{im}+\left(1-\psi\right)\sigma_{ih}\right)+\left(1-\theta\right)\sigma_{im} && (15) \\
&= \gamma\sigma_{im}+\left(\gamma-1\right)\sigma_{ih} && (16) \\
\theta &= \frac{1-\gamma}{1-\frac{1}{\psi}} && (17)
\end{aligned}
$$

when $i=m$,

$$
E_t\left(r_{m,t+1}\right)-r_{f,t+1}+\frac{\sigma_m^2}{2}=\gamma\sigma_m^2+\left(\gamma-1\right)\sigma_{mh}
$$

Unforecastability of market returns ($\sigma_{mh}=0$),implies an estimate of risk aversion of around 2(0.0575/0.0315).

So some puzzles are fixed but a new problem arises.

- **The elasticity of intertemporal substitution (EIS) is a parameter of central importance in determining the link between macroeconomics and finance.**

  The EIS determines

- the comovement between consumption and real interest rates over the business cycle and hence the power of monetary policy in smoothing fluctuations in aggregate demand (see, for example, Woodford, 2003, chapter 4);

- the importance of the macroeconomic effects of capital income taxation (King and Rebelo, 1990)

- the importance of the burden of government debt or unfunded social security (Hall, 1988).

- As recently pointed out by Guvenen(2003)**, calibrated models and estimated Euler equation deliver opposite views on this parameter**.

- The consistency of calibrated dynamic macroeconomic models with aggregate data requires a large value of the EIS (Kydland and Prescott(1978))

- Direct estimates of the EIS from the first order conditions for the solution of the consumer's intertemporal optimization problem deliver much lower values: Hall (1988) Campbell and Mankiw(1989), Yogo(2004)

3.2 Relaxing additivity over states

Habits: Costantinides and Campbell and Cochrane(1999)

3.3 Long-run Consumption Growth

Parker and Julliard(2005) Rather than using the single period moment condition, use the multiperiod moment condition

$$1 = E_t \left[\beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} R^{rf}_{t,t+1} R^{rf}_{t+1,t+2} ... R^{rf}_{t+k-1,t+k} \right]$$

which is a moment condition robust to measurement error in consumption and simple "mistakes" by consumers.

The paper by HHL show that

- **the Recursive Epstein-Zin-Weil Utility variety produces a model in which asset returns at date t+1 are priced by their exposure to such "long-run consumption" risk.** Parker-Julliard find that this model accounts for the value premium.
- Bansal-Yaron(2005) also argue that average returns of value vs. growth stocks can be understood by different covariance with long-run consumption growth. In fact they examine long-run covariances of earnings with consumption, rather than returns. **HHL show that results on the differences between value and growth stocks depend crucially on wether one includes a time-trend in the regression of earnings on consumption**.

3.4 The Long-Run Risk Model

Consider again Epstein-Zin function

$$
\begin{aligned}
U_t &= \left\{ (1-\delta) C_t^{\frac{1-\gamma}{\delta}} + \delta \left(E_t \left(U_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}} \\
\theta &= \frac{1-\gamma}{1-\frac{1}{\psi}}
\end{aligned}
$$

The intertemporal budget constraint for a representative agent can be written as follows:

$$
W_{t+1} = (1 + R_{a,t+1}) (W_t - C_t)
$$

The first order condition for optimisation are

$$
\begin{aligned}
E_t \left[M_{t+1} (1 + R_{i,t+1}) \right] &= 1 \\
E_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left\{ \frac{1}{(1 + R_{a,t+1})} \right\}^{1-\theta} ((1 + R_{i,t+1})) \right] &= 1 \\
E_t \left[\exp \left\{ \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{a,t+1} + r_{i,t+1} \right\} \right] &= 1
\end{aligned}
$$

log-linearization implies that:

$$
\begin{aligned}
r_{a,t+1} &= k_{0,a} + k_{1,a} z_{t+1} - z_t + \Delta c_{t+1} \\
r_{m,t+1} &= k_{0,m} + k_{1,m} dp_{t+1} - dp_t + \Delta d_{t+1} \\
z_t &= \log \left(\frac{C_t}{P_t} \right)
\end{aligned}
$$

To capture long-run risks, consumption and the dividend growth rates are modeled to contain a small persistent predictable component, x_t, while fluctuating economic uncertainty is introduced via a time-varying volatility process for the cash-flows:

$$
\begin{aligned}
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\Delta c_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
\Delta d_{t+1} &= \mu_d + x_t + \varphi_d \sigma_t u_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \nu \left(\sigma_t^2 - \sigma^2 \right) + \sigma_w w_{t+1}
\end{aligned}
$$

note that all errors are uncorrelated and there exist only one source of time-varying uncertainty.

To characterize the behaviour of asset returns we need a solution for z_t and dp_t in terms of the relevant state variables that are x_t and σ_t^2.

The approximate solution for z_t has the form:

$$
\begin{aligned}
z_t &= A_0 + A_1 x_t + A_2 \sigma_t^2 \\
dp_t &= A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2 \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-k_1\rho} \\
A_{1,m} &= \frac{\phi-\frac{1}{\psi}}{1-k_{1,m}\rho} \\
A_2 &= \frac{0.5\left[\left(\theta-\frac{\theta}{\psi}\right)^2+\left(\theta A_1 k_1 \phi_e\right)^2\right]}{\theta\left(1-k_1 v_1\right)}
\end{aligned}
$$

substituing the solutions in the Euler Equation we have:

$$
E_t\left(m_{t+1}\right) = m_0 - \frac{1}{\psi}x_t + \frac{\left(\frac{1}{\psi}-\gamma\right)(\gamma-1)}{2}\left(1+\left(\frac{k_1\phi_e}{1-k_1\rho}\right)^2\right)\sigma_t^2
$$

$$
\begin{aligned}
m_{t+1} - E_t\left(m_{t+1}\right) &= -\lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} - \lambda_{m,w}\sigma_w w_{t+1} \\
\lambda_{m,\eta} &= \gamma \\
\lambda_{m,e} &= \left(\gamma-\frac{1}{\psi}\right)\left(\frac{k_1\phi_e}{1-k_1\rho}\right) \\
\lambda_{m,w} &= \left(\gamma-\frac{1}{\psi}\right)\left[\frac{k_1\left(1+\left(\frac{k_1\phi_e}{1-k_1\rho}\right)^2\right)}{2\left(1-k_1 v_1\right)}\right]
\end{aligned}
$$

3.5 Stock Returns and Cointegration between Consumption and Wealth

Lettau and Ludvigson concentrate on the intertemporal budget constraint

$$
c_t - w_t = E_t\left[\sum_{j=1}^{\infty}\rho^j\left(r_{m,t+j}-\Delta c_{t+j}\right)\right] + \frac{\rho k}{1-\rho}
$$

The study the role of fluctuations in aggregate consumption-wealth ratio for predicting stock returns by using the intertemporal budget constraint together with some proxy for $c_t - w_t$.

To illustrate how such a proxy could be found note that, following Campbell(1996), the log of total wealth can be approximated as:

$$w_t = va_t + (1 - v) h_t$$

where v is a constant of linearization, equal to the average share of asset holdings in total wealth, a_t is the log of asset holdings and h_t is the log of human capital. While we have available data for financial wealth, the measurement of h_t is not immediate. To find an empirical counterpart of this variable consider that labour income can be interpreted as a dividend on human capital (see Julliard(2004)):

$$1 + R_{h,t+1} = \frac{H_{t+1} + Y_{t+1}}{H_t}$$

Log-linearizing this relation around the steady state human capital-labor income ratio ($\frac{Y}{H} = \frac{1}{\rho_h} - 1$) we have:

$$r_{h,t+1} = (1 - \rho_h) k_h + \rho_h (h_{t+1} - y_{t+1}) - (h_t - y_t) + \Delta y_{t+1}$$

By solving this relation forward and by imposing the transversality condition we have:

$$h_t = y_t + \sum_{i=1}^{\infty} \rho_h^{i-1} (\Delta y_{t+i} - r_{h,t+i}) + k_h$$

so the log of human capital to income ratio is determined by discounted sum of future labour income growth and human capital returns.

Consistently with our linearization for wealth, the total return on wealth can be approximated by:

$$r_{m,t} = vr_{a,t} + (1 - v) r_{h,t} + k_r$$

By substituting all these relationships in the intertemporal budget constraint we have:

$$\begin{aligned} c_t - va_t - (1 - v) y_t &= E_t \left[\sum_{j=1}^{\infty} \rho^j (vr_{a,t+j} + (1 - v) r_{h,t+j}) - \Delta c_{t+j} \right] + k + \\ & (1 - v) \sum_{j=1}^{\infty} E_t \rho_h^{j-1} (\Delta y_{t+j} - r_{h,t+j}) \end{aligned} \tag{18}$$

which implies cointegration between c_t, a_t,and y_t and that disequilibrium in consumption can predict returns on wealth.

In fact LL assume that total consumption is proportional to consumption of non-durables and services $c_t = \lambda c_{n,t}$, then they derive the following cointegrating vector:

$$
\begin{aligned}
c_{n,t} &= c^{*}_{n,t} \\
c^{*}_{n,t} &= 0.61 + 0.31 a_t + 0.59 y_t
\end{aligned}
$$

to find that $\left(c_{n,t} - c^{*}_{n,t}\right)$ is a good predictor of stock market returns.

4 Extending LL with PV models.

In principle one could concentrate on the consumption function rather than on the intertemporal budget constraint. If we combine (18) with the first order conditions we have:

$$\begin{aligned} c_t - va_t - (1-v)\, y_t &= (1-\psi)\, E_t \left[\sum_{j=1}^{\infty} \rho^j \left(v r_{a,t+j} + (1-v)\, r_{h,t+j} \right) \right] + k + \\ & (1-v) \sum_{j=1}^{\infty} E_t \rho_h^{j-1} \left(\Delta y_{t+j} - r_{h,t+j} \right) \end{aligned}$$

The solved-out consumption function (**??**) shows that the log consumption-wealth ratio is a constant plus $(1-\psi)$ times the discounted value of expected future returns on invested wealth. The EIS parameter can be identified and estimated from (**??**) given the availability of some proxy for future expected returns. Values of the EIS ψ lower than one imply that the income effect of higher returns dominates the substitution effect, while if ψ is greater than one, then the substitution effect dominates and the consumption-wealth ratio falls when expected returns rise. The combination of the intertemporal budget constraints with the first order condition of the consumer optimization problem under Eptein-Zin-Weil preferences makes the relation between excess consumption and expected long-term returns tighter than in the intertemporal budget constraints. Moreover, it is now explicit that the correlation between consumption and long-horizon returns depends on the combined effect of income and substitution effects. A positive relation implies that the income effect dominates, this is what Lettau and Ludvigson meant when stating "...**If expected consumption growth is not too volatile**, stationary deviations from the shared trend among these three variables produce movements in the consumption–aggregate wealth ratio and predict future asset returns...". Clearly the evidence in LL is suggestive of a value for the EIS smaller than one, but the estimation of the linearized intertemporal budget constraint cannot be helpful in reconciling the available conflicting evidence on the empirical value of such parameter.

Solving out for expected consumption growth allows the estimation of the intertemporal elasticity of substitution and provides an immediate interpretation of the correlation between excess-consumption and long-horizon returns on the market portfolio. Empirical estimation of (**??**) is the natural step to take at this stage. OF course in order to identify and estimate ψ, the problem of non-observability of $r_{h,t+j}$ must be solved.

To illustrate how this method could be implemented, in the simplest possible case, assume

$$r_{h,t+j} = r_{a,t+j} + u_{t+j}$$

we then have:

$$c_t - va_t - (1-v)\,y_t \;=\; (1-\psi)\,E_t\left[\sum_{j=1}^{\infty}\rho^j\,(r_{a,t+j})\right] + k + \epsilon_t$$

$$\epsilon_t = \sum_{j=1}^{\infty} E_t \rho_h^{j-1}\,(\Delta y_{t+j} - r_{h,t+j})$$

The strategy for identifying and estimating ψ comes in two steps. We first estimate a cointegrating relation between c_t, a_t,and y_t.Such a cointegrating relation is implied by the intertemporal budget constraint, that defines the consumption-wealth ratio as a stationary variable. We then proceed to estimate the following stationary VAR[1]:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{u}_t \tag{19}$$

$$\mathbf{X}_t = \begin{bmatrix} r_{m,t} \\ (c_t - c_t^*) \\ \Delta y_t \\ \Delta a_t \end{bmatrix}.$$

(19) is constructed by considering the stationary VAR representation of a cointegrated system proposed by Campbell and Shiller(1987) and formally derived in Mellander et al.(1993). In practice, we adopt the same VAR estimated by LL and augment it by another stationary variable, the quarterly returns on financial wealth.

The consumption function (**??**) puts a set of restrictions on the VAR that can be exploited to estimate the parameter to our interest. In fact, we have:

$$\mathbf{e}'_{cay} X_t = (1-\psi)\,E_t\left[\sum_{j=1}^{\infty}\rho^j \mathbf{e}'_r A^j X_t\right] \tag{20}$$

where $\mathbf{e}'_{cay}$, $\mathbf{e}'_r$, are selector vectors for cay, $r_{m,t}$, (i.e. row vectors of the length of the vector $\mathbf{X}$, all of which elements are zero except for the 2nd element of $\mathbf{e}'_{cay}$ and the first element of $\mathbf{e}'_r$,which are unity). Since the above expression has to hold for general z_t, and, given stationarity of the VAR, the sum converges, it must be the case that:

$$\mathbf{e}'_{cay} = (1-\psi)\,\mathbf{e}'_r \rho A (I - \rho A)^{-1} \tag{21}$$

which implies:

$$\mathbf{e}'_{cay}(I - \rho A) = (1-\psi)\,\mathbf{e}'_r \rho A \tag{22}$$

[1] We adopt a first order representation of our VAR, if the estimated VAR is of higher order all following results are applicable to the stacked representation of the VAR

by imposing the restrictions on the cointegrated VAR, conditionally upon ρ, ψ is identified and it can be estimated in the restricted VAR. The estimation procedure considers jointly the forward looking consumption function and a VAR used to generate projections of the relevant variables and it avoids the problem of generated regressors that would be encountered by a two-step procedure in which future expected variable are projected first and then they are substituted in the forward-looking consumption function to estimate the parameters of interest.

4.1 Potential Problems

- $c_t = \lambda c_{n,t}$
- $r_{h,t+j} = r_{a,t+j} + u_{t+j}$
- $c_t = \gamma_c c_{1,t} + (1-\gamma_c) c_{2,t}$ liquidity constrained agents (Attanasio and Vissing Jorgensen)

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