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Network formation with sequential demands

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Abstract. This paper introduces a non-cooperative game-theoretic model of sequential network formation, in which players propose links and demand payoffs. Payoff division is therefore endogenous. We show that if the value of networks satisfies size monotonicity, then each and every equilibrium network is efficient. The result holds not only when players make absolute participation demands, but also when they are allowed to make link-specific demands.

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1 Introduction

We analyze the formation process of a cooperation structure (or network) as a non-cooperative game, where players move sequentially. The main difference between this paper and the seminal work in this area by Aumann and Myerson (1988) is that we are interested in situations in which it is impossible to pre-assign a fixed imputation to each cooperation structure, i.e., situations in which the distribution of payoffs is *endogenous*.[1] Indeed, the formation of international cooperation networks, and, more generally, of any market network, occurs

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[1] Slikker and Van Den Nouweland (1998) studied a link formation game with endogenous payoff division but with a simultaneous-move framework.

through a bargaining process, in which the *demand* of a payoff for participation is a crucial variable.

The most important theoretical debate stemming from Aumann and Myerson (1988) is about the potential conflict between efficiency and stability of networks. In the example of sequential network formation game studied by Aumann and Myerson the specific imputation rule that they consider (the Myerson value) determines an inefficient equilibrium network. The implication of their paper is therefore that *not all* fixed allocation rules are compatible with efficiency, even if the game is sequential. Jackson and Wolinsky (1996) consider value functions depending on the communication structure rather than on the set of connected players and demonstrate that efficiency and stability are indeed incompatible under fairly reasonable assumptions (anonymity and component balancedness) on the fixed imputation rules. Their approach is axiomatic, and hence their result does not have direct connections with the Aumann and Myerson result, which was obtained in a specific extensive form game. The strong conclusion of Jackson and Wolinsky is that *no* fixed allocation rule would ensure that at least one stable graph is efficient for every value function.[2] Dutta and Mutuswami (1997) show, on the other hand, that a mechanism design approach (where the allocation rules themselves are the mechanisms to play with) can help reconcile efficiency and stability. In particular, they solve the impossibility result highlighted by Jackson and Wolinsky by imposing the anonimity axiom only on the equilibrium network. With a similar mechanism design approach, one could probably find fixed allocation rules that lead to efficient network formation in sequential games like the one of Aumann and Myerson. However, since in many situations of market network formation there is no mechanism designer who can select the "right" allocation mechanism, we are here interested to ask what happens to the conflict between efficiency and stability discussed above when payoff division is endogenous.

The main result of this paper is that, if the value function satisfies size monotonicity (i.e., if the efficient networks connect all players in some way), then the sequential network formation process with endogenous payoff division leads *all* equilibria to be efficient (Theorem 2). As shown in Example 2, there exist value functions satisfying size monotonicity for which *no* allocation rule can eliminate inefficient equilibria when the game is simultaneous move, nor with the Jackson and Wolinsky concept of stability. So our efficiency result could not be obtained without the sequential structure of the game. We will also show (see Example 3) that the sequential structure alone, without endogenous payoff division, would not be sufficient.

In the game that we most extensively analyze, we assume that players propose links and formulate a single *absolute* demand, representing their final payoff demand. This is representative of situations such as the formation of economic unions, in which negotiations are multilateral in nature, and each player (country) makes an absolute claim on the total surplus from cooperation. We will show

[2] See also Jackson and Watts (1996) and Qin (1996).

that the result that all equilibria are efficient extends to the case in which players attach to each proposed link a separate payoff demand.

The next section describes the model and presents the link formation game. Section 3 contains the analysis of the Subgame Perfect Equilibria of the game, the main results, and a discussion of them. Section 4 presents the extension to link-specific demands, and Sect. 5 concludes.

2 The model

2.1 Graphs and values

Let $N = \{1, \ldots, n\}$ be a finite set of players. A graph g is a set L of links (non-directed segments) joining pairs of players in N (nodes). The graph containing a link for every pair of players is called complete graph, and is denoted by g^N. The set G of all possible graphs on N is then $\left\{g : g \subseteq g^N\right\}$. We denote by ij the link that joins players i and j, so that if $ij \in g$ we say that i and j are directly connected in the graph g. For technical reasons, we will say that each player is always connected to himself, i.e. that $ii \in g$ for all $i \in N$ and all $g \in G$. We will denote by $g + ij$ the graph obtained adding the link ij to the graph g, and by $g - ij$ the graph obtained removing the link ij from g.

Let $N(g) \equiv \{i : \exists j \in N \text{ s.t. } ij \in g\}$. Let $n(g)$ be the cardinality of $N(g)$. A *path* in g connecting i_1 and i_k is a set of nodes $\{i_1, i_2, \ldots, i_k\} \subseteq N(g)$ such that $i_p i_{p+1} \in g$ for all $p = 1, \ldots, k-1$.

We say that the graph $g' \subset g$ is a *component* of g if

1. for all $i \in N(g')$ and $j \in N(g')$ there exists a path in g' connecting i and j;
2. for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies that $ij \in g'$.

So defined, a component of g is a maximal connected subgraph of g. In what follows we will use the letter h to denote a component of g (obviously, when all players are indirectly or directly connected in g the graph g itself is the unique component of g). Note that according to the above definition, each isolated player in the graph g represents a component of g. The set of components of g will be denoted by $C(g)$. Finally, $L(g)$ will denote the set of links in g.

To each graph $g \subseteq g^N$ we associate a value by means of the function $v : G \to R_+$. The real number $v(g)$ represents the aggregate utility produced by the set of agents N organized according to the graph (or network) g. We say that a graph g^* is *efficient* with respect to v if $v(g^*) \geq v(g)\ \forall g \subseteq g^N$. $G^*(v)$ will denote the set of efficient networks relative to v.

We restrict the analysis to *anonymous* and additive value functions, i.e., such that $v(g)$ does not depend on the identity of the players in $N(g)$ and such that the value of a graph is the sum of the values of its components.

2.2 The link formation game

We will study a sequential game $\Gamma(v)$, in which agents form links and formulate payoff demands. In this section we consider the benchmark case in which each agent's demand consists of a positive real number, representing his demanded payoff in the game.

In the formulation of the game $\Gamma(v)$, it will be useful to refer to some additional definitions. A pre-graph on N is a set A of directed arcs (directed segments joining two players in N). The arc from player i to player j is denoted by a_i^j. The set of arcs A uniquely induces the graph

$$g(A) \equiv \left\{ ij \in g^N : a_i^j \in A \text{ and } a_j^i \in A \right\}.$$

2.2.1 Players, actions, and histories

In the game $\Gamma(v)$ the set of players $N = \{1, \ldots, i, \ldots, n\}$ is exogenously ordered by the function $\rho : N \to N$. We use the notation $i \leq j$ as equivalent to $\rho(i) \leq \rho(j)$. Players sequentially choose actions according to the order ρ. An action x_i for player i is a pair (a_i, d_i), where a_i is a vector of arcs sent by i to some subset of players in $N \backslash i$ and $d_i \in [0, D]$ is i's payoff demand, where D is some positive finite real number.[3]

A history $x = (x_1, \ldots, x_n)$ is a vector of actions for each player in N. We will use the notation (borrowed from Harris 1985)

$$\lambda_i x \equiv (x_1, \ldots, x_{i-1})$$

to identify a subgame. We denote by X the set of possible histories, by $\lambda_i X$ the set of possible histories before player i and by X_i the set of possible actions for player i.

2.2.2 From histories to graphs

Players' actions induce graphs on the set N as follows. Firstly, we assume that at the beginning no links are formed, i.e., the game starts from the empty graph $g = \{\emptyset\}$. The history x generates the graph $g(x)$ according to the following rule. Let $A(x) \equiv (a_1, \ldots, a_n)$ be the arcs sent by the players in the history x.

– If h is a component of $g(A(x))$ and h is *feasible given* x, i.e., if

$$\sum_{i \in N(h)} d_i \leq v(h), \tag{1}$$

then $h \in C(g(x))$;

– If h is a component of $g(A(x))$ and (1) is violated, then $h \notin C(g(x))$ and $i \in C(g(x))$ for all $i \in N(h)$;

– If h is not a component of $g(A(x))$, then $h \notin C(g(x))$.

[3] Assuming an upper bound on demands is without loss of generality, since one could always set $D = v(g^*)$ without affecting any of the equilibria of the game.

In words, the component h forms as the outcome of the history x if and only if the arcs sent in x generate h and the demands of the players in $N(h)$ are compatible, in the sense that they do not exceed the value produced by the component h.

2.2.3 Payoffs and strategies

The payoff of player i is defined as a function of the history x. Letting $h_i(x) \in C(g(x))$ denote the component of $g(x)$ containing i, player i gets

$$P_i(x) = \begin{array}{ll} d_i & \textit{if } \sum_{j \in h_i(x)} d_j \leq v(h_i(x)) \\ 0 & \textit{otherwise}. \end{array} \tag{2}$$

This implies that we allow for free disposal.

A strategy for player i is a function $\sigma_i : \lambda_i X \to X_i$. A strategy profile for $\Gamma(v)$ is a vector of functions $\sigma = (\sigma_1, \ldots, \sigma_n)$. A Subgame Perfect Equilibrium (henceforth SPE) for $\Gamma(v)$ is defined as follows. For any subgame $\lambda_i x$, let $\sigma\,|\lambda_i x$ denote the restriction of the strategy profile σ to the subgame. A strategy profile σ^* is a SPE of $\Gamma(v)$ if for every subgame $\lambda_i x$ the profile $\sigma^*\,|\lambda_i x$ represents a Nash Equilibrium. We will denote by $f(\lambda_i x)$ a SPE path of the subgame $\lambda_i x$, i.e., equilibrium continuation histories after $\lambda_i x$. We will only consider equilibria in pure strategies.

3 Equilibrium

In this section we analyze the set of SPE of the game $\Gamma(v)$. We first show that SPE always exist. We then study the efficiency properties of SPE. Finally, we illustrate by example what is the role of the two main features of $\Gamma(v)$, namely the sequential structure and the endogeneity of payoff division, for the efficiency result.

3.1 Existence of equilibrium

Since the game $\Gamma(v)$ is not finite in the choice of payoff demands, we need to establish existence of a SPE (see the Appendix for the proof).

Theorem 1. *The game $\Gamma(v)$ always admits Subgame Perfect Equilibria in pure strategies.*

3.2 Efficiency properties of equilibria

This section contains the main result of the paper: *all* the SPE of $\Gamma(v)$ induce an efficient network. We obtain this result for a wide class of value functions, satisfying a weak "superadditivity" condition, that we call *size monotonicity*. We first provide the definition and some discussion of this condition, then we prove our main result. We then analyze the role of each feature of our game (sequentiality and endogenous payoff division) and of size monotonicity in obtaining our result, and discuss the latter in the framework of the efficiency-stability debate related to Aumann and Myerson (1988) and Jackson and Wolinsky (1996) seminal contributions.

Definition 1. *The link ij is critical for the graph g if $ij \in g$ and $\#C(g) > \#C(g - ij)$.*

In words, a link is critical for a graph if by removing it we increase the number of components. Intuitively, a critical link is essential for the component it belongs to in the sense that without it that component would split in two different components.

Definition 2. *The value function v satisfies size monotonicity if and only if for all graphs g and critical link $ij \in g$*

$$v(g) > v(g - ij).$$

Size monotonicity requires that merging components in the "minimal" way strictly increases the value of the graph. By "minimal" we mean here that such merging occurs through a single additional link. This condition is trivially satisfied when additional links always increase the value of the graph, leading to an efficient fully connected graph. However, this condition is also compatible with cases in which "more" communication (more connected players) originates more value, but, for a fixed set of players that are communicating, this value decreases with the number of links used to communicate. Value functions exhibiting congestion in the number of links within components satisfy this assumption. The extreme case is represented by value functions such that the efficient graph consists of a single path connecting all players, or the star graph, with one player connected with all other players and no other pair of players directly linked (minimally connected graphs). One example that would originate such value functions is the symmetric connection model studied in Jackson and Wolinsky (1996), with a cost of maintaining links for each player, which is a strictly convex and increasing function of the number of maintained links.

The next lemma formally proves one immediate implication of size monotonicity, i.e., that all players are (directly or indirectly) connected.

Lemma 1. *Let v satisfy size monotonicity. All efficient graphs are connected, i.e., if g is efficient then $C(g) = \{g\}$ and $N(g) = N$.*

Proof. Consider a graph g such that $C(g) = \{h_1, \ldots, h_p\}$, with $p > 1$. Then let $i \in h_1$ and $j \in h_2$ ($ij \notin g$). The link ij is a critical link according to Definition 1, so that, by size monotonicity of v, we have that $v(g) < v(g + ij)$, implying that g is not efficient. **QED.**

We now state our main theorem, proving that size monotonicity is a sufficient condition for *all* SPE to be efficient.

Theorem 2. *Let v satisfy size monotonicity. Every SPE of $\Gamma(v)$ leads to an efficient network.*

We prove the theorem in two steps. We first prove by an induction argument in step 1 that if a given history is not efficient and satisfies a certain condition on payoff demands, then some player has a profitable deviation. Then, in step 2, we show that if some history x such that $g(x) \notin G^*$ is a SPE, then the condition on payoff demands introduced in step 1 would be satisfied, which implies that there exists a profitable deviation from *any* history that leads to an inefficient network.

The proof relies on two lemmas, the first characterizing equilibrium payoffs and the second characterizing equilibrium graphs.

Lemma 2. *Let v satisfy size monotonicity. For any arbitrary history of $\Gamma(v)$, $\lambda_m x$, the continuation equilibrium payoff for player m, $P_m(f(\lambda_m x))$, is strictly positive, for all $m = 1, \ldots, n-1$.*

Proof. Recall that n is the last player in the order of play ρ, and let $m < n$ be any player moving before n. Consider an arbitrary history $\lambda_m x$. In order to prove that the continuation equilibrium payoff is strictly positive for player m, let us show that there exists $\varepsilon > 0$ such that if player m plays the action $x_m = \left(a_m^n, \varepsilon\right)$, then it is a dominant strategy for player n to reciprocate m's arc and form some feasible component h with $mn \in h$.

Suppose first that $\varepsilon = 0$, so that, at the arbitrary history $\lambda_m x$, player m chooses $x_m = (a_m^n, 0)$.

We want to show that there cannot be an equilibrium continuation history $f(\lambda_m x, x_m)$ such that, denoting the history $(\lambda_m x, x_m, f(\lambda_m x, x_m))$ by $\hat{x}$, $h_m(\hat{x}) = mm$ (i.e., where m is alone even though she demands 0). Suppose this is the case, and let $\hat{x}_n = (a_n, d_n)$ be a strategy for player n such that $a_n^m \notin a_n$. Let $h_n(\hat{x})$ be the component including n if this continuation history is played. Denote by h_n' the component obtained by adding the link mn to $h_n(\hat{x})$. By size monotonicity,

$$v(h_n(\hat{x})) < v(h_n').$$

If the component $h_n(\hat{x})$ is feasible, the component h_n' is feasible too, for some demand $d_n + \delta > d_n$ of player n.[4] It follows that it is dominant for n to reciprocate m's arc and get a strictly greater payoff. So $\hat{x}$ cannot be an equilibrium continuation payoff.

Consider then $x_m(\varepsilon) \equiv (a_m^n, \varepsilon)$ with $\varepsilon > 0$.

Consider the continuation history $\hat{x}(\varepsilon) = f(\lambda_m x, x_m(\varepsilon))$, with

$$\hat{x}(\varepsilon) = (\lambda_m x, x_m(\varepsilon), f(\lambda_m x, x_m(\varepsilon))),$$

[4] If $h_n(\lambda_m x, x_m, \hat{x})$ is not feasible, then either there exists some positive demand d_n' for player n such that $\sum_{i \in N(h_n') \setminus n} d_i + d_n' = v(h_n')$ or player n could just reciprocate player m's arc and demand $d_n' = v(mn) > 0$ (this last inequality by size monotonicity).

and $\hat{x}_n = (a_n, d_n)$ such that $a_n^m \notin a_n$. Let $h_n(\hat{x}(\varepsilon))$ be the component that includes n given $\hat{x}(\varepsilon)$. Let again $h_n'(\varepsilon) \equiv mn \cup h_n(\hat{x}(\varepsilon))$. Define

$$\delta_{\min} \equiv \min_{\varepsilon \geq 0} v\left(h_n'(\varepsilon)\right) - v\left(h_n(\hat{x}(\varepsilon)\right) > 0$$

where the strict inequality comes form size monotonicity.
Let $0 < \varepsilon < \delta_{\min}$.
If $h_n(\hat{x}(\varepsilon))$ is feasible, then $h_n'(\varepsilon)$ is feasible too, for some positive additional demand of player n. Thus, it is possible for player n to demand a strictly higher payoff than d_n (this because $\varepsilon < \delta_{\min}$).[5] Therefore a positive payoff is always attainable by any player $m < n$, at any history. **QED.**

Lemma 3. *Let v satisfy size monotonicity. Let x be a SPE history of the game $\Gamma(v)$. In the induced graph $g(x)$ all players are connected, i.e., $C(g(x)) = \{g(x)\}$ and $N(g(x)) = N$.*

Proof. Suppose that $C(g(x)) = \{h_1, \ldots, h_k\}$ with $k > 1$. Let again n be the last player in the ordering ρ. Note first that there must be some component h_p such that $n \notin h_p$, since otherwise the assumption that $k > 1$ would be contradicted. Also, note that by Lemma 2, x being an equilibrium implies that[6]

$$\sum_{i \in N(h_p)} d_i = v(h_p) \quad \forall p \in \{1, \ldots, k\}.$$

Let us then consider h_p and the last player m in $N(h_p)$ according to the ordering ρ. Let $\hat{x}_m(\varepsilon) = \left(a_m \cup a_m^n, d_m + \varepsilon\right)$, with continuation history $f(\lambda_m x, \hat{x}_m(\varepsilon))$. Let

$$\hat{x}(\varepsilon) = (\lambda_m x, \hat{x}_m(\varepsilon), f(\lambda_m x, \hat{x}_m(\varepsilon)))$$

and let $h_n(\hat{x}(\varepsilon))$ be the component including n in $g(\hat{x}(\varepsilon))$. Suppose first that $mn \notin h_n(\hat{x}(\varepsilon))$ and $in \in h_n(\hat{x}(\varepsilon))$ for some $i \in N(h_p)$. Note first that if some player $j > m$ is in $h_n(\hat{x}(\varepsilon))$, then by Lemma 2 $h_n(\hat{x}(\varepsilon))$ is feasible given x_n, and since player m is getting a higher payoff than under x, the action $\hat{x}_m(\varepsilon)$ is a profitable deviation for him. We therefore consider the case in which no player $j > m$ is in $h_n(\hat{x}(\varepsilon))$, and $h_n(\hat{x}(\varepsilon))$ is not feasible. In this case, it is a feasible strategy for player n, who is getting a zero payoff under x_n, to reciprocate only player m's arc and form the component h_n' such that, by size monotonicity,

$$v(h_n'(\varepsilon)) > v(h_p).$$

[5] If instead $h_n(\lambda_m x, x_m(\epsilon_m), \hat{x}_m(\epsilon_m))$ is not feasible, then either there exists some positive demand d_n' such that $\sum_{i \in N(h_n'(\varepsilon_m)) \setminus n} d_i + d_n' = v(h_n'(\varepsilon_m))$ or player n could just reciprocate player m's arc and demand $d_n' = v(mn) - \varepsilon_m > 0$ (this last inequality again by size monotonicity).

[6] Note that there cannot be any equilibrium where the last player demands something unfeasible: since in every equilibrium the last player obtains a zero payoff, one could think that she could then demand anything, making the complete graph unfeasible, but this would entail a deviation by one of the previous players, who would demand ϵ less, in order to make n join in the continuation equilibrium. Thus, the unique equilibrium demand of player n is 0.

If ε is small enough we get

$$v(h_n'(\varepsilon)) - v(h_p) > \varepsilon$$

which implies that reciprocating only player m's arc and demanding $d_n = v(h_n'(\varepsilon)) - v(h_p) - \varepsilon > 0$ is a profitable deviation for player n.

Thus, we can restrict ourselves to the case in which $in \notin h_n(\hat{x}(\varepsilon))$ for all $i \in N(h_p)$. Let $h_n'(\varepsilon)$ be obtained by adding the link mn to $h_n(\hat{x}(\varepsilon))$. By size monotonicity

$$v\left(h_n'(\varepsilon)\right) - v\left(h_n(\hat{x}(\varepsilon))\right) > 0.$$

Let also

$$\delta_{\min} \equiv \min_{\varepsilon \geq 0} \left[v\left(h_n'(\varepsilon)\right) - v\left(h_n(\hat{x}(\varepsilon))\right)\right] > 0.$$

Consider a demand ε such that $0 < \varepsilon < \delta_{\min}$. As in the proof of Lemma 2, we claim that if player m demands ε, then it is dominant for player n to reciprocate player m's link and form the component $h_n'(\varepsilon)$. Note first that, given that $0 < \varepsilon_m < \delta_{\min}$, if $h_n(\hat{x}(\varepsilon)$ is feasible, then $h_n'(\varepsilon)$ is feasible for some positive additional demand (w.r.t. d_n) of player n. If instead $h_n(\hat{x}(\varepsilon)$ was not feasible, then player n would be getting a zero payoff, and this would be strictly dominated by reciprocating m's arc and getting a payoff of

$$\left[v\left(h_n'(\varepsilon)\right) - v\left(h_n(\hat{x}(\varepsilon))\right)\right] - \varepsilon$$

which, again by the fact that $\varepsilon < \delta_{\min}$, is strictly positive. **QED.**

Proof of Theorem 2.

Step 1. Induction argument.

Induction Hypothesis (H): Let x be an arbitrary history such that $g(x) \notin G^*$. Let m be the first player in the ordering ρ such that there is no x^* such that (1) $\lambda_{m+1}x^* = \lambda_{m+1}x$ and (2) $g(x^*) \in G^*$. Let x be such that

$$\sum_{i=1}^{m} d_i \leq v\left(g(x)\right) - \sum_{i=m+1}^{n} d_i.$$

Then there exists some $\varepsilon > 0$ and action $x_m^* = \left(a_m^*, d_m + \varepsilon\right)$ that induce a continuation history $f\left(\lambda_m x, x_m^*\right)$ such that, denoting by x^* the history $(\lambda_m x, x_m^*, f(\lambda_m x, x_m^*))$, $g(x^*) \in G^*$ and $\sum_{i=1}^{n} \hat{d}_i = v(g(x^*))$.

(H) true for player n: Let $x_n = (a_n, d_n)$. Let player m, as defined in (H), be n. In words, this means that n could still induce the efficient graph by deviating to some other action. Formally, there exist some arcs a_n^* and a demand d_n' such that $g\left(x_1, \ldots, x_{n-1}, a_n^*, d_n'\right) \in G^*$ and, therefore, such that $v\left(g\left(x_1, \ldots, x_{n-1}, a_n^*, d_n'\right)\right) > v(g(x))$. By (H)

$$\sum_{i=1}^{n} d_i \leq v\left(g(x)\right)$$

and by size monotonicity all players are connected in $g\left(x_1,\ldots,x_{n-1},a_n^*,d_n'\right)$. These two facts imply that player n can induce the efficient graph and demand $d_n' = d_n + \varepsilon_n$ with

$$\varepsilon_n = \left[v\left(g^*\right) - v\left(g\left(x\right)\right)\right] > 0.$$

(H) true for player $m+1$ implies (H) true for player m: Suppose again that x is an inefficient history and that m is the first player in x such that the action a_m is not compatible with efficiency in the sense of assumption (H). Let a_m^* be some action compatible with efficiency and let $x_m^*(\varepsilon) = \left(a_m^*, d_m + \varepsilon\right)$. Let also $f\left(\lambda_m x, x_m^*(\varepsilon)\right)$ represent the corresponding continuation history, and $x^*(\varepsilon) = \left(\lambda_m x, x_m^*(\varepsilon), f\left(\lambda_m x, x_m^*(\varepsilon)\right)\right)$. We need to show that there exists $\varepsilon > 0$ such that $g\left(x^*(\varepsilon)\right) \in G^*$. Note first that in the history $x^*(\varepsilon)$, the first player k such that a_k is not compatible with efficiency must be such that $k > m$. Since by (H)

$$\sum_{i=1}^{m} d_i \leq v\left(g\left(x\right)\right) - \sum_{i=m+1}^{n} d_i$$

there exists an $\varepsilon > 0$ such that

$$\sum_{i=1}^{m-1} d_i + d_m + \varepsilon < v\left(g^*\right) - \sum_{i=m+1}^{n} d_i.$$

Thus, if player m plays $x_m^*(\varepsilon)$, player $(m+1)$ faces a history $\left(\lambda_m x, x_m^*(\varepsilon)\right)$ that satisfies the inductive assumption (H). Suppose now that player $(m+1)$ optimally plays some action x_{m+1} such that no efficient graph is compatible (in the sense of assumption (H)) with the history $(\lambda_m x, x_m(\varepsilon), x_{m+1})$. Then, by (H) we know there would be a deviation for player $(m+1)$, contradicting the assumption that x_{m+1} is part of the continuation history at $(\lambda_m x, x_m(\varepsilon))$. Thus, we know that player $(m+1)$ will optimally play some strategy x_{m+1}^* such that the continuation history $f\left(\left(\lambda_m x, x_m(\varepsilon), x_{m+1}^*\right)\right)$ induces a feasible efficient graph.

Step 2. We now show that the induction argument can be applied to each candidate SPE history x of $\Gamma(v)$ such that $v(g(x)) < v(g^*)$ (which we want to rule out). This is shown to imply that the first player m (such that there does not exist x^* such that $\lambda_{m+1}x^* = \lambda_{m+1}x$ and $v\left(g\left(x^*\right)\right) = v(g^*)$) has a profitable deviation.

Note first that by Lemma 3 if x is a SPE history then all players are connected. This, together with Lemma 2, directly implies that

$$\sum_{i=1}^{n} d_i = v\left(g\left(x\right)\right)$$

or, equivalently, that

$$\sum_{i=1}^{m} d_i = v\left(g\left(x\right)\right) - \sum_{i=m+1}^{n} d_i$$

for all $m = 1,\ldots,n$. It follows that the induction argument can be applied to all inefficient SPE histories to conclude that the first player whose action is not compatible with efficiency in the sense of assumption (H) has some action $x_m^*(\varepsilon) = \left(a_m^*, d_m + \varepsilon\right)$ such that $\varepsilon > 0$ and such that the induced graph $g(x^*(\varepsilon)) \in G^*$ is feasible, where, as usual, $x^*(\varepsilon) = \left(\lambda_m x, x_m^*(\varepsilon), f\left(\lambda_m x, x_m^*(\varepsilon)\right)\right)$. Since $g(x^*(\varepsilon))$ is feasible, then the action $x_m^*(\varepsilon)$ represents a deviation for player m, proving the theorem. **QED.**

The efficiency theorem extends to the case in which the order of play is random, i.e., in which each mover only knows a probability distribution over the identity of the subsequent mover. This is true because the value function is assumed to satisfy anonymity. Another important remark about the role of the order of play regards the asymmetry of equilibrium payoffs: for any given order of play the equilibrium payoffs are clearly asymmetric, since the last mover always obtains 0. However, if *ex ante* all orders of play have the same probability, then the expected equilibrium payoff is $E(P_i(g(x(\rho)))) = \frac{v(g^*)}{n}\ \forall i$.

3.3 Discussion

In this section we want to discuss our result in the framework of the recent literature debate on the possibility of reconciling efficiency and stability in the process of formation of networks. As we pointed out in the introduction, this debate has been initiated by two seminal papers: Aumann and Myerson (1988) have shown that if the Myerson value is imposed as a fixed imputation rule, then forward looking players forming a networks through sequential link formation can induce inefficient networks. The value function they consider is obtained from a traditional coalitional form game. Jackson and Wolinksy (1996) obtained a general impossibility result considering value functions that depend on the communication structure rather than only on the set of connected players. This incompatibility has been partially overcome by Dutta and Mutuswami (1997) who show that it disappears if component balancedness and anonymity are required only on stable networks.

We first note that the *size monotonicity* requirement of Theorem 2 in the present paper is compatible with the specific value function for which Jackson and Wolinsky show that no anonymous and component balanced imputation rule exists such that at least one stable graph is efficient. In this sense, we can conclude that in our game the aforementioned conflict between efficiency and stability does not appear. Since however imputation rules of the type considered by Dutta and Mutuswami allow for efficient and stable networks, our game can be considered as another way to overcome that conflict.

The real novelty of our efficiency result is therefore the fact that *all subgame perfect equilibria of our game are efficient*. In the rest of this section we will show that both the sequential structure of the game and the endogeneity of the final imputation rule are ”tight” conditions for the result, as well as the size monotonicity requirement. Indeed, we first show that relaxing size monotonicity generates inefficient equilibria. We then construct a value function for which all fixed component balanced and anonymous imputation rules generate at least one inefficient stable graph in the sense of Jackson and Wolinsky. The same is shown for a game of endogenous payoff division in which agents move simultaneously. We finally show that sequentiality alone does not generate our result, since no fixed component balanced and anonymous imputation rule exists such that all subgame perfect equilibria are efficient.

3.3.1 Eliminating size monotonicity

The next example shows that if a value function v does not satisfy size monotonicity, then the SPE of $\Gamma(v)$ may induce an inefficient network.

Example 1. Consider a four-player game with the following value function:

$$\begin{aligned} v(h) &= 9 \text{ if } N(h) = N \\ v(h) &= 8 \text{ if } \#N(h) = 3 \text{ and } \#L(h) = 2; \\ v(h) &= 5 \text{ if } \#N(h) = 2; \\ v(h) &= 0 \text{ otherwise}. \end{aligned}$$

The efficient network is one with two separate links. We show that the history x such that

$$\begin{aligned} x_1 &= \big((a_1^2, a_1^3, a_1^4), 3\big) \\ x_2 &= \big((a_2^1, a_2^3, a_2^4), 3\big) \\ x_3 &= \big((a_3^2, a_3^4), 3\big) \\ x_4 &= (a_4^3, 0) \end{aligned}$$

is a SPE of the game $\Gamma(v)$, leading to the inefficient graph $(12, 23, 34)$.

1. **Player 4:** given that at the history $\lambda_4 x$ we have $d_1 + d_2 + d_3 = 9$, player 4 optimally reciprocates the arc of player 3.
2. **Player 3:** sending just a_3^2 or a_3^1 or both, would let player 3 demand at most $d_3 = 2$; forming a link just with player 4 would allow player 3 to demand at most $d_3 = 3$, since player 4 would have at that node the outside option of going with the first two movers.
3. **Player 2:** If $d_2 > d_1 = 3$, then player 3 has the outside option of just reciprocating the arc of player 1 and demand $d_3 = 3$. Thus, $d_2 > 3$ is not a profitable deviation for player 2. In terms of arcs, note first that if player 2 sends just a_2^1 then $d_2 \leq 2$, given that $d_1 = 3$. Suppose now that player 2 sends arcs only to 1 and 4 demanding $d_2 = 3 + \epsilon$. In this case player 3 would react by sending an arc just to player 4, demanding $3 + \epsilon - \delta$ ($\epsilon > \delta > 0$), which 4 would optimally reciprocate.
4. **Player 1:** We just check that player 1 could not demand $d_1 = 3 + \epsilon > 3$. If he does, then player 2 can "underbid" by a small δ, as in the argument above, so that player 3 and/or 4 would always prefer to reciprocate links with player 2.

This example has shown that when size monotonicity is violated then inefficient equilibria may exist. The intuition for the failure of Theorem 2 when v is not size monotonic can be given as follows. By Lemma 1, under size monotonicity all efficient graphs are connected (though not necessarily fully connected). It follows that the gains from efficiency can be shared among all players in equilibrium (since efficiency requires all players to belong to the same component). When size monotonicity fails, however, the efficient graph may consist of more than one component. It becomes then impossible to share the gains from efficiency among all players, since side payments across components are not allowed in the game $\Gamma(v)$. It seems reasonable to conjecture that it would be possible to conceive a game form allowing for such side payments and such that all equilibria are efficient even when size monotonicity fails.

3.3.2 The role of sequentiality

The next example displays a value function satisfying size monotonicity, and serves the purpose of demonstrating the crucial role of the sequential structure of our game for the result that *all* equilibria are efficient. In fact, neither using the stability concept of Jackson and Wolinsky, nor with a simultaneous move game, it is possible to eliminate all inefficient equilibria.

Example 2. Consider a four-player game with the following value function:

$$
\begin{aligned}
v(h) = 1 \quad & \quad \textit{if } \#N(h) = 2; \\
2 \quad & \quad \textit{if } \#N(h) = 3; \\
20 \quad & \quad \textit{if } \#N(h) = 4 \textit{ and } \#L_i = 2\ \forall i; \\
24 \quad & \quad \textit{if } h = g^N; \\
4 \quad & \quad \textit{otherwise}.
\end{aligned}
$$

This value function satisfies size monotonicity, and the only two connected networks with value greater than 4 are the complete graph and the one where each player has two links.

Let us first show that the inefficient network with value equal to 20 is stable, in the sense of Jackson and Wolinsky (1996), *for every allocation rule* satisfying anonymity and component balancedness. To see this, note that in such network anonymity implies that each player would receive 5, which is greater than anything achievable by either adding a new link or severing one ($5 > 4$). Along the same line it can be proved that the complete (efficient) graph is stable.

Similarly, even if we allow payoff division to be endogenous, a simultaneous move game would always have an equilibrium profile leading to the inefficient network with value equal to 20. To see this, consider a simultaneous move game where every player announces at the same time a set of arcs and a demand (keeping all the other features of the game as in $\Gamma(v)$). Consider a strategy profile in which every player demands 5 and sends only two arcs, in a way that every arc is reciprocated. It is clear that any deviation in terms of arcs (less or more) induces a network with value 4, and hence the deviation cannot be profitable.

On the other hand, given the sequential structure of $\Gamma(v)$, the inefficient networks are never equilibria, and the intuition can be easily obtained through the example above: calling σ the strategy profile leading to the inefficient network discussed above, the first mover can deviate by sending all arcs and demanding more than 5, since in the continuation game he expects the third arc will be reciprocated and the complete graph will be formed.

3.3.3 The role of endogenous payoff division

Having shown the crucial role of sequentiality, the next task is to show the relevance of the other innovative aspect of $\Gamma(v)$, namely, endogenous payoff division. Consider a game $\Gamma(v, Y)$ that is like $\Gamma(v)$ but for the fact that the action space of each player only includes the set of possible arcs he could send, and no payoff demand can be made. The imputation rule Y (of the type considered in Jackson and Wolinsky 1996) determines payoffs for each network. We can now show by example that there are some value functions that satisfy size monotonicity for which no allocation rule satisfying anonymity and component balancedness can eliminate all inefficient networks from the set of equilibrium outcomes of $\Gamma(v, Y)$.

Proposition 1. *There exists value functions satisfying size monotonicity and such that every fixed imputation rule Y satisfying anonymity and component balancedness induces at least one inefficient equilibrium in the associated sequential game $\Gamma(v, Y)$.*

Proof. By Example.

Example 3. Consider a three-player game $\Gamma(v, Y)$ with the following value function:[7]

$$\begin{aligned} v(12) &= v(23) = v(13) = 1; \\ v(12, 23) &= v(13, 12) = v(13, 23) = 1 + \varepsilon > 1; \\ v(12, 13, 23) &= 1. \end{aligned}$$

Given anonymity of Y, the only payoff distribution if the complete graph forms is $P_i(g^N) = \frac{1}{3}$. Similarly, if $h = ij$, then both i and j must receive $\frac{1}{2}$. If $h = (ij, jk)$, then let us call x the payoff to i and k and y the payoff to the pivotal player, j, with $(2x + y = 1 + \epsilon)$. Let ϵ be small, so that $\frac{1+\epsilon}{3} < \frac{1}{2}$.

1. If $y \geq \frac{1}{2}$, the first mover cannot send one arc only. If he sends an arc only to the second mover, then player 2's best response is to send two arcs and get y; if he sends an arc to the third mover only, the second mover does the

[7] This value function was used in Jackson and Wolinsky (1996) to get their impossibility result under the axiomatic approach discussed in the previous section.

same, and the third mover gets y. So, if the first mover sends only one arc his payoff is $\frac{1+\epsilon-y}{2} < \frac{1}{3}$. By sending both arcs, player 1 would end up forming the complete graph and obtaining $\frac{1}{3}$, which makes the complete graph an equilibrium network.

2. If $y < \frac{1}{2}$, note that there always exists an equilibrium continuation history leading to the graph (12) if player 1 sends the arc only to player 2. Thus, if $x < \frac{1}{2}$, player 1 cannot get as much as $\frac{1}{2}$ on any other network, and sending an arc only to player 2 will therefore be an equilibrium strategy. If on the contrary $x \geq \frac{1}{2}$, there could be an incentive for player 1 to form the efficient graph and get x. However, it can be easily checked that in this case, the following strategy profile is an equilibrium:

$$\begin{aligned}
\sigma_2 &= a_1^2 \\
\sigma_2 &= \begin{cases} \sigma_2\left(a_1^2, a_1^3\right) = \left(a_2^1, a_2^3\right) \\ \sigma_2\left(a_1^3\right) = \left(a_2^1, a_2^3\right) \\ \sigma_2\left(a_1^2\right) = \left(a_2^1\right) \end{cases} \\
\sigma_3 &= \begin{cases} \sigma_3\left(a_1^2, a_1^3, a_2^1, a_2^3\right) = \sigma_3\left(a_1^2, a_1^3, a_2^1\right) = \sigma_3\left(a_1^2, a_2^1\right) = a_3^1 \\ \sigma_3\left(a_1^2, a_1^3, a_2^3\right) = a_3^2 \\ \sigma_3\left(a_1^3, a_2^3, a_2^1\right) = \sigma_3\left(a_1^3, a_2^3\right) = a_3^2 \\ \sigma_3\left(a_1^3, a_2^1\right) = \sigma_3\left(a_1^2, a_2^1, a_2^3\right) = \sigma_3\left(a_1^2, a_2^3\right) = \left(a_3^1, a_3^2\right). \end{cases}
\end{aligned}$$

In words, there are optimal strategies that support the pair (12) as a SPE equilibrium. **QED.**

4 Link-specific demands

Consider now a variation of the game, $\Gamma_1(v)$, which differs from $\Gamma(v)$ in that players can attach payoff demands on each arc they send, rather than demanding just one aggregate payoff from the whole component. Player $i's$ demand d_i is a vector of real positive numbers, one for each arc sent in the vector a_i. We describe how payoffs depend on histories in $\Gamma_1(v)$ on the basis of the formal description of the game $\Gamma(v)$:

1. The feasibility condition given in (1) is replaced by:

$$\sum_{i \in N(h)} \sum_{j: ij \in h} d_i^j \leq v(h); \tag{3}$$

2. The payoff for player i in the component $h \in C(g(x))$ is given by

$$P_i(x) = \begin{array}{ll} \sum_{j \neq i: ij \in h} d_i^j & \textit{if } L(h(x)) \neq \emptyset \\ 0 & \textit{otherwise} \end{array} \tag{4}$$

(instead of (2)). In words, the payoff for player i from history x would be equal to the sum of the link-specific demands made by i to the members of her component whom she is directly linked to.

The same efficiency result as the one obtained in Theorem 2 can be obtained for the game $\Gamma_1(v)$. Proofs are found in the appendix.

Lemma 4. *Let v satisfy size monotonicity. Let $\lambda_m x$ be an arbitrary history of the game $\Gamma_1(v)$. Then $P_i(f(\lambda_m x)) > 0$ for all $i = 1, ..., n-1$.*

Lemma 5. *Let v satisfy size monotonicity. Let x be a SPE history of the game $\Gamma_1(v)$. In the induced graph $g(x)$ all players are connected, i.e., $C(g(x)) = \{g(x)\}$ and $N(g(x)) = N$.*

Theorem 3. *Let v satisfy size monotonicity. Every SPE of $\Gamma_1(v)$ leads to an efficient network.*

5 Conclusions

This paper provides an important result for all the situations in which a communication network forms in the absence of a mechanism designer: if players sequentially form links and bargain over payoffs, the outcome is an efficient network. This result holds as long as disaggregating components *via* the removal of "critical" links lowers the aggregate value of the network. In other words, efficiency arises whenever more communication is good, at least when it is obtained with the minimal set of links. We have shown this result by proving that all the subgame perfect equilibria of a sequential link formation game, in which the relevant players demand absolute payoffs, lead to efficient networks. On the other hand, endogenous payoff division is not sufficient to obtain optimality when the optimal network has more than one component. Allowing for link-specific demands we obtain identical results.

Appendix

Proof of Theorem 1. We prove the theorem by showing that every player's maximization problem at each subgame has a solution. Using the notation introduced in the previous sections, we show that for each player m and history x, there exists an element $x_m \in X_m$ maximizing m's payoff given the continuation histories originating at $(\lambda_m x, x_m)$. Since the choice set X_m is given by the product set $A_m \times [0, D]$, where the finite set A_m is the set of vectors of arcs that player m can choose to send to other players in the game, it suffices to show that we can associate with each vector of arcs $a_m \in A_m$ a maximal feasible demand $d_m(a_m)$.

Suppose not. Then, given a_m, $\forall d_m \exists \varepsilon > 0$ such that $(d_m + \varepsilon)$ is feasible. This, together with the fact that the set $[0, D]$ is compact, imply that there exists some demand $\overline{d}_m(a_m)$ which is not feasible given a_m and which is the limit of some sequence of feasible demands $(d_m^p)_{p=1,...,\infty}$. We prove the theorem by contradicting this conclusion.

First, we denote by $\overline{x}$ a continuation history given $(a_m, \overline{d}_m(a_m))$, and, for all p, we denote by $\overline{x}(p)$ a continuation history given $(a_m, (d_m^p))$. For all p, feasibility of d_m^p implies that player m belongs to some component h_m^p such that

$$v(h_m^p) \geq \sum_{\substack{i \in N(h_m^p) \\ i = m}} d_i(p) + d_m^p. \tag{5}$$

We claim that as $d_m^p \to \overline{d}_m(a_m)$ (5) remains satisfied for some component $\overline{h}_m$. Suppose first that there exists p such that the component h_m^p is the same for all $p \geq p$. We proceed by induction.

Induction Hypothesis: Consider the history x and the histories $x(p)$, $p \geq \overline{p}$, the history identical to x but for player m's demand which is d_m^p. If x_i is the best response of player n at the subgame $\lambda_n x(p)$ for all $p \geq \overline{p}$ then x_i is a best response of player n at $\lambda_n x$.

Player n: At the subgame $\lambda_n x(p)$ player n can either optimally join a component including m or not join any component including m. In the first case, his payoff by not joining m's component with action $\overline{x}_n(p)$ is weakly greater than the one he gets by joining with any action $x_n(p)$:

$$P_n(\lambda_n x(p), \overline{x}_n(p)) \geq P_n(\lambda_n x(p), x_n(p))$$

Bringing m's demand to the limit does not change the above inequality.

In the second case, player n's payoff is maximized by joining a component including m with action $\overline{x}_n(p)$:

$$P_n(\lambda_n x(p), \overline{x}_n(p)) \geq P_n(\lambda_n x(p), x_n(p))$$

We can apply the same limit argument in this case, by noting that at the limit condition (5) remains satisfied.

True for player $k+1$ implies true for player k: Assume that the induction hypothesis is satisfied for all players $k+1, \ldots, n$. Then, the continuation histories after the subgames $\lambda_{k+1} x(p)$ and $\lambda_{k+1} x$ are the same. Player k's optimal choice $\overline{x}_k(p)$ at $\lambda_k x(p)$ satisfies the following condition for all $x_k \in X_k$:

$$P_k(\lambda_k x(p), \overline{x}_k(p), f(\lambda_k x(p), \overline{x}_k(p))) \geq P_k(\lambda_k x(p), x_k(p), f(\lambda_k x(p), x_k(p))).$$

Since we have argued that by the induction hypothesis that

$$f(\lambda_k x(p), \overline{x}_k(p)) = f(\lambda_k x, \overline{x}_k(p));$$
$$f(\lambda_k x(p), x_k(p)) = f(\lambda_k x(p), x_k(p)),$$

we conclude that at the limit

$$P_k(\lambda_k x, \overline{x}_k(p), f(\lambda_k x, \overline{x}_k(p))) \geq P_k(\lambda_k x, x_k(p), f(\lambda_k x(p), x_k(p))).$$

This means that $\overline{x}_k(p)$ is still optimal at $\lambda_k x$. Moreover, the feasibility condition (5) still holds whenever player k was joining a component including m. This concludes the induction argument.

The above argument directly implies that if component h_m^p is still feasible at the limit, so that the demand $\overline{d}_m(a_m)$ is itself feasible.

Finally, suppose that there exists no p such that the component $\overline{h}_m^p$ is the same for all $p \geq \overline{p}$. In this case, since the set of possible components to which

m can belong to given a_m is finite, for each possible such component h we can associate a subsequence $\{d_m(h)\}_{p=1,\ldots,\infty} \to \overline{d}_m(a_m)$. The feasibility condition applied to each component h implies that for all h:

$$v(h) \geq \sum_{\substack{i \in N(h) \\ i = m}} d_i + d_m(h).$$

We can apply the above induction argument to this case by considering some converging subsequence, thereby showing that there exists some feasible component $\overline{h}_m$ induced by the demand $\overline{d}_m(a_m)$. **QED.**

Proof of Lemma 4. Let n be the last player in the ordering ρ and let $m < n$. Consider an arbitrary history $\lambda_m x$. We show that there exists a demand $d_m^n > 0$ such that if player m plays the action $x_m = \left(a_m^n, d_m^n\right)$ then it is a dominating strategy for player n to reciprocate m's arc and form some feasible component h with $mn \in h$.

For a given $d_m^n > 0$, let $x_m\left(d_m^n\right) = \left(a_m^n, d_m^n\right)$, and consider again the continuation history $\hat{x}\left(d_m^n\right) = f\left(\lambda_m x, x_m\left(d_m^n\right)\right)$. Let also $x_n = (a_n, d_n)$[8] be a strategy for player n such that $a_n^m \notin a_n$. Let $h(n, d_m^n)$ be the component that includes n if x_n is played at the history $\lambda_n \hat{x}\left(d_m^n\right)$ and $h'(n, d_m^n)$ be the component obtained by adding the link mn to $h(n, d_m^n)$. Define

$$\delta_{\min} \equiv \min_{d_m^n > 0} \left\{ v\left(h'(n, d_m^n)\right) - v\left(h(n, d_m^n)\right) \right\} > 0,$$

where the last inequality comes form size monotonicity. Let now $0 < d_m^n < \delta_{\min}$. Note first that if $h(n, d_m^n)$ is feasible, then $h'(n, d_m^n)$ is feasible for some positive demand d_n^m of player n. Thus, player n can get a strictly higher payoff than under x_n (this because $\varepsilon < \delta_{\min}$). If instead $h(n, d_m^n)$ is not feasible, then either there exists some positive demand d_n^m for player n such that

$$\sum_{i \in N(h'(n, d_m^n)) \setminus n} \sum_{j : ij \in h'(n, d_m^n))} d_i^j + d_n^m = v(h'(n, d_m^n))$$

or player n could just reciprocate player m's arc and demand her $d_n^m = v(mn) - d_m^n > 0$ (this last inequality again follows from size monotonicity). It follows that it is dominant for n to reciprocate m's arc and get a strictly positive payoff. **QED.**

Proof of Lemma 5. Suppose that $C(g(x)) = \{h_1, \ldots, h_k\}$ with $k > 1$. Let again n be the last player in the ordering ρ. Note first that there must be some component h_p such that $n \notin h_p$, since otherwise the assumption that $k > 1$ would be contradicted. Also, note that by Lemma 4, x being an equilibrium implies that for all $p = 1, \ldots, k$

$$\sum_{i \in N(h_p)} \sum_{j : ij \in h_p} d_i^j = v(h_p).$$

[8] Recall that in game $\Gamma_2(v)$ d_n is a vector, with as many dimensions as the number of arcs sent by n.

Let us then consider h_p and the last player m in $N(h_p)$ according to the ordering ρ. Let $\hat{x}_m\left(d_m^n\right) = \left(a_m \cup a_m^n, d_m \cup d_m^n\right)$, with continuation history $\hat{x}\left(d_m^n\right) = f\left(\lambda_m x, \hat{x}_m\left(d_m^n\right)\right)$. Let $h(n, d_m^n)$ be the component including n in $g(\hat{x}\left(d_m^n\right))$. Suppose first that $mn \notin h(n, d_m^n)$ and $in \in h(n, d_m^n)$ for some $i \in N(h_p)$. Consider then the demand

$$\hat{d}_m^n < \min_{j \in N(h_p)} \left\{d_j^n\right\}.$$

Let now player m play $\hat{d}_m^n$. Suppose that still $in \in N(h(n, d_m^n))$ for some $i \in N(h_p)$. Then it would be a profitable deviation for player n to reciprocate the arc sent by m instead of the arc sent by some other player $i \in N(h_p)$, to which a demand $d_i^n > d_m^n$ is attached.

Suppose now that $in \notin N(h(n, d_m^n))$ for all $i \in N(h_p)$. Let $h'(n, d_m^n)$ be obtained by adding the link mn to $h(n, d_m^n)$. By size monotonicity

$$v\left(h'(n, d_m^n)\right) - v\left(h(n, d_m^n)\right) > 0.$$

Now let

$$\delta_{\min} \equiv \min_{d_m^n \geq 0} \left[v\left(h'(n, d_m^n)\right) - v\left(h(n, d_m^n)\right)\right] > 0.$$

Consider now a demand $0 < d_m^n < \delta_{\min}$. As in the proof of Lemma 4, we claim that it is dominant for player n to reciprocate player m's link and form a feasible component. Note first that, given that $0 < d_m^n < \delta_{\min}$, if $h(n, d_m^n)$ is feasible, then $h(n, d_m^n)$ is feasible for some positive demand d_n^m of player n. If instead $h(n, d_m^n)$ was not feasible, then player n would be getting a zero payoff, and this would be strictly dominated by reciprocating m's arc and getting a payoff of $\left[v\left(h'(n, d_m^n)\right) - v\left(h(n, d_m^n)\right)\right] - d_m^n$, which, again by the fact that $d_m^n < \delta_{\min}$, is strictly positive. **QED.**

Proof of Theorem 3. We proceed by first showing by induction, in step 1, that if a given history is not efficient and satisfies a certain condition on payoff demands, then some player has a profitable deviation. In step 2 we establish that if a history x, leading to an inefficient graph, was SPE, then it would have to satisfy the condition on payoff demands described in step 1, which implies that there exists a profitable deviation from *any* such history x leading to an inefficient graph.

Step 1. Induction Argument.

Induction Hypothesis (H): Let x be an arbitrary history such that $g(x) \notin G^*$. Let m be the first player in the ordering ρ such that there is no x^* such that (1) $\lambda_{m+1}x^* = \lambda_{m+1}x$ and (2) $g(x^*) \in G^*$. Let x be such that

$$\sum_{i=1}^{m} \sum_{j:ij \in N(h(i))} d_i^j \leq v\left(g\left(x\right)\right) - \sum_{i=m+1}^{n} \sum_{j:ij \in N(h(i))} d_i^j.$$

Then there exists some $\varepsilon_m > 0$ such that the action $x_m^* = \left(a_m^*, d_m + \varepsilon_m\right)$ induces a history $\hat{x} = f\left(\lambda_m x, x_m^*\right)$ such that $g(\hat{x}) \in G^*$ and $\sum_{i=1}^{n} \sum_{j:ij \in N(h(i))} \hat{d}_i^j = v(g(\hat{x}))$.

(H) true for player n: Let $x_n = (a_n, d_n)$. By assumption (H), there exists some arcs a_n^* such that $g\left(\lambda_n a, a_n^*\right) \in G^*$ and, therefore, such that $v\left(g\left(\lambda_n a, a_n^*\right)\right) > v(g(x))$. By (H)

$$\sum_{i=1}^{n} \sum_{j:ij \in N(h(i))} d_i^j \leq v(g(x));$$

Moreover, by size monotonicity all players are connected in $g\left(\lambda_n a, a_n^*\right)$.[9] These two facts imply that player n can induce the efficient graph and demand the vector $d_n + \varepsilon_n$, where

$$\sum_{i \in N(g^*):in \in g^*} \varepsilon_n^i = \left[v\left(g\left(\lambda_n a, a_n^*\right)\right) - v(g(x))\right] > 0.$$

(H) true for player $m+1$ implies (H) true for player m: Suppose again that x is an inefficient history and that m is the first player in x such that the action a_m is not compatible with efficiency (in the sense of assumption (H)). Let a_m^* be some vector of arcs compatible with efficiency and let $x_m^*(\varepsilon) = \left(a_m^*, d_m + \varepsilon\right)$. Let $x^*(\varepsilon) \equiv f\left(\lambda_m x, x_m^*(\varepsilon)\right)$ represent the relative continuation history. We need to show that there exists $\varepsilon > 0$ such that $g(x^*(\varepsilon)) \in G^*$. Note first that in the history $x^*(\varepsilon)$ the first player k such that a_k is not compatible with efficiency must be such that $k > m$. Also, since by (H)

$$\sum_{i=1}^{m} \sum_{j:ij \in N(h(i))} d_i^j \leq v(g(x)) - \sum_{i=m+1}^{n} \sum_{j:ij \in N(h(i))} d_i^j$$

there exists an $\varepsilon_m > 0$ such that

$$\sum_{i=1}^{m-1} \sum_{j:ij \in N(h(i))} d_i^j + \sum_{j:mj \in N(h(m))} \left(d_m^j + \varepsilon_m\right) < v\left(g^*\right) - \sum_{i=m+1}^{n} \sum_{j:ij \in N(h(i))} d_i^j.$$

Thus, if player m plays $x_m^*(\varepsilon_m)$, player $m+1$ faces a history $\left(\lambda_m x, x_m^*(\varepsilon_m)\right)$ that satisfies the inductive assumption (H). Suppose now that player $m+1$ optimally plays some action x_{m+1} such that no efficient graph is compatible (in the sense of assumption (H)) with the history $\left(\lambda_m x, x_m^*(\varepsilon_m), x_{m+1}\right)$. Then, by (H) we know there would be a deviation for player $m+1$, contradicting the assumption that x_{m+1} is part of the continuation history at $\left(\lambda_m x, x_m^*(\varepsilon_m)\right)$. Thus, we know that player $m+1$ will optimally play some strategy x_{m+1}^* such that the continuation history $f\left(\left(\lambda_m x, x_m^*(\varepsilon_m), x_{m+1}^*\right)\right)$ induces a feasible efficient graph.

Step 2. We now show that the induction argument can be applied to each SPE history x of $\Gamma_2(v)$ such that $g(x) \notin G^*$. This is shown to imply that the first player m such that there is no x^* such that $\lambda_{m+1}x^* = \lambda_{m+1}x$ and $g(x^*) \in G^*$ has a profitable deviation.

[9] $\lambda_i a$ constitutes a slight abuse of notation, describing the history of arcs sent before the turn of player i.

Note first that by Lemma 5 if x is a SPE history then all players are connected. This, together with Lemma 4, directly implies that

$$\sum_{i=1}^{n} \sum_{j:ij\in N(h(i))} d_i^j = v\left(g\left(x\right)\right)$$

or, equivalently, that

$$\sum_{i=1}^{m} \sum_{j:ij\in N(h(i))} d_i^j = v\left(g\left(x\right)\right) - \sum_{i=m+1}^{n} \sum_{j:ij\in N(h(i))} d_i^j$$

for all $m = 1, \ldots, n$. It follows that the induction argument can be applied to all inefficient SPE histories, to conclude that the first player whose action is not compatible with efficiency in the sense of (H), has some action $x_m^*\left(\varepsilon_m\right) = \left(a_m^*, d_m + \varepsilon_m\right)$ such that $\varepsilon_m > 0$ and such that the induced graph $g\left(f\left(\lambda_m x, x_m^*(\varepsilon_m)\right)\right) \in G^*$ is feasible. Since $g\left(f\left(\lambda_m x, x_m^*\right)\right)$ is feasible, then the action $x_m^*\left(\varepsilon_m\right)$ represents a deviation for player m, proving the theorem. **QED.**

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