Modelling Returns

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Modelling returns

The (naive) log random walk (LRW) hypothesis on the evolution of prices states that, prices evolve approximately according to the stochastic difference equation:

$$\ln P_t = \mu\Delta + \ln P_{t-\Delta} + \epsilon_t$$

where the 'innovations' ϵ_t are assumed to be uncorrelated across time ($cov(\epsilon_t; \epsilon_{t'}) = 0 \quad \forall t \neq t'$), with constant expected value 0 and constant variance $\sigma^2\Delta$.
Consider what happens over a time span of, say, 2Δ.

$$\ln P_t = 2\mu\Delta + \ln P_{t-2\Delta} + \epsilon_t + \epsilon_{t-\Delta} = \ln P_{t-2\Delta} + u_t$$

having set $u_t = \epsilon_t + \epsilon_{t-\Delta}$.

Modelling returns

Consider now the case in which the time interval is of the length of 1-period. If we take prices as inclusive of dividends we can write the following model for log-returns

$$r_{t,t+1} = \mu + \sigma\epsilon_t$$
$$\epsilon_t = i.i.d.(0,1)$$

$$E(r_{t,t+n}) = E(\sum_{i=1}^{n} r_{t+i,t+i-1}) = \sum_{i=1}^{n} E(r_{t+i,t+i-1}) = n\mu$$

$$Var(r_{t,t+n}) = Var(\sum_{i=1}^{n} r_{t+i,t+i-1}) = \sum_{i=1}^{n} Var(r_{t+i,t+i-1}) = n\sigma^2$$

Monte-Carlo simulation

- given some estimates of the unknown parameters in the model (μ σ in our case).
- an assumption is made on the distribution of ϵ_t.
- The an artificial sample for ϵ_t of the length matching that of the available can be computer simulated.
- The simulated residuals are then mapped into simulated returns via μ, σ.
- This exercise can be replicated N times (and therefore a Monte-Carlo simulation generates a matrix of computer simulated returns whose dimension are defined by the sample size T and by the number of replications N).
- The distribution of model predicted returns can be then constructed and one can ask the question if the observed data can be considered as one draw from this distribution.

do exactly like in Monte-Carlo but rather than using a theoretical distribution for ϵ_t use their empirical distribution and resample from it with reimmission.

Simulation can be used for several tasks,

- provide statistical evidence of the capability of the model to replicate the data
- derive the distribution of returns to implement VaR
- assess statistical properties of estimators

Stocks for the long-run

The fact that, under the LRW, the expected value grows linearly with the length of the time period while the standard deviation (square root of the variance) grows with the square root of the number of observations, has created a lot of discussion

We have three flavors of the “stocks for the long run” argument. The first and the second are a priori arguments depending on the log random walk hypothesis or something equivalent to it, the third is an a posteriori argument based on historical data.

Stocks for the long-run

Assume that single period (log) returns have (positive) expected value μ and variance σ^2. Moreover, assume for simplicity that the investor requires a Sharpe ratio of say S out of his-her investment. Under the above hypotheses, plus the log random walk hypothesis, the Sharpe ratio over n time periods is given by

$$S = \frac{n\mu}{\sqrt{n}\sigma} = \sqrt{n}\frac{\mu}{\sigma}$$

so that, if n is large enough, any required value can be reached.

Stocks for the long-run

Another way of phrasing the same argument, when we add the hypothesis of normality on returns, is that, there any given probability α and any given required return C there is always an horizon for which the probability for n period return less than C is less than α.

$$\Pr\left(R^p < C\right) = \alpha.$$

$$\Pr\left(R^p < C\right) = \alpha \Longleftrightarrow \Pr\left(\frac{R^p - n\mu}{\sqrt{n}\sigma} < \frac{C - n\mu}{\sqrt{n}\sigma}\right) = \alpha$$

$$\Longleftrightarrow \Phi\left(\frac{C-\mu_p}{\sigma_p}\right) = \alpha,$$

$$C = n\mu + \Phi^{-1}\left(\alpha\right)\sqrt{n}\sigma$$

But $n\mu + \Phi^{-1}\left(\alpha\right)\sqrt{n}\sigma$, for $\sqrt{n} > \frac{1}{2}\frac{\Phi^{-1}(\alpha)}{\mu}\sigma$ is an increasing function in n so that for any α and any chosen value C, there exists a n such that from that n onward, the probability for an n period return less than C is less than α.

Stocks for the long-run

Note, however, that the value of n for which this lower bound crosses a given C level is the solution of

$$n\mu + \Phi^{-1}(\alpha)\sqrt{n}\sigma \geq C$$

In particular, for $C = 0$ the solution is

$$\sqrt{n} \geq -\frac{\Phi^{-1}(\alpha)\,\sigma}{\mu}$$

Consider now the case of a stock with σ/μ ratio for one year is of the order of 6. Even allowing for a large α,say 0.25, so that $\Phi^{-1}(\alpha)$ is near minus one , the required n shall be in the range of 36 which is only slightly shorter than the average working life.
As a matter of fact, based on the analysis of historical prices and risk adjusted returns, stocks have been almost always a good long run investment.