Stock and Bonds Returns. An Introduction

Carlo Favero

Returns

Consider an asset that does not pay any intermediate cash income (a zero-coupon bons, such as a Treasury Bill, or a share in a company that pays no dividends). Let P_t be the price of the security at time t.
The **linear or simple** return between times t and $t-1$ is defined as:

$$R_t = P_t / P_{t-1} - 1$$

The **log** return is defined as:

$$r_t = \ln(P_t / P_{t-1})$$

Note that, while P_t means "price at time t", r_t is a a shorthand for "return between time $t-1$ and t"

Returns

The two definitions of return yield different numbers when the ratio between consecutive prices is far from 1.
Consider the Taylor formula for $\ln(x)$ for x in the neighbourhood of 1:

$$\ln(x) = \ln(1) + (x-1)/1 - (x-1)^2/2 + ...$$

if we truncate the series at the first order term we have:

$$\ln(x) \cong 0 + x - 1$$

so that if x is the ratio between consecutive prices, then for x close to one the two definitions give similar values. Note however that $\ln(x) \leq x - 1$. In fact $x - 1$ is equal to and tangent to $\ln(x)$ in $x = 1$ and above it anywhere else.

Multi-period returns

Define the simple multi-period return between time t and t+n as:

$$
\begin{aligned}
R_{t,t+n} &= P_{t+n}/P_t - 1 \\
&= \frac{P_{t+n}}{P_{t+n-1}}\frac{P_{t+n-1}}{P_{t+n-2}}...\frac{P_{t+1}}{P_t} - 1 \\
&= \prod_{i=1}^{n}(1 + R_{t+i,t+i-1}) - 1
\end{aligned} \tag{1}
$$

in the case of log returns we have instead:

$$
\begin{aligned}
r_{t,t+n} &= \ln\left(P_{t+n}/P_t\right) \\
&= \ln\left(\frac{P_{t+n}}{P_{t+n-1}}\frac{P_{t+n-1}}{P_{t+n-2}}...\frac{P_{t+1}}{P_t}\right) \\
&= \sum_{i=1}^{n} r_{t+i,t+i-1}
\end{aligned} \tag{2}
$$

Annualized returns

annualized returns the constant annual rate of return equivalent to the multiperiod returns to an of an investment in asset i over the period t, t+n. In the case of simple returns we have

$$\left(1+R^{A}_{t,t+n}\right)^{n} = 1+R_{t,t+n} = \prod_{i=1}^{n}\left(1+R_{t+i,t+i-1}\right)$$

$$R^{A}_{t,t+n} = \left(\prod_{i=1}^{n}\left(1+R_{t+i,t+i-1}\right)\right)^{\frac{1}{n}} - 1$$

Consider now log returns:

$$nr^{A}_{t,t+n} = r_{t,t+n} = \sum_{i=1}^{n} r_{t+i,t+i-1}$$

$$r^{A}_{t,t+n} = \frac{1}{n}\sum_{i=1}^{n} r_{t+i,t+i-1}$$

Working with Returns

Consider the value of a buy and hold portfolio of invested in shares of k different companies, that pay no dividend, at time t be:

$$V_t = \sum_{i=1}^{k} n_i P_{it}$$

The simple one-period return of the portfolio shall be a linear function of the returns of each stock.

$$R_t = \frac{V_t}{V_{t-1}} - 1 = \sum_{i=1..k} \frac{n_i P_{it}}{\sum_{j=1..k} n_j P_{jt-1}} - 1$$

$$= \sum_{i=1..k} \frac{n_i P_{it-1}}{\sum_{j=1..k} n_j P_{jt-1}} \frac{P_{it}}{P_{it-1}} - 1 =$$

$$= \sum_{i=1..k} w_{it}(R_{it} + 1) - 1 = (\sum_{i=1..k} w_{it} R_{it} + \sum_{i=1..k} w_{it} 1) - 1 = \sum_{i=1}^{k} w_{it} R_{it}$$

Working with Returns

log returns are not additive in the cross-section but they are additive when we consider the time-series of returns

$$r_t = \ln(\frac{V_t}{V_{t-1}})$$
$$= \ln(\frac{\sum_{i=1}^{k} n_i P_{it-1}}{\sum_{i=1}^{k} n_i P_{it-1}} \frac{P_{it}}{P_{it-1}}) = \ln(\sum_{i=1}^{k} w_{it} \exp(r_{it}))$$

$$r_{t,t+n} = \sum_{i=1}^{n} r_{t+i,t+i-1}$$

Note that additivity in the time-series does not apply to simple returns.

Stock Returns and the dynamic dividend growth model

consider the one-period total holding returns in the stock market, that are defined as follows:

$$H_{t+1}^{s} \equiv \frac{P_{t+1} + D_{t+1}}{P_t} - 1 = \frac{P_{t+1} - P_t + D_{t+1}}{P_t} = \frac{\Delta P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t}, \qquad (3)$$

Dividing both sides by $\left(1 + H_{t+1}^{s}\right)$ and multiplying both sides by P_t / D_t we have:

$$\frac{P_t}{D_t} = \frac{1}{\left(1 + H_{t+1}^{s}\right)} \frac{D_{t+1}}{D_t} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right).$$

Taking logs we have:

$$p_t - d_t = -r_{t+1}^{s} + \Delta d_{t+1} + \ln\left(1 + e^{p_{t+1} - d_{t+1}}\right)$$

Stock Returns and the dynamic dividend growth model

Taking a first-order Taylor expansion of the last term about the point $\bar{P}/\bar{D} = e^{\bar{p}-\bar{d}}$:

$$\begin{aligned}\ln\left(1+e^{p_{t+1}-d_{t+1}}\right) &\simeq \ln(1+e^{\bar{p}-\bar{d}}) + \frac{e^{\bar{p}-\bar{d}}}{1+e^{\bar{p}-\bar{d}}}[(p_{t+1}-d_{t+1})-(\bar{p}-\bar{d})] \\ &= -\ln(1-\rho)-\rho\ln\left(\frac{1}{1-\rho}-1\right)+\rho(p_{t+1}-d_{t+1}) \\ &= \kappa+\rho(p_{t+1}-d_{t+1})\end{aligned}$$

where

$$\rho \equiv \frac{e^{\bar{p}-\bar{d}}}{1+e^{\bar{p}-\bar{d}}} = \frac{\bar{P}/\bar{D}}{1+(\bar{P}/\bar{D})} < 1 \qquad \kappa \equiv -\ln(1-\rho)-\rho\ln\left(\frac{1}{1-\rho}-1\right).$$

Total stock market returns can then be written as:

$$r^s_{t+1} = \kappa+\rho\left(p_{t+1}-d_{t+1}\right)+\Delta d_{t+1}-\left(p_t-d_t\right),$$

Stock Returns and the dynamic dividend growth model

By *forward* recursive substitution one obtains:

$$(p_t - d_t) = \kappa \sum_{j=1}^{m} \rho^{j-1} + \sum_{j=1}^{m} \rho^{j-1} (\Delta d_{t+j} - r^s_{t+j}) + \rho^m (p_{t+m} - d_{t+m}) .$$

Under the assumption that there can be no rational bubbles, i.e., that

$$\lim_{m \longrightarrow \infty} \rho^m (p_{t+m} - d_{t+m}) = 0,$$

$$(p_t - d_t) = \frac{\kappa}{1-\rho} + \sum_{j=1}^{m} \rho^{j-1} \left(\Delta d_{t+j} - r^s_{t+j} \right) .$$

Zero-Coupon Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{\left(1 + Y_{t,T}\right)^{T-t}},$$

Taking logs of the left and the right-hand sides of the expression for $P_{t,T}$, and defining the continuously compounded *yield*, $y_{t,T}$, as $\log(1 + Y_{t,T})$, we have the following relationship:

$$p_{t,T} = -\left(T - t\right) y_{t,T},$$

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$\begin{aligned} r^T_{t,t+1} &\equiv p_{t+1,T} - p_{t,T} = -\left(T - t - 1\right) y_{t+1,T} + \left(T - t\right) y_{t,T} \\ &= y_{t,T} - \left(T - t - 1\right)\left(y_{t+1,T} - y_{t,T}\right), \\ &= \left(T - t\right) y_{t,T} - \left(T - t - 1\right)\left(y_{t+1,T}\right) \end{aligned}$$

Coupon Bonds

The relationship between price and yield to maturity of a constant coupon (C) bond is given by:

$$P^c_{t,T} = \frac{C}{\left(1+Y^c_{t,T}\right)} + \frac{C}{\left(1+Y^c_{t,T}\right)^2} + \dots + \frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}.$$

To measure the length of time that a bondholder has invested money for we need to introduce the concept of duration:

$$D^c_{t,T} = \frac{\frac{C}{\left(1+Y^c_{t,T}\right)} + 2\frac{C}{\left(1+Y^c_{t,T}\right)^2} + \dots + (T-t)\frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}}{P^c_{t,T}}$$

$$= \frac{C\sum_{i=1}^{T-t}\frac{i}{\left(1+Y^c_{t,T}\right)^i} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}}{P^c_{t,T}}.$$

Coupon Bonds

The one-period return on a coupon bond can be approximated as follows:

$$r^c_{t+1} = D^c_{t,T} y^c_{t,T} - \left(D^c_{t,T} - 1\right) y^c_{t+1,T},$$

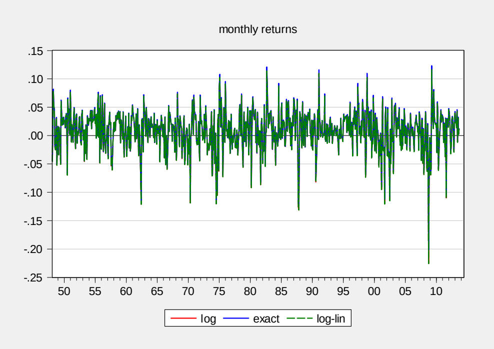
where in turn the duration can be computed as

$$D^c_{t,T} = \frac{1 - \left(1 + Y^c_{t,T}\right)^{-(T-t)}}{1 - \left(1 + Y^c_{t,T}\right)^{-1}},$$

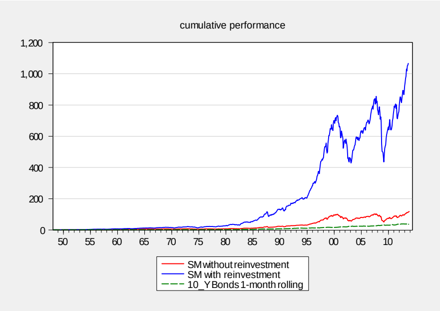
Graphics

- time series graphics of returns (change of prices)
- time series graphics of portfolio perfomance (level of prices)
- scatter plots
- multiple graphs
- density estimates (histograms)

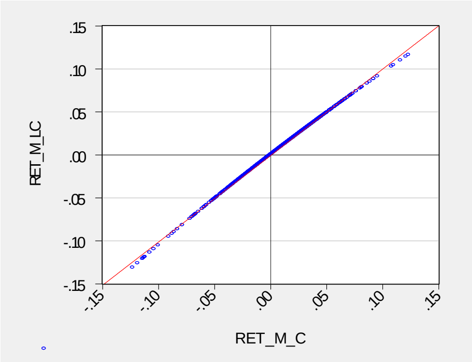
Time-series graphs (returns)



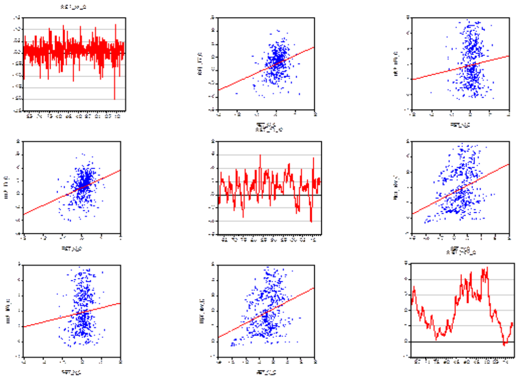
Time-series graphs (portfolios)



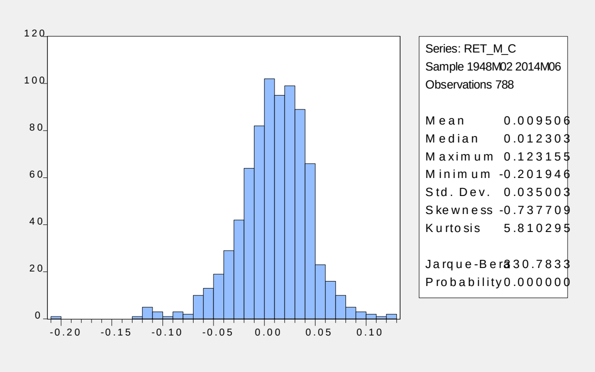
Scatter-plots



Multiple-graphs



Histograms


120
100
80
60
40
20
0
-0.20
-0.15
-0.10
-0.05
0.00
0.05
0.10
Series: RET_M_C
Sample 1948M02 2014M06
Observations 788
Mean 0.009506
Median 0.012303
Maximum 0.123155
Minimum -0.201946
Std. Dev. 0.035003
Skewness -0.737709
Kurtosis 5.810295
Jarque-Bera 330.7833
Probability 0.000000

Matrix Representation of the data

A matrix is a double array of i rows and j columns, whose generic element can be written as a_{ij}, it is a convenient way of collecting simultaneusly information on the time-series and the cross-section of returns:

$$A = \begin{bmatrix} a_{11} & . & . & a_{1j} \\ & & & \\ a_{i1} & & & a_{ij} \end{bmatrix}, 0 = \begin{bmatrix} 0 & . & . & 0 \\ & & & \\ 0 & & & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & . & . & 0 \\ & & & \\ 0 & & & 1 \end{bmatrix}$$

Matrix Operations

- Transposition $a'_{ij} = a_{ji}$
- Addition: For A and B nxm $(a+b)_{ij} = a_{ij} + b_{ij}$
- Multiplication: For A nxm and B mxp $(ab)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$
- Inversion for non-singular A nxn, A^{-1} satisfies $A^{-1}A = AA^{-1} = I$

A Static Asset Allocation Problem with Constant Expected Returns

Let's denote with $\mathbf{r}$ the random vector of linear total returns from time t to time T from a given menu of N risky assets for interval $[t, T]$,
$\mathbf{r} \sim \mathcal{D}(\boldsymbol{\mu}, \Sigma)$
Given a degree of risk aversion λ, a standard *mean-variance* description of this allocation problem is the following:

$$\max_{\mathbf{w}} \left(1 - \mathbf{w}'\mathbf{e}\right) r^f + \mathbf{w}'\boldsymbol{\mu} - \frac{1}{2}\lambda(\mathbf{w}'\Sigma\mathbf{w})$$

where $E[\mathbf{r}] = \left(1 - \mathbf{w}'\mathbf{e}\right) r^f + \mathbf{w}'\boldsymbol{\mu} = r^f + \mathbf{w}'\left(\boldsymbol{\mu} - r^f\mathbf{e}\right)$ and $Var[\mathbf{r}] = \mathbf{w}'\Sigma\mathbf{w}$.

A Static Asset Allocation Problem with Constant Expected Returns

first-order conditions (FOCs) are necessary and sufficient and define the following system of N linear equations in N unknowns, the portfolio weights $\mathbf{w} \in \mathcal{R}^N$:

$$(\boldsymbol{\mu} - r^f \mathbf{e}) - \lambda \Sigma \mathbf{w} = \mathbf{0}.$$

Solving the FOCs yields:

$$\mathbf{w} = \frac{1}{\lambda} \Sigma^{-1} \left(\boldsymbol{\mu} - r^f \mathbf{e} \right),$$

Consider now the special case in which $\mathbf{w}'\mathbf{e} = 1$, that is no investment in the riskfree bond is allowed. The optimal portfolio in this case is the famous *tangency portfolio*:

$$\mathbf{e}'\mathbf{w} = \frac{1}{\lambda} \mathbf{e}' \Sigma^{-1} \left(\boldsymbol{\mu} - r^f \mathbf{e} \right) = 1 \Longrightarrow \lambda = \mathbf{e}' \Sigma^{-1} \left(\boldsymbol{\mu} - r^f \mathbf{e} \right)$$

$$\mathbf{w}^T = \frac{\Sigma^{-1} \left(\boldsymbol{\mu} - r^f \mathbf{e} \right)}{\mathbf{e}' \Sigma^{-1} \left(\boldsymbol{\mu} - r^f \mathbf{e} \right)},$$

A Static Asset Allocation Problem with Constant Expected Returns

(1) The weights in the tangency portfolio do not depend on the risk aversion parameter λ.
(2) Given that the optimal risky portfolio is uniquely determined, the tangency portfolio must then coincide with the market portfolio. Agents maximize their utility by taking a linear combination of the market portfolio and and the risk-free securities. Note that in this case we can express the return on any portfolio in the following way:

$$\begin{aligned} r^p &= (1-\beta)\, r^f + \beta r^M \\ r^p - r^f &= \beta \left(r^p - r^M \right) \end{aligned}$$

and the CAPM holds.

A Static Asset Allocation Problem with Constant Expected Returns

(3) Efficient portfolios are those with the highest expected return for a given level of risk.If we summarize the expected return-risk (mean-variance) properties of the feasible portfolios in a plot with portfolio expected return, $\boldsymbol{\mu}^P$, on the vertical axis and portfolio standard-deviation, σ^P, on the horizontal axis, then all efficient portfolios can be represented as points in the space $\left(\sigma^P, \boldsymbol{\mu}^P\right)$ and the efficient frontier is the line that connects all these points. given the properties of the tangency portfolios, weights for all portfolios on the efficient frontier are obtained by inputing different values for the risk-free rate in the expression for optimal weights
(4) the CER implies that the tangency portfolios and the efficient frontier do not depend on the horizon at which returns are defined.