Macroeconomics Sequence, Block I

Introduction to Consumption Asset Pricing

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October 21, 2016

The Lucas' Tree Model

- This is a 'general equilibrium model' where instead of deriving properties of the allocation given the prices, Lucas uses the fundamentals to derive prices.
- We will not analyze the general equilibrium aspects of this model (next sequence of lectures)
- We can still derive testable restrictions on prices without computing the level of prices (e.g., arbitrage)
  $\Rightarrow$ Consumption CAPM
- Such Predictions are not always supported by the data (e.g. the Equity Premium Puzzle)

The Lucas' Tree Model II

- Consider a large number of identical consumers, with v-M-N utility.
- Each agent owns shares of $k \geq 1$ productive forever lasting assets in fixed supply (the trees).
- All tree are identical and produce random quantities $\{d_t\}$ of a single perishable consumption in all time periods (the dividends). We can consider different type of trees.
- There is hence a common shock driving the dividends, which is uninsurable (aggregate uncertainty)
- Agents can trade one bond and one risky asset

$$\max_{\{c_t, b_{t+1}, s_{t+1}\}} \quad \mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t.} \qquad \text{NPGC and}$$

$$c_t + \frac{b_{t+1}}{1+r_t} + p_t s_{t+1} \leq b_t + (p_t + d_t)\, s_t; \; s_0, \; b_0 \text{ given.}$$

Pricing Assets: The Stochastic Discount Factor

- From the first order conditions we obtain:

$$
\begin{aligned}
u'(c_t^*) &= \mathbf{E}_t \left[\beta(1+r_t)u'(c_{t+1}^*)\right] \\
p_t u'(c_t^*) &= \mathbf{E}_t \left[\beta\left(p_{t+1}+d_{t+1}\right)u'(c_{t+1}^*)\right] \\
\Rightarrow p_t &= \mathbf{E}_t \left[\left(p_{t+1}+d_{t+1}\right)\frac{\beta u'(c_{t+1}^*)}{u'(c_t^*)}\right] = \mathbf{E}_t \left[\left(p_{t+1}+d_{t+1}\right) SDF_{t+1}\right] \\
&= \mathbf{E}_t \left(p_{t+1}+d_{t+1}\right)\mathbf{E}_t \frac{\beta u'(c_{t+1}^*)}{u'(c_t^*)} + Cov_t\left(p_{t+1}+d_{t+1}, \frac{\beta u'(c_{t+1}^*)}{u'(c_t^*)}\right)
\end{aligned}
$$

- SDF= Stochastic discount factor or ‘pricing kernel’. Same factor used to price all assets. A risk ‘irrelevant’ if not correlated to SDF

- Define the return of the risky asset: $\frac{p_{t+1}+d_{t+1}}{p_t} = 1+\rho_{t+1}$, we get

$$
\begin{aligned}
\mathbf{E}_t \left[(1+\rho_{t+1})\frac{\beta u'(c_{t+1}^*)}{u'(c_t^*)}\right] &= \mathbf{E}_t \left[(1+r_t)\frac{\beta u'(c_{t+1}^*)}{u'(c_t^*)}\right] = 1 \\
\Rightarrow \quad & \mathbf{E}_t \left[(\rho_{t+1}-r_t) SDF_{t+1}\right] = 0.
\end{aligned}
$$

Asset Pricing Models

- Recall the condition

$$p_t = \mathbf{E}_t \left[\left(p_{t+1} + d_{t+1} \right) SDF_{t+1} \right]$$

or the condition:

$$\mathbf{E}_t \left[(\rho_{t+1} - r_t) SDF_{t+1} \right] = 0$$

- All asset pricing models are based on 'some' SDF.
- All asset pricing models amount to alternative ways to connecting the stochastic discount factor to the data
- Different manipulations to the two above conditions will stress different implications of the theory, and suggest different empirical strategies to bring the model to the data
- The Consumption Capital Asset Pricing Model (CCAPM), in its basic form, postulates the presence of a representative agent and uses its MRS as SDF

Consumption CAPM and the Security Market Line I

- Since r_t is known at t: the previous first order condition can be rewritten as

$$\mathbf{E}_t \rho_{t+1} = r_t + \frac{Cov_t\left(\rho_{t+1}, SDF_{t+1}\right)}{Var_t(SDF_{t+1})}\left(-\frac{Var_t(SDF_{t+1})}{\mathbf{E}_t(SDF_{t+1})}\right)$$

- Call 'the market' (m) an asset which is perfectly negatively correlated with marginal utility, i.e. $u'(c^*_{t+1}) = -\nu m_{t+1}$.
- Intuitively, this is what drives the movements in aggregate consumption (aggregate shocks)
- Clearly:

$$\mathbf{E}_t\left[SDF_{t+1}\right] = -\nu \frac{\beta}{u'(c^*_t)} \mathbf{E}_t\left[m_{t+1}\right]$$

and hence

$$\frac{Cov_t\left(\rho_{t+1},\ SDF_{t+1}\right)}{\mathbf{E}_t\left[SDF_{t+1}\right]} = \frac{Cov_t(\rho_{t+1}, m_{t+1})}{\mathbf{E}_t\left[m_{t+1}\right]}.$$

Consumption CAPM and the Security Market Line II

From the FOC we hence have:

$$
\begin{aligned}
(1) \qquad \mathbf{E}_t m_{t+1} &= r_t + \frac{Cov_t\left(m_{t+1}, SDF_{t+1}\right)}{Var_t(SDF_{t+1})}\left(-\frac{Var_t(SDF_{t+1})}{\mathbf{E}_t(SDF_{t+1})}\right) \\
&= r_t + \frac{Cov_t\left(m_{t+1}, m_{t+1}\right)}{Var_t(m_{t+1})}\left(\frac{Var_t(m_{t+1})}{\mathbf{E}_t(m_{t+1})}\right) \\
(2) \qquad \mathbf{E}_t \rho_{t+1} &= r_t + \frac{Cov_t\left(\rho_{t+1}, m_{t+1}\right)}{Var_t(m_{t+1})}\left(\frac{Var_t(m_{t+1})}{\mathbf{E}_t(m_{t+1})}\right) \\
(1)+(2) \Rightarrow \mathbf{E}_t \rho_{t+1} - r_t &= \frac{Cov_t(\rho_{t+1}, m_{t+1})}{Var_t\left(m_{t+1}\right)}\left(\mathbf{E}_t m_{t+1} - r_t\right)
\end{aligned}
$$

Where $\beta^i := \frac{Cov_t(\rho^i_{t+1}, m_{t+1})}{Var_t(m_{t+1})}$ is called 'the Beta' of the asset ρ^i, since it would be the coefficient of the regression of the excess return of asset: $\rho^i_{t+1} - r_t$ on the excess return of the market: $m_{t+1} - r_t$.

Test to CAPM model: identify an asset m that can act as 'the market' and test if all assets in the market can be 'beta-priced'.

The Security Market Line (SML)

$$\mathbf{E}\rho^i = r + \beta^i (\mathbf{E}m - r) + \alpha^i$$

- The slope of the SML (the risk premium) is $\mathbf{E}m - r$.
- β^i is the non-diversifiable risk associated to the asset i
- α^i is an ‘abnormal’ (compared to the CAPM) return

Figure:

Mean-Standard Deviation Frontier I

Recall again that the pricing formulas imply:

$$0 = E_t[SDF_{t+1}(\rho_{t+1} - r_t)].$$

Hence

$$\mathbf{E}_t\left[\rho_{t+1} - r_t\right]\mathbf{E}_t\left[SDF_{t+1}\right] = -Corr_t(\rho_{t+1}, SDF_{t+1})\sigma_t(\rho_{t+1})\sigma_t(SDF_{t+1}),$$

where $\sigma_t(\cdot)$ is the standard deviation operator. Since $|Corr_t(\rho_{t+1}, SDF_{t+1})| \leq 1$ mean and variance of each asset obey:

$$\left|\mathbf{E}_t\left[\rho_{t+1} - r_t\right]\right| \leq \frac{\sigma_t(SDF_{t+1})}{\mathbf{E}_t\left[SDF_{t+1}\right]}\sigma_t(\rho_{t+1}).$$

The mean-standard deviation frontier is the set of returns satisfying the above condition with equality.

The slope of the mean-standard deviation frontier depends on the (normalized) volatility of the SDF: $\frac{\sigma_t(SDF_{t+1})}{\mathbf{E}_t[SDF_{t+1}]}$ (coeff. of variation)

Mean-Standard Deviation Frontier II

- It can be describe for any set of assets
- When we have a risk free asset the frontier is linear, in general is a hyperbolic curve

Figure

Sharpe Ratios

Previous condition based on the SDF, which is not observable.

The Sharpe Ratio is the ration between the mean of the excess return (with respect to r) to standard deviation of an asset

$$\frac{\mathbf{E}_t \rho_{t+1} - r_t}{\sigma_t(\rho_{t+1})}.$$

From the condition we derived above in the beta model, we have

$$\frac{\mathbf{E}_t \rho_{t+1} - r_t}{\sigma_t(\rho_{t+1})} = -\frac{Corr_t\left(\rho_{t+1}, SDF_{t+1}\right)}{\mathbf{E}_t(SDF_{t+1})}\sigma_t(SDF_{t+1}) = -\frac{Corr_t\left(\rho_{t+1}, m_{t+1}\right)}{\mathbf{E}_t(m_{t+1})}\sigma_t(m_{t+1})$$

with m the 'market' portfolio.

Since $-Corr_t\left(\rho_{t+1}, SDF_{t+1}\right) \leq 1$, the slope of the mean-standard deviation frontier $\frac{\sigma_t(SDF_{t+1})}{\mathbf{E}_t[SDF_{t+1}]}$ is the largest available Sharpe ratio.

Pricing assets with no Bubbles

Finally, we focus on equilibria such that some asymptotic conditions on prices are satisfied and get

$$p_t = \mathbf{E}_t \left[\sum_{n=0}^{\infty} \frac{\beta^n u'(c^*_{t+n})}{u'(c^*_t)} d_{t+n} \right]$$

Now use the law of iterate expectations and obtain

$$\frac{1}{1 + r_{t+n}} = \mathbf{E}_t \frac{\beta u'(c^*_{t+n+1})}{u'(c^*_{t+n})}$$

we get

$$p_t = \mathbf{E}_t \left[\sum_{n=0}^{\infty} \frac{d_{t+n}}{\prod_{k=0}^{n} (1 + r_{t+k})} \right]$$

When $r_t = r$ we obtain

$$p_t = \mathbf{E}_t \left[\sum_{n=0}^{\infty} \frac{d_{t+n}}{(1 + r)^n} \right]$$

The Equity Premium Puzzle I

Assume CRRA (evidence is for DARA)

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \beta = \frac{1}{1+\theta}.$$

Time dimension : Intertemporal Substitution

Euler's equation for Bond imposes an upper bound for γ

$$\begin{aligned} 1 = \beta(1+r)\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma} &= \frac{1+r}{1+\theta}\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma} \\ &\simeq 1 + r - \theta - \gamma g_c \end{aligned}$$

$$\text{or} \qquad r \simeq \theta + \gamma g_c$$

The remuneration for waiting (r) should compensate impatience (θ) and the lack of intertemporal consumption smoothing (γg_c)

With $r = 5\%$, $g = 2\%$, and θ should be positive (impatience).

Then γ should be less than 2.5 (risk-free rate puzzle)

Recall precautionary savings! But $\sigma_c^2 \approx (0.01)^2$

The Equity Premium Puzzle II: Risk dimension

Recall the first order conditions

$$\mathbf{E}_t \left[(\rho_{t+1} - r_t) \left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right] = 0$$

or

$$\mathbf{E}_t \rho_{t+1} = r_t + \gamma Cov_t \left(\rho_{t+1}, g_c \right)$$

- Mehra and Prescott (1985) found that in the data the premium between stock and bond was 6% (Short-Term Debt (T-Bil) $\mathbf{E} r_t = 1\%$ and Stock Exchange Index (NYSE) $\mathbf{E} \rho_t = 7\%$ period: 1889-1978). With Covariance between g and ρ of a bit more than 0.002, γ should be at least 25. (Correlation puzzle)
- The equity premium puzzle is sometimes interpreted as a failure of the complete market model.
- Incomplete markets? (Kruger and Lustig, 2006).
- Perhaps Kreps-Porteus preferences or other non-separabilities?

Lucas' Objection

No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it.

The Hansen and Jagannathan (1991) Bound

- Recall again, the formula for pricing one-period assets:

  $$p_t = \boldsymbol{E}_t[SDF_{t+1}(p_{t+1} + d_{t+1})] \iff 0 = \boldsymbol{E}_t[SDF_{t+1}(\rho_{t+1} - r_t)].$$

  where SDF_{t+1} is the one-period pricing kernel or SDF, and $\rho_{t+1} - r_t$ the excess return of the asset.
- For CRRA preferences, SDF_{t+1} is:

  $$SDF_{t+1} = \beta \left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma}$$

- The Risk Free rate is the conditional mean of the SDF:

  $$\frac{1}{1 + r_t} = \boldsymbol{E}_t[SDF_{t+1}] = \boldsymbol{E}_t \left[\beta \left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right]$$

  HJ bound the set of SDF that can price a set I of assets:

$$\forall i \in I \quad \frac{|\boldsymbol{E}_t[\rho^i_{t+1} - r_t]|}{\sigma_t(\rho^i_{t+1})} \leq \frac{\sigma_t(SDF_{t+1})}{\boldsymbol{E}_t[SDF_{t+1}]} \approx \gamma \sigma_t(\Delta \ln c^*_{t+1}) \qquad \text{(HJ)}$$

Derivation of HJ Bounds set

- They derive the mean variance frontier for discount factors
- Suppose we do not have r_t and not have or know the SDF
- This procedure allows for incomplete markets
- If we knew the SDF, we would have

$$\frac{1}{\boldsymbol{E}_t[SDF_{t+1}]} = 1 + r_t.$$

- Recall that we can trace the frontier using Sharpe ratios
- For each r_t we can compute the largest Sharpe ratios

$$\hat{R}\left(r_t\right) := \max_{\rho^i \text{ traded}} \frac{|\boldsymbol{E}_t[\rho^i_{t+1} - r_t]|}{\sigma_t(\rho^i_{t+1})}.$$

- This interest rate will be interpreted as the result of $r_t = \frac{1}{\boldsymbol{E}_t[m_{t+1}]} - 1$ for a hypothetical SDF m_{t+1}.
- To be a valid SDF for I, from (HJ), its variance must solve

$$\sigma(m_{t+1}) \geq \hat{\sigma}(m_{t+1}) := \hat{R}\left(\frac{1}{\boldsymbol{E}_t[m_{t+1}]} - 1\right) \boldsymbol{E}_t[m_{t+1}].$$

Figure: Slope in Mean-variance Frontiere and HJ bounds

The Equity Premium Puzzle I

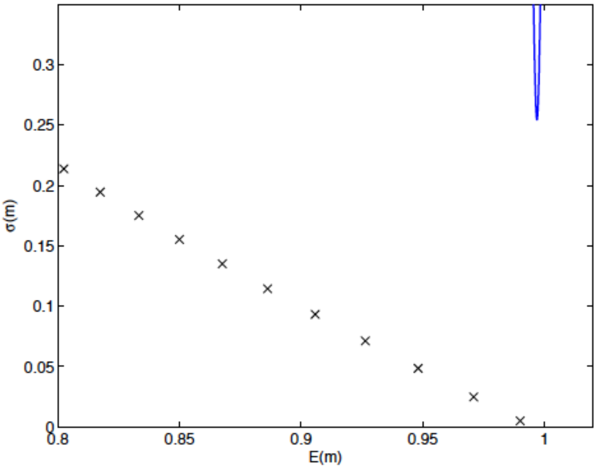

Figure: Solid line: Hansen-Jagannathan volatility bounds for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948-2006. Crosses: Mean and standard deviation for intertemporal marginal rate of substitution for CRRA time separable preferences. The coefficient of relative risk aversion, γ takes on the values 1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and the discount factor β=0.995.

Epstein-Zin or Kreps-Porteus (KP) Preferences

Introduction

- One typical assumption when uncertainty is introduced is to use only one utility. This is based on the assumption that agents objective over their lifetime is the sum of his expected utility at each periods.
- This makes the objective function additive in the two dimensional space state-date.

$$U(c_0; \tilde{c}_1) := u(c_0) + \mathbf{E}[u(\tilde{c}_1)],$$

  where we consider $\tilde{c}_1$ as a random variable with realization c_1.
- In this model, the degree of aversion to consumption fluctuations over time and the degree of aversion to risk are identical and equal to $-u''/u'$.

The Kreps-Porteus (KP) Observation

- The model is a direct extension of the additive model shown before. It is written as

$$U(c_0; \tilde{c}_1) := u_0(c_0) + u_1(v^{-1}(\mathbf{E}v(\tilde{c}_1)));$$

  where $u_0; u_1$ and v are three increasing functions.
- Let $Ce(\tilde{c}_1)$ be the *certainty equivalent* functional. For all random variables $\tilde{c}_1$ it is defined as

$$v(Ce(\tilde{c}_1)) \equiv \mathbf{E}v(\tilde{c}_1).$$

- U can then be rewritten as

$$U(c_0; \tilde{c}_1) = u_0(c_0) + u_1(Ce(\tilde{c}_1)).$$

Formal Interpretation

- We see that the lifetime utility is computed by performing two different operations
- First, one computes the certainty equivalent Ce of the future uncertain consumption $\tilde{c}_1$ by using utility function v. This is done in an a-temporal context. Thus, the concavity of v measures the degree of *risk aversion* alone.
- Second, one evaluates the lifetime utility by summing up the utility of the current consumption and the utility of the future certainty equivalent consumption, using functions u_0 and u_1.
- Because all uncertainty has been removed in this second operation, the concavity of these two functions are related to preferences for *consumption smoothing over time*.

Particular Cases

- If v and u_1 are identical, we are back to the standard additive model.
- But this model is much richer than the standard additive model because of its ability to disentangle preferences with respect to risk and time.
- Suppose for example that v is the identity function, but u_0 and $u_1 = u_0$ are concave. In that case, the agent is willing to smooth the expected consumption over time, despite he is risk neutral.
- At the opposite side of the spectrum, we can imagine a risk-averse agent who is indifferent toward (certainty equivalent) consumption smoothing. This would be the case if v is concave, but u_0 and u_1 are linear.

Disentangling Risk and Time

TEST: This is your first day of a 3 week summer vacation.

You have already your airplane an so. Ready to leave for the Maldives. You also know that today the CEO will decide whether to fire or not your most heated boss.
Do you want to know before coming back from your vacation whether the boss has been fired or you will just wait to see who is in the office when your are back to work?

Behavioural Interpretation of the Model I

- In the classical case with $u_1 = v$, the timing of the resolution of the uncertainty does not matter for the consumer.
- Suppose that the consumption plan under consideration be $(c_0; \tilde{c}_1)$; where $\tilde{c}_1$ is random. This consumption plan is completely exogenous and cannot be modified by the agent.
- If the realization of $\tilde{c}_1$ is *NOT expected* to be known before $t = 1$, the lifetime expected utility would be measured by $u_0(c_0) + \mathbf{E}u_1(\tilde{c}_1)$.
- Suppose alternatively that the realization of $\tilde{c}_1$ is expected to be known at $t = 0$, but that we cannot revise the level of saving after this observation. Conditional to c_1 (known); the lifetime utility is $u_0(c_0) + u_1(c_1)$. Ex ante, before knowing the realized value of $\tilde{c}_1$; the expected lifetime utility is

$$\mathbf{E}[u_0(c_0) + u_1(\tilde{c}_1)] = u_0(c_0) + \mathbf{E}[u_1(\tilde{c}_1)].$$

- Thus, in this case, agents are INDIFFERENT about the timing of the resolution of the uncertainty.

Behavioural Interpretation of the Model II

- With Kreps-Porteus preferences, if c_1 is NOT expected to be known before $t = 1$, the lifetime utility of the agent is measured according to $u_0(c_0) + u_1(Ce(\tilde{c}_1))$, with $v(Ce(\tilde{c}_1)) = \mathbf{E}v(\tilde{c}_1)$.
- On the contrary, suppose that the realization of c_1 is observed at $t = 0$. Then, because obviously $Ce(\tilde{c}_1) = c_1$ under certainty, the lifetime utility conditional to c_1 is $u_0(c_0) + u_1(c_1)$; as in the classical case.
- Ex ante, the lifetime utility of the agent equals $\mathbf{E}[u_0(c_0) + u_1(\tilde{c}_1)]$
- This is generally not equal to $u_0(c_0) + u_1(Ce(\tilde{c}_1))$. We conclude that an agent with Kreps-Porteus preferences is in general NOT INDIFFERENT to the timing of the resolution of uncertainty

Preferences for Early Resolution of Uncertainty

- We say that an agent has preferences for an EARLY resolution of uncertainty (PERU) if he prefers to observe c_1 at date $t = 0$ than at $t = 1$, whatever the distribution of $\tilde{c}_1$. This is the case when

$$\mathbf{E}[u_0(c_0) + u_1(\tilde{c}_1)] > u_0(c_0) + u_1(Ce(\tilde{c}_1));$$

  or, equivalently, when

$$u_1^{-1}(\mathbf{E}u_1(\tilde{c}_1)) > Ce(\tilde{c}_1) = v^{-1}(\mathbf{E}v(\tilde{c}_1)).$$

- In words, PERU requires that the certainty equivalent be always larger when using function u_1 than when using function v.
- This is true if and only if u_1 is LESS CONCAVE than v.

Recursive Utility under Uncertainty

- In applications (and the exercise) it is typically assumed that the decision maker have preferences over uncertain consumption lotteries which are represented by sequence of functions V_t defined recursively by

$$V_t \equiv U(c_t, \mathbf{E}_t \tilde{V}_{t+1})$$

where $U(.,.)$ has the following functional form:

$$U(c, \mathbf{E}\tilde{V}) \equiv \frac{\left[(1-\beta)c^{1-\rho} + \beta\left[1+(1-\beta)(1-\gamma)(\mathbf{E}\tilde{V})\right]^{\frac{1-\rho}{1-\gamma}}\right]^{\frac{1-\gamma}{1-\rho}} - 1}{(1-\beta)(1-\gamma)},$$

with $\rho \in \Re^+$, $\gamma \in \Re^+$, and $\beta \in (0,1)$.

- One can show that $-1/\rho$ is a measure of constant intertemporal elasticity of substitution (IES) for deterministic variations in consumption, and γ is the constant coefficient of relative risk aversion (CRRA) for static gambles.

The Equity Premium Puzzle II

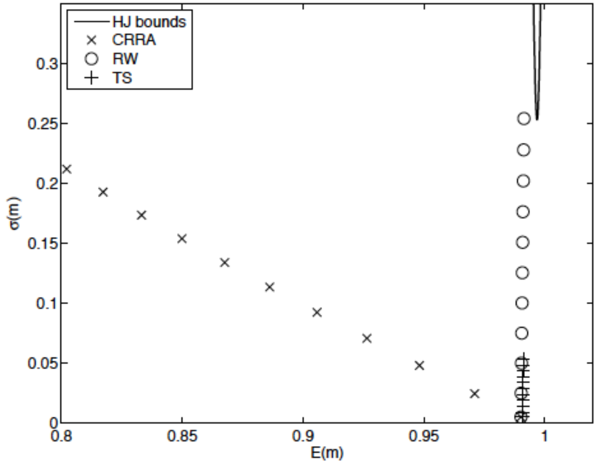

Figure: *Solid line:* Hansen-Jagannathan volatility bounds for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948-200+. *Circles:* Epstein-Zin preferences with random walk consumption. *Pluses:* Epstein-Zin preferences and trend stationary consumption. *Crosses:* CRRA time separable preferences. The coefficient of relative risk aversion, γ takes on the values 1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and the discount factor β=0.995.

The Welfare Cost of the Business Cycles

Hp: Log consumption is distributed normally with a trend:

$$\ln c_t \sim N\left(a + gt - \frac{1}{2}\sigma_c^2,\ \sigma_c^2\right).$$

Removing variability increase welfare, how much?

$$\mathbf{E}_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^t \frac{\left(\left(1+\lambda\right)c_t\right)^{1-\gamma}}{1-\gamma}\right] = \sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^t \frac{\left(Ae^{gt}\right)^{1-\gamma}}{1-\gamma}.$$

$$\lambda \approx \frac{1}{2}\gamma\sigma_c^2.$$

$\gamma \in (1, 4)$ and $\sigma_c = 0.032$ implies $\lambda \leq 0.002.$

- How much by some macroeconomic policy? RBC suggests that only the 30% of variance in business cycles frequency can be reduced by demand management policies.

- Krussel and Smith move away from the representative agent