Macroeconomics Sequence, Block I

Transversality Vs No-Ponzi Game Condition

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The general framework (deterministic)

$$
\begin{aligned}
V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad (1) \\
s.t.\ x_0 &\in X \\
x_{t+1} &\in \Gamma(x_t) \quad \text{for all } t.
\end{aligned}
$$

Time invariant function F, and correspondence Γ; $\beta \in [0, 1)$. We assume Γ to be non empty for all $x \in X$.

Recall the Euler Variational approach: the Transversality Condition

The Transversality Condition

Proposition Assume F is bounded, continuous, concave, and differentiable. Moreover assume Γ has a compact and convex graph. (i) If the (interior) sequence $\{x_t^*\}_{t=1}^{\infty}$ with $x_{t+1}^* \in int\Gamma(x_t^*)$ for any $t = 0, 1, 2, ...$ satisfies

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0 \text{ for } t = 0, 1, ... \qquad (2)$$

and for any other feasible sequence $\{x_t\}_{t=0}^{\infty}$ we have

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \geq 0, \qquad (3)$$

then $\{x_t^*\}_{t=1}^{\infty}$ is an optimal sequence.
(ii) If in addition $F_1(x, x') > 0$ for any $x, x' \in intX$ and $X \subseteq \mathbb{R}_+^l$, the condition (3) can be substituted by

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_T^* \leq 0.$$

General Interpretation

The transversality condition requires any alternative trajectory $\{x_t\}$ satisfying

$$\lim_{t\to\infty} \beta^t F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) < 0$$

to be infeasible.

If given $\{x_t^*\}$ it is impossible to reduce the limit value of the optimal stock by choosing $x_t \neq x_t^*$

(except by incurring in an infinite loss because $\{x\}$ is not feasible)

Then the value of the capital has been exhausted along the trajectory, and $\{x_t^*\}$ must be optimal as long there are no finite period gains (the Euler condition).

Proof of Transversality

(i) We are done if we can show that for any feasible path we have

$$\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t F(x_t^*, x_{t+1}^*) \geq \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t F(x_t, x_{t+1}),$$

From the concavity and differentiability of F we have

$$F(x_t, x_{t+1}) \leq F(x_t^*, x_{t+1}^*) + F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_2(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*)$$

Multiplying by β^t and summing up the first T terms one gets

$$\sum_{t=0}^{T}\beta^t F(x_t, x_{t+1}) \leq \sum_{t=0}^{T}\beta^t F(x_t^*, x_{t+1}^*) + D_T, \qquad (4)$$

where $D_T = \sum_{t=0}^{T}\beta^t\left[F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_2(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*)\right].$

Proof (Continued)

We want to show that in (4) $\lim_{T\to\infty} D_T \leq 0$.

We can rearrange the terms and obtain that

$$D_T = \sum_{t=0}^{T-1} \beta^t \left[F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*)\right](x_{t+1} - x_{t+1}^*) + \\ -\beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*).$$

Euler conditions (2) guarantee that the fist $T-1$ terms go to zero:

$$\lim_{T\to\infty} D_T = -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \leq 0.$$

the last inequality is implied by the transversality condition.
(ii) if $F_1 > 0$ and $x_T \geq 0$,

$$\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_T - x_T^*) \geq -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*) x_T^* \geq 0.$$

Q.E.D.

No Ponzi Games vs Transversality

Consider a consumer facing a constant path of income and can buy and sell in all future markets at price p_t.

$$\text{Problem I (Arrow-Debreu):} \quad \max_{\{c_t\}_{t=0}^{\infty}} \sum_t \beta^t u(c_t)$$

$$\text{s.t.} \quad : \quad c_t \geq 0, \text{ for all } t, \text{ and}$$

$$\sum_t p_t c_t \leq \sum_t p_t y,$$

Suppose, we have $p_t = \beta^t$. Clearly, $c_t^* = y$ for all t.
Now, denote by b_t the level of debt at period t : $(b_0 = 0)$. The agent can borrow and lend. $p_t = \beta^t$ corresponds to $1 + r = \frac{1}{\beta}$.

$$\text{Problem II (Radner):} \quad \max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_t \beta^t u(c_t)$$

$$\text{s.t.} \quad : \quad c_t \geq 0, \text{ and}$$

$$c_t + b_{t+1} \leq (1 + r) b_t + y, \text{ for all } t.$$

No Ponzi Games (Continued)

Recall $1 + r = \frac{1}{\beta}$. Multiply by β^t the BC in period t, and rearranging the sequence of per-period budget constraints, for all T

$$\sum_{t=0}^{T} \beta^t c_t \leq \sum_{t=0}^{T} \beta^t y - \beta^T b_{T+1}.$$

A necessary condition for having the same solution in Problems I and II is to impose $\lim_{T\to\infty} \beta^T b_{T+1} = 0$ for all sequences.

Since the agent would never over-save, it suffices to require

$$\lim_{T\to\infty} p_T b_{T+1} = \lim_{T\to\infty} \beta^T b_{T+1} \geq 0, \tag{5}$$

The usual form of the NPG.

NB: The NPG is imposed as additional condition on Problem II.

Transversality

The transversality condition is an optimality condition, not a constraint.

Since in this spefic problem, the optimal path is $b_t^* = 0$ for all t, the transversality condition for this problem will be

$$\lim_{T \to \infty} \beta^T u'\left(c_T^*\right) b_{T+1} \geq 0. \tag{6}$$

Note: The NPG condition does not contemplates the (subjective) marginal utility of the agent. The transversality does. Here β is the subjective discount factor of the agent not a price in the BC.

Sometimes one could get confused between the two conditions:

1. When u' is finite one can disregard it in the transversality.
2. The equilibrium price of period t consumption goods takes the value $p_t = \beta^t \frac{u'(c_t^*)}{u'(c_0)}$.
    - Recall the NPG condition: $\lim_{T \to \infty} p_T b_{T+1} \geq 0$. It clearly resembles to condition (6) when $u'\left(c_0\right) = 1$ (normalization).