Stock and Bonds Returns. An Introduction

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Returns

Consider an asset that does not pay any intermediate cash income (a zero-coupon bons, such as a Treasury Bill, or a share in a company that pays no dividends). Let P_t be the price of the security at time t.
The **linear or simple** return between times t and $t-1$ is defined as:

$$R_t = P_t / P_{t-1} - 1$$

The **log** return is defined as:

$$r_t = \ln(P_t / P_{t-1})$$

Note that, while P_t means "price at time t", r_t is a a shorthand for "return between time $t-1$ and t"

Returns

The two definitions of return yield different numbers when the ratio between consecutive prices is far from 1.
Consider the Taylor formula for $\ln(x)$ for x in the neighbourhood of 1:

$$\ln(x) = \ln(1) + (x-1)/1 - (x-1)^2/2 + ...$$

if we truncate the series at the first order term we have:

$$\ln(x) \cong 0 + x - 1$$

so that if x is the ratio between consecutive prices, then for x close to one the two definitions give similar values. Note however that $\ln(x) \leq x - 1$. In fact $x - 1$ is equal to and tangent to $\ln(x)$ in $x = 1$ and above it anywhere else.

Multi-period returns

Define the simple multi-period return between time t and t+n as:

$$R_{t,t+n} = P_{t+n}/P_t - 1 = \frac{P_{t+n}}{P_{t+n-1}}\frac{P_{t+n-1}}{P_{t+n-2}}\ldots\frac{P_{t+1}}{P_t} - 1 = \prod_{i=1}^{n}\left(1 + R_{t+i,t+i-1}\right) - 1 \tag{1}$$

in the case of log returns we have instead:

$$r_{t,t+n} = \ln\left(P_{t+n}/P_t\right) = \ln\left(\frac{P_{t+n}}{P_{t+n-1}}\frac{P_{t+n-1}}{P_{t+n-2}}\ldots\frac{P_{t+1}}{P_t}\right) = \sum_{i=1}^{n} r_{t+i,t+i-1} \tag{2}$$

Annualized returns

annualized returns the constant annual rate of return equivalent to the multiperiod returns to an of an investment in asset i over the period t, t+n. In the case of simple returns we have

$$\left(1+R^A_{t,t+n}\right)^n = 1+R_{t,t+n} = \prod_{i=1}^{n}\left(1+R_{t+i,t+i-1}\right)$$

$$R^A_{t,t+n} = \left(\prod_{i=1}^{n}\left(1+R_{t+i,t+i-1}\right)\right)^{\frac{1}{n}} - 1$$

Consider now log returns:

$$nr^A_{t,t+n} = r_{t,t+n} = \sum_{i=1}^{n} r_{t+i,t+i-1}$$

$$r^A_{t,t+n} = \frac{1}{n}\sum_{i=1}^{n} r_{t+i,t+i-1}$$

Working with Returns

Consider the value of a buy and hold portfolio of invested in shares of k different companies, that pay no dividend, at time t be:

$$V_t = \sum_{i=1}^{k} n_i P_{it}$$

The simple one-period return of the portfolio shall be a linear function of the returns of each stock.

$$\begin{aligned} R_t &= \frac{V_t}{V_{t-1}} - 1 = \sum_{i=1..k} \frac{n_i P_{it}}{\sum_{j=1..k} n_j P_{jt-1}} - 1 \\ &= \sum_{i=1..k} \frac{n_i P_{it-1}}{\sum_{j=1..k} n_j P_{jt-1}} \frac{P_{it}}{P_{it-1}} - 1 = \end{aligned}$$

$$= \sum_{i=1..k} w_{it}(R_{it}+1) - 1 = (\sum_{i=1..k} w_{it} R_{it} + \sum_{i=1..k} w_{it} 1) - 1 = \sum_{i=1}^{k} w_{it} R_{it}$$

Working with Returns

log returns are not additive in the cross-section but they are additive when we consider the time-series of returns

$$
\begin{aligned}
r_t &= \ln(\frac{V_t}{V_{t-1}}) \\
&= \ln(\frac{\sum_{i=1}^{k} n_i P_{it-1}}{\sum_{i=1}^{k} n_i P_{it-1}} \frac{P_{it}}{P_{it-1}}) = \ln(\sum_{i=1}^{k} w_{it} \exp(r_{it}))
\end{aligned}
$$

$$
r_{t,t+n} = \sum_{i=1}^{n} r_{t+i,t+i-1}
$$

Note that additivity in the time-series does not apply to simple returns.

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Stock Returns and the dynamic dividend growth model

consider the one-period total holding returns in the stock market, that are defined as follows:

$$H_{t+1}^{s} \equiv \frac{P_{t+1} + D_{t+1}}{P_t} - 1 = \frac{P_{t+1} - P_t + D_{t+1}}{P_t} = \frac{\Delta P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t}, \qquad (3)$$

Dividing both sides by $\left(1 + H_{t+1}^{s}\right)$ and multiplying both sides by P_t / D_t we have:

$$\frac{P_t}{D_t} = \frac{1}{\left(1 + H_{t+1}^{s}\right)} \frac{D_{t+1}}{D_t} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right).$$

Taking logs we have:

$$p_t - d_t = -r_{t+1}^{s} + \Delta d_{t+1} + \ln\left(1 + e^{p_{t+1} - d_{t+1}}\right)$$

Stock Returns and the dynamic dividend growth model

Taking a first-order Taylor expansion of the last term about the point $\bar{P}/\bar{D} = e^{\bar{p}-\bar{d}}$:

$$
\begin{aligned}
\ln\left(1+e^{p_{t+1}-d_{t+1}}\right) &\simeq \ln(1+e^{\bar{p}-\bar{d}}) + \frac{e^{\bar{p}-\bar{d}}}{1+e^{\bar{p}-\bar{d}}}\left[(p_{t+1}-d_{t+1})-(\bar{p}-\bar{d})\right] \\
&= -\ln(1-\rho) - \rho\ln\left(\frac{1}{1-\rho}-1\right) + \rho(p_{t+1}-d_{t+1}) \\
&= \kappa + \rho(p_{t+1}-d_{t+1})
\end{aligned}
$$

where

$$
\rho \equiv \frac{e^{\bar{p}-\bar{d}}}{1+e^{\bar{p}-\bar{d}}} = \frac{\bar{P}/\bar{D}}{1+(\bar{P}/\bar{D})} < 1 \qquad \kappa \equiv -\ln(1-\rho) - \rho\ln\left(\frac{1}{1-\rho}-1\right).
$$

Total stock market returns can then be written as:

$$
r^{s}_{t+1} = \kappa + \rho\left(p_{t+1}-d_{t+1}\right) + \Delta d_{t+1} - \left(p_t - d_t\right),
$$

Stock Returns and the dynamic dividend growth model

By *forward* recursive substitution one obtains:

$$(p_t - d_t) = \kappa \sum_{j=1}^{m} \rho^{j-1} + \sum_{j=1}^{m} \rho^{j-1}(\Delta d_{t+j} - r^s_{t+j}) + \rho^m (p_{t+m} - d_{t+m}) .$$

Under the assumption that there can be no rational bubbles, i.e., that

$$\lim_{m \longrightarrow \infty} \rho^m (p_{t+m} - d_{t+m}) = 0,$$

$$(p_t - d_t) = \frac{\kappa}{1-\rho} + \sum_{j=1}^{m} \rho^{j-1} \left(\Delta d_{t+j} - r^s_{t+j}\right) .$$

Stock Returns and the dynamic dividend growth model

If we decompose future variables into their expected component and the unexpected one (an error term) we can write the relationship between the dividend-yield and the returns one-period ahead and over the long-horizon as follows:

$$r_{t+1}^{s} = \kappa + \rho E_t \left(p_{t+1} - d_{t+1} \right) + E_t \Delta d_{t+1} - \left(p_t - d_t \right) + \rho u_{t+1}^{pd} + u_{t+1}^{\Delta d}$$

$$\sum_{j=1}^{m} \rho^{j-1} r_{t+j}^{s} = \frac{\kappa}{1-\rho} + \sum_{j=1}^{m} \rho^{j-1} E_t \left(\Delta d_{t+j} \right) - \left(p_t - d_t \right) + \rho^m E_t \left(p_{t+m} - d_{t+m} \right) +$$

$$\rho^m u_{t+m}^{pd} + \sum_{j=1}^{m} \rho^{j-1} u_{t+j}^{\Delta d}$$

Note that when the price dividends ratio is a noisy process, such noise dominates the variance of one-period returns but the picture is different when we consider long-horizon returns.

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Zero-Coupon Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{\left(1 + Y_{t,T}\right)^{T-t}},$$

Taking logs of the left and the right-hand sides of the expression for $P_{t,T}$, and defining the continuously compounded *yield*, $y_{t,T}$, as $\log(1 + Y_{t,T})$, we have the following relationship:

$$p_{t,T} = -\left(T - t\right) y_{t,T},$$

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$\begin{aligned} r^T_{t,t+1} &\equiv p_{t+1,T} - p_{t,T} = -\left(T - t - 1\right) y_{t+1,T} + \left(T - t\right) y_{t,T} \\ &= y_{t,T} - \left(T - t - 1\right)\left(y_{t+1,T} - y_{t,T}\right), \end{aligned}$$

Coupon Bonds

The relationship between price and yield to maturity of a constant coupon (C) bond is given by:

$$P_{t,T}^{c} = \frac{C}{\left(1 + Y_{t,T}^{c}\right)} + \frac{C}{\left(1 + Y_{t,T}^{c}\right)^{2}} + ... + \frac{1 + C}{\left(1 + Y_{t,T}\right)^{T-t}}.$$

To measure the length of time that a bondholder has invested money for we need to introduce the concept of duration:

$$D_{t,T}^{c} = \frac{\frac{C}{\left(1+Y_{t,T}^{c}\right)} + 2\frac{C}{\left(1+Y_{t,T}^{c}\right)^{2}} + ... + (T-t)\frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^{c}}$$

$$= \frac{C\sum_{i=1}^{T-t} \frac{i}{\left(1+Y_{t,T}^{c}\right)^{i}} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^{c}}.$$

Coupon Bonds

Note that when a bond is floating at par we have:

$$
\begin{aligned}
D_{t,T}^{c} &= Y_{t,T}^{c}\sum_{i=1}^{T-t}\frac{i}{\left(1+Y_{t,T}^{c}\right)^{i}}+\frac{(T-t)}{(1+Y_{t,T})^{T-t}} \\
&= Y_{t,T}^{c}\frac{\left((T-t)\frac{1}{1+Y_{t,T}^{c}}-(T-t)-1\right)\frac{1}{\left(1+Y_{t,T}^{c}\right)^{T-t+1}}+\frac{1}{1+Y_{t,T}^{c}}}{\left(1-\frac{1}{1+Y_{t,T}^{c}}\right)^{2}}+\frac{(T-t)}{(1+Y_{t,T})^{T}} \\
&= \frac{1-\left(1+Y_{t,T}^{c}\right)^{-(T-t)}}{1-\left(1+Y_{t,T}^{c}\right)^{-1}},
\end{aligned}
$$

because when $|x|<1$,

$$
\sum_{k=0}^{n}kx^{k}=\frac{(nx-n-1)\,x^{n+1}+x}{(1-x)^{2}}.
$$

Coupon Bonds

Applying the log-linearization of one-period returns to a coupon bond we have:

$$
\begin{aligned}
p_{c,t,T} - c &= -r^c_{t+1} + k + \rho \left(p_{c,t+1,T} - c \right) \\
r^c_{t+1} &= k + \rho p_{c,t+1,T} + (1 - \rho)\, c - p_{c,t,T}.
\end{aligned}
$$

When the bond is selling at par, $\rho = (1 + C)^{-1} = \left(1 + Y^c_{t,T}\right)^{-1}$. Solving this expression forward to maturity delivers:

$$
p_{c,t,T} = \sum_{i=0}^{T-t-1} \rho^i \left(k + (1 - \rho)\, c - r^c_{t+1+i} \right).
$$

Coupon Bonds

The log yield to maturity $y^c_{t,T}$ satisfies an expression with the same structure:

$$
\begin{aligned}
p_{c,t,T} &= \sum_{i=0}^{T-t-1} \rho^i \left(k + (1-\rho)\, c - y^c_{t,T}\right) = \frac{1-\rho^n}{1-\rho} \left(k + (1-\rho)\, c - y^c_{t,T}\right) \\
&= D^c_{t,T} \left(k + (1-\rho)\, c - y^c_{t,T}\right).
\end{aligned}
$$

By substituting this expression back in the equation for linearized returns we have the expression

$$
r^c_{t+1} = D^c_{t,T} y^c_{t,T} - \left(D^c_{t,T} - 1\right) y^c_{t+1,T},
$$

that illustrates the link between continuously compounded returns and duration.

A simple model of the term structure

Consider the relation between the return on a riskless one period short-term bill, r_t, and a long term bond bearing a coupon C, the one-period return on the long-term bond $H_{t,T}$ is a non-linear function of the yield to maturity $R_{t,T}$. Shiller (1979) introduces the *linearization discussed above in* the neighborhood $R_{t,T} = R_{t+1,T} = \bar{R} = C$:

$$H_{t,T} \simeq \frac{R_{t,T} - \gamma_T R_{t+1,T}}{1 - \gamma_T}$$

$$\gamma_T = \left\{ 1 + \bar{R} \left[1 - 1/(1 + \bar{R})^{T-t-1} \right]^{-1} \right\}^{-1}$$

$$\lim_{T \longrightarrow \infty} \gamma_T = \gamma = 1/(1 + \bar{R})$$

In this case, by equating one-period risk-adjusted returns, we have:

$$E \left[\frac{R_{t,T} - \gamma R_{t+1,T}}{1 - \gamma} \mid I_t \right] = r_t + \phi_{t,T}$$

A simple model of the term structure

From the above expression, by recursive substitution, we have:

$$R_{t,T} = R^*_{t,T} + E[\Phi_T \mid I_t] = \frac{1-\gamma}{1-\gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

where $\Phi_{t,T}$ is the term premium over the whole life of the bond:

$$\Phi_{t,T} = \frac{1-\gamma}{1-\gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^j \phi_{t+j,T}$$

For long-bonds, when $T-t$ is very large, we have :

$$R_{t,T} = R^*_{t,T} + E[\Phi_T \mid I_t] = (1-\gamma) \sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

Subtracting the risk-free rate from both sides of this equation we have:

$$S_{t,T} = R_{t,T} - r_t = \sum_{j=1}^{T-1} \gamma^j E[\Delta r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

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Modelling returns

The (naive) log random walk (LRW) hypothesis on the evolution of prices states that, prices evolve approximately according to the stochastic difference equation:

$$\ln P_t = \mu\Delta + \ln P_{t-\Delta} + \epsilon_t$$

where the 'innovations' ϵ_t are assumed to be uncorrelated across time ($cov(\epsilon_t; \epsilon_{t'}) = 0 \quad \forall t \neq t'$), with constant expected value 0 and constant variance $\sigma^2\Delta$.
Consider what happens over a time span of, say, 2Δ.

$$\ln P_t = 2\mu\Delta + \ln P_{t-2\Delta} + \epsilon_t + \epsilon_{t-\Delta} = \ln P_{t-2\Delta} + u_t$$

having set $u_t = \epsilon_t + \epsilon_{t-\Delta}$.

Modelling returns

Consider now the case in which the time interval is of the length of 1-period. If we take prices as inclusive of dividends we can write the following model for log-returns

$$
\begin{aligned}
r_{t,t+1} &= \mu + \sigma \epsilon_t \\
\epsilon_t &= i.i.d.(0,1)
\end{aligned}
$$

$$E(r_{t,t+n}) = E(\sum_{i=1}^{n} r_{t+i,t+i-1}) = \sum_{i=1}^{n} E(r_{t+i,t+i-1}) = n\mu$$

$$Var(r_{t,t+n}) = Var(\sum_{i=1}^{n} r_{t+i,t+i-1}) = \sum_{i=1}^{n} Var(r_{t+i,t+i-1}) = n\sigma^2$$

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Monte-Carlo simulation

- given some estimates of the unknown parameters in the model (μ σ in our case).
- an assumption is made on the distribution of ϵ_t.
- The an artificial sample for ϵ_t of the length matching that of the available can be computer simulated.
- The simulated residuals are then mapped into simulated returns via μ, σ.
- This exercise can be replicated N times (and therefore a Monte-Carlo simulation generates a matrix of computer simulated returns whose dimension are defined by the sample size T and by the number of replications N).
- The distribution of model predicted returns can be then costruncted and one can ask th question if the observed data can be considered as one draw from this distribution.

do exactly like in Monte-Carlo but rather than using a theoretical distribution for ϵ_t use their empirical distribution and resample from it with reimmission.

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Stocks for the long-run

The fact that, under the LRW, the expected value grows linearly with the length of the time period while the standard deviation (square root of the variance) grows with the square root of the number of observations, has created a lot of discussion

We have three flavors of the "stocks for the long run" argument. The first and the second are a priori arguments depending on the log random walk hypothesis or something equivalent to it, the third is an a posteriori argument based on historical data.

Stocks for the long-run

Assume that single period (log) returns have (positive) expected value μ and variance σ^2. Moreover, assume for simplicity that the investor requires a Sharpe ratio of say S out of his-her investment. Under the above hypotheses, plus the log random walk hypothesis, the Sharpe ratio over n time periods is given by

$$S = \frac{n\mu}{\sqrt{n}\sigma} = \sqrt{n}\frac{\mu}{\sigma}$$

so that, if n is large enough, any required value can be reached.

Stocks for the long-run

Another way of phrasing the same argument, when we add the hypothesis of normality on returns, is that, there any given probability α and any given required return C there is always an horizon for which the probability for n period return less than C is less than α.

$$\Pr\left(R^p < C\right) = \alpha.$$

$$\Pr\left(R^p < C\right) = \alpha \Longleftrightarrow \Pr\left(\frac{R^p - n\mu}{\sqrt{n}\sigma} < \frac{C - n\mu}{\sqrt{n}\sigma}\right) = \alpha$$

$$\Longleftrightarrow \Phi\left(\frac{C - \mu_p}{\sigma_p}\right) = \alpha,$$

$$C = n\mu + \Phi^{-1}\left(\alpha\right)\sqrt{n}\sigma$$

But $n\mu + \Phi^{-1}\left(\alpha\right)\sqrt{n}\sigma$, for $\sqrt{n} > \frac{1}{2}\frac{\Phi^{-1}(\alpha)}{\mu}\sigma$ is an increasing function in n so that for any α and any chosen value C, there exists a n such that from that n onward, the probability for an n period return less than C is less than α.

Stocks for the long-run

Note, however, that the value of n for which this lower bound crosses a given C level is the solution of

$$n\mu + \Phi^{-1}(\alpha)\sqrt{n}\sigma \geq C$$

In particular, for $C = 0$ the solution is

$$\sqrt{n} \geq -\frac{\Phi^{-1}(\alpha)\,\sigma}{\mu}$$

Consider now the case of a stock with σ/μ ratio for one year is of the order of 6. Even allowing for a large α,say 0.25, so that $\Phi^{-1}(\alpha)$ is near minus one , the required n shall be in the range of 36 which is only slightly shorter than the average working life.
As a matter of fact, based on the analysis of historical prices and risk adjusted returns, stocks have been almost always a good long run investment.