Macroeconomics: Block I

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The q Theory of Investment

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Introduction

- Investment is expenditures by firms on equipment and structures.
- For individual plants, investment is simply the expenditure required to adjust its stock of capital.
- Capital includes all equipment and structures the plant uses.
- The plant combines capital with other inputs, such as labor and energy, to produce goods or services.
- When an extraction company acquires diesel engines, it is investing in equipment.
- When an automobile manufacturer builds a new warehouse, it is investing in structures.

Neoclassical Theory

- Consider the problem of an infinitely lived firm that in every period chooses how much to invest, i.e. how much to add to its stock of productive capital.
- We assume that the firm owns capital.
- Denote the production function of the firm by $f(k_t)$, the level of technology of the firm at time t by z_t,
- p_t is the price of a unit of investment good or the unit price of capital goods as (the price of the final good is normalized to one) we have

$$\begin{aligned} V_0^* &= \max_{\{i_t, k_{t+1}\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[z_t f(k_t) - p_t i_t\right] \\ & s.t. \\ k_{t+1} &= (1-\delta) k_t + i_t; \\ k_{t+1} &\geq 0, \text{ for all } t; \ k_0 \text{ given.} \end{aligned} \tag{1}$$

Euler Equation

- We now derive the Euler equation for the problem.
- We are looking for a feasible deviation from the optimal *interior* program $\{i_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, where interiority simple requires $k_{t+1}^* > 0$ for all t.
- In the spirit of the Euler variational approach, the perturbation is aimed at changing k_{t+1}^* (and i_t^*, i_{t+1}^*), while keeping unchanged all k_s^* for $s \neq t+1$, in particular both k_t^* and k_{t+2}^*.

Euler variational approach

- Let ε any real number in an open neighborhood O of zero (in order to keep feasibility)
- For each ε, the perturbed plan $\{\hat{\imath}_t^{\varepsilon}, \hat{k}_{t+1}^{\varepsilon}\}_{t=0}^{\infty}$ is constructed from $\{i_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ as follows: $\hat{k}_{t+1}^{\varepsilon} = k_{t+1}^* + \varepsilon$, and $\hat{k}_s^{\varepsilon} = k_s^*$ for $s \neq t+1$.
- Such perturbation implies: $\hat{\imath}_t^{\varepsilon} = i_t^* + \varepsilon$ and $\hat{\imath}_{t+1}^{\varepsilon} = i_{t+1}^* - (1-\delta)\,\varepsilon$ and $\hat{\imath}_s^{\varepsilon} = i_s^*$ for $s \neq t, t+1$.
- If we denote by $\hat{V}_0(\varepsilon)$ the value associated to the perturbed plan for each $\varepsilon \in O$, the optimality of the original plan implies $\hat{V}_0(\varepsilon) \leq V_0^*$ for all $\varepsilon \in O$, and $\hat{V}_0(0) = V_0^*$.
- Stated in other terms, $\varepsilon = 0$ is the optimal solution to

$$\max_{\varepsilon \in O} \; \hat{V}_0(\varepsilon).$$

The first order condition is $\hat{V}_0'(0) = 0$.

The Euler Equation

- Since k_s^* are untouched, both for $s \leq t$ and $s \geq t+2$ the derivative with respect to ε of all terms are zero but period t and $t+1$ returns. We hence have:

$$\begin{aligned}\hat{V}_0'(\varepsilon) &= \frac{d}{d\varepsilon}\left(\frac{1}{1+r}\right)^t \Big\{ z_t f(k_t^*) - p_t(i_t^* + \varepsilon) + \\ &+ \left(\frac{1}{1+r}\right)\left(z_{t+1} f(k_{t+1}^* + \varepsilon) - p_{t+1}\left(i_{t+1}^* - (1-\delta)\varepsilon\right)\right)\end{aligned}$$

- The condition $\hat{V}_0'(0) = 0$ hence delivers :

$$p_t = \frac{1}{1+r}\left[z_{t+1} f'(k_{t+1}^*) + p_{t+1}(1-\delta)\right].$$

- It is sometimes called the Jorgenson's optimal investment condition.

The q-Theory of Investment

- The neoclassical model has a couple of drawbacks.
- First, when firms are heterogeneous in the marginal product of capital. Then all investment in the economy will take place in the firms with the highest marginal product of capital. Starting from the very top and down. This is clearly a counterfactual.
- Second, current investment is independent of future marginal products of capital.
- We want a theory that makes firms willing to smooth investment over time.
- We make costly to invest or disinvest large amounts of capital at once: $\Rightarrow$ *adjustment costs*.

The Problem

The problem of the firm hence specializes to

$$
\begin{aligned}
V_0^* &= \max_{\{i_t, k_{t+1}\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[z_t f\left(k_t\right) - p_t\left(i_t + \phi\left(i_t, k_t\right)\right)\right] \\
&\quad s.t. \\
k_{t+1} &= (1-\delta)\, k_t + i_t; \qquad \left(\text{multiplier} \quad \frac{\lambda_t}{(1+r)^t}\right) \\
k_{t+1} &\geq 0, \text{ for all } t; \; k_0 \text{ given.}
\end{aligned}
$$

Optimal Conditions

Standard Kuhn-Tucker theory:

$$
\begin{aligned}
i_t &: \quad p_t \left(1 + \phi_1' \left(i_t^*, k_t^*\right)\right) = \lambda_t^* \\
k_{t+1} &: \quad \frac{1}{1+r} \left[z_{t+1} f' \left(k_{t+1}^*\right) - p_{t+1} \phi_2' \left(i_t^*, k_t^*\right) + (1-\delta) \lambda_{t+1}^*\right] = \lambda_t^*.
\end{aligned}
$$

- The (costate) variable λ_t^* represents the *value* of the marginal contribution of capital to profits (the period t shadow price).
- The first condition just equates costs to returns of a marginal unit of investment.
- The second is similar to the Jorgenson's optimal investment condition with λ partially replacing p and a different productivity (it includes adjustment costs reduction)

The q Theory

- Now define $q_t = \frac{\lambda_t^*}{p_t}$ the same marginal value normalized by the market price of capital.
- From the FOC we obtain

$$1 + \phi_1' \left(i_t^*, k_t^* \right) \equiv g \left(i_t^*, k_t^* \right) = q_t.$$

Denoting by h the inverse function of g conditional on k, we obtain

$$i_t^* = h \left(q_t, k_t^* \right),$$

with $h(1, k) = 0$ whenever $\phi_1' (0, k) = 0$ for all k.

1. q_t is a "sufficient statistic" for fixed investment
2. the firm makes positive investment if and only if $q_t > 1$.
3. Third, how much investment changes with q depends on the slope of h, hence on the *convexity of the adjustment cost function* ϕ.

Marginal versus Average q

- Hayashi (1982) showed that under four key conditions the shadow price q_t (the *marginal-q*) corresponds to the ratio between the value of the firm V_t^* divided by the replacement cost of capital $p_t k_t$.
- The latter ratio is often called Tobin's *average-q*: $q^a = \frac{V_t^*}{p_t k_t}$.
- Such conditions are:

(i) the production function and the adjustment cost function are homogeneous of degree one, i.e. they display constant returns to scale;

(ii) the capital goods are all homogeneous and identical; and

(iii) the stock market is efficient, i.e. the stock market price of the firm equals the discounted present value of all future dividends;

(iv) and the firms operates in a competitive environment, i.e. it takes as given prices and wages;

Employment Dynamics under uncertainty

- state variable: stock of workers in a firm n_t :

$$n_{t+1} = (1-\delta)\, n_t + h_t,$$

- δ is the exogenous separation rate
- h_t is the *gross* employment variation in period t.

$$\begin{aligned}
W_0^* &= \max_{\{n_{t+1},h_t\}} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[z_t f(n_t) - w_t n_t - \phi(h_t) \right] \right] \\
& \quad s.t. \\
n_{t+1} &= (1-\delta)\, n_t + h_t; \\
n_{t+1} &\geq 0, \text{ for all } t; \; n_0 \text{ given.}
\end{aligned}$$

$$\phi(h) = \begin{cases} hH & \text{if } h > 0 \\ 0 & \text{if } h = 0 \\ -hF & \text{if } h < 0. \end{cases}$$

... Employment Dynamics Continued

- w analogous to the *user or rental cost* of capital $(r+\delta)$ in the previous model.
- λ_t^* is the shadow value of labor:

$$\lambda_t^* = \mathbf{E}_t \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^s \frac{\left[z_{t+s} f'\left(n_{t+s}\right) - w_{t+s}\right]}{1+r},$$

or

$$\lambda_t^* = \frac{z_t f'\left(n_t\right) - w_t}{1+r} + \frac{1-\delta}{1+r} \mathbf{E}_t \left[\lambda_{t+1}^*\right]. \tag{2}$$

- We have

$$-F \leq \lambda_t^* \leq H$$

with $\lambda_t^* = H$ if $h_t > 0$ and $\lambda_t^* = -F$ if $h_t < 0$.

Stochastic Steady State

- Assume $w_t = \bar{w}$ and $z_t \in \{z_\ell, z_h\}$ with transition matrix Π.
- Steady state with $\lambda_h^* = H$ while $\lambda_\ell^* = -F$.
- From (2) we have

$$
\begin{aligned}
\lambda_h^* &= H = \frac{z_\ell f'(\underline{n}) - \bar{w}}{1+r} + \frac{1-\delta}{1+r}\mathbf{E}\left[\lambda'; \ell\right] \\
\lambda_h^* &= H = \frac{z_h f'(n_h) - \bar{w}}{1+r} + \frac{1-\delta}{1+r}\mathbf{E}\left[\lambda'; h\right] \\
\lambda_\ell^* &= -F = \frac{z_\ell f'(n_\ell) - \bar{w}}{1+r} + \frac{1-\delta}{1+r}\mathbf{E}\left[\lambda'; \ell\right].
\end{aligned}
$$

NB: The hiring/firing decision h can take several values

- In order to have a non-degenerate steady state we must have

$$\mathbf{E}\left[\lambda'; h\right] = \pi_{hh}\lambda_h^* + \pi_{h\ell}\lambda_\ell^*.$$

Q: What is the detailed expression for $\mathbf{E}\left[\lambda'; \ell\right]$?