Macroeconomics Sequence, Block I

Markov Chains and Stochastic DP

Nicola Pavoni

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The general framework (deterministic)

$$
\begin{aligned}
V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad (1) \\
s.t.\ x_0 &\in X \\
x_{t+1} &\in \Gamma(x_t) \quad \text{for all } t.
\end{aligned}
$$

Time invariant function F, and correspondence Γ; $\beta \in [0,1)$. We assume Γ to be non empty for all $x \in X$.

Recall that the BPO is equivalent to the possibility of writing the value function V^* as Bellman Functional Equation:

$$
V(x_0) = \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta V(x_1), \qquad (2)
$$

The Stochastic Optimal Growth Model I

Consider the stochastic version of the optimal growth model:

$$\begin{aligned} V^*(k_0, z_0) &= \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(f(k_t, z_t) - k_{t+1}\right) \right] \\ \text{s.t.} \quad 0 &\leq k_{t+1} \leq f(k_t, z_t),\ k_0 \text{ and } z_0 \text{ given.} \end{aligned}$$

This model with persistent shocks and non inelastic labor supply has been used in the Real Business Cycles literature to study the effects of technological shocks on aggregate variables such as consumption, investment and employment.

$\Rightarrow$ Next Lessons.

The Stochastic Optimal Growth Model II

The Bellman equation is

$$V(k,z) = \max_{0 \le k' \le z f(k)} \; u(f(k,z) - k') + \beta \mathbf{E}\left[V(k', z') \mid z \right]$$

Simple case: $\{z_t\}_{t=0}^{\infty}$ are i.i.d., $u(c) = \ln c$, $f(k,z) = z k^{\alpha}$, $(\delta = 1)$,

$\Rightarrow$ The optimal policy function is

$$k_{t+1} = \alpha \beta z k_t^{\alpha}$$

and the value function is

$$V(k,z) = A(z) + \frac{\alpha}{1 - \beta\alpha} \ln k.$$

$\Rightarrow$ Exercise.

Stochastic Dynamic Programming with finite states

Let the set of states be

$$Z = \{z_1,\ z_2,\ ...,\ z_N\}.$$

The Bellman equation is

$$V(x, z_i) = \sup_{x' \in \Gamma(x, z_i)} F(x, x', z_i) + \beta \sum_{j=1}^{N} \pi_{ij} V(x', z_j),\ \forall i$$

$\mathbf{V}(x) = (V(x, z_1), ..., V(x, z_N)) \Rightarrow$ (Vector) Bellman Operator:

$$(T\mathbf{V})(x) = \begin{cases} \sup_{x' \in \Gamma(x, z_1)} F(x, x', z_1) + \beta \sum_{j=1}^{N} \pi_{1j} V(x', z_j) \\ \ldots \\ \sup_{x' \in \Gamma(x, z_N)} F(x, x', z_N) + \beta \sum_{j=1}^{N} \pi_{Nj} V(x', z_j) \end{cases}$$

Markov Chains I

Transition probabilities:

$$\pi_{ij} = \Pr\left\{ z' = z_j \mid z = z_i \right\}, \; i, j = 1, 2, ..., N.$$

probability of the system to move to state z_j when current state is z_i.

$$\pi_{ij} \geq 0, \text{ and } \sum_{j=1}^{N} \pi_{ij} = 1 \text{ for } i = 1, 2, ..., N,$$

π belong to the $N-$dimensional simplex Δ^N.

Transition matrix, Markov matrix, or stochastic matrix.

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & ... & \pi_{1N} \\ \pi_{21} & \pi_{22} & ... & ... \\ ... & ... & \pi_{ij} & ... \\ \pi_{N1} & ... & ... & \pi_{NN} \end{bmatrix}$$

Markov Chains II

Probability distribution over the state in period t is

$$p^t = \left(p_1^t, p_2^t, ...p_N^t\right),$$

$\Rightarrow$ distribution over the states in period $t+1$

$$p^t\Pi = \left(p_1^{t+1}, p_2^{t+1}, ...p_N^{t+1}\right),$$

where $p_j^{t+1} = \sum_{i=1}^{N} p_i^t \pi_{ij}, \ j = 1, 2, ..., N$.

Example:

$$p_j^1 = \sum_{i=1}^{N} p_i^0 \pi_{ij}.$$

Examples: Social mobility, Migration.
$p_i^0 =$ stock of people in period zero, $p_j^1 =$ stock of people in period one, $p_i^0 \pi_{ij} =$ flow of people moving from state i into state j.

Markov Chains III

If the current state is z_i with certainty

$\Rightarrow$ initial (degenerate) distribution $p^t = e_i = (0,\ ...,\ 1,\ ...,\ 0)$

$\Rightarrow$ $p^{t+1} =$ the $i-$th row of Π : $e_i\Pi = (\pi_{i1},\ \pi_{i2}, ...\pi_{iN})$.

Additivity (Chapman-Kolmogorov Th.):
$p^{t+n} = p^t\Pi^n = p^t(\Pi \cdot \Pi \cdot ...\Pi)$.

Stationary Distributions I: Existence

Is there a stationary distribution, that is a probability distribution p^* with the property $p^* = p^*\Pi$?

Theorem 18 Given a stochastic matrix Π, there always exists at least one stationary distribution p^* such that $p^* = p^*\Pi$.

Proof: Existence requires the solution of the system of equations $p^*(I - \Pi) = 0$, or

$$(I - \Pi')p^* = 0.$$

That is, we are done if Π' admits (at least) one eigenvalue $\lambda = 1$ with associated eigenvector p^*. It does because is a stochastic matrix:

$$\lambda p^* = \Pi' p^*$$

Q.E.D.

Stationary Distributions II: Uniqueness

When can we say that p^* is unique?

Theorem 19 Assume that $\pi_{ij} > 0$ for all $i, j = 1, 2, \ldots N$. There exists a limiting distribution p^* such that

$$p_j^* = \lim_{n\to\infty} \pi_{ij}^{(n)},$$

where $\pi_{ij}^{(n)}$ is the (i, j) element of the matrix Π^n. And p_j^* are the unique nonnegative solutions of the following system of equations

$$\begin{aligned} p_j^* &= \sum_{k=1}^{N} p_k^* \pi_{kj}; \text{ or } p^* = p^*\Pi; \text{ and} \\ \sum_{j=1}^{N} p_j^* &= 1. \end{aligned}$$

Proof of Uniqueness

The mapping

$$\begin{aligned} T_\Pi &: \Delta^N \to \Delta^N \\ T_\Pi p &= p\Pi \end{aligned}$$

defines a contraction on the (complete) metric space $\left(\Delta^N, |\cdot|_N\right)$ where

$$|x|_N \equiv \sum_{i=1}^{N} |x_i| .$$

(Exercise) Q.E.D.

Related Concepts

- Absorbing, Recurrent, and Transient states
- Regular and Ergodic Markov Chains

General Stochastic Dynamic Programming

The Bellman functional equation in the general stochastic case is

$$V(k,z) = \sup_{k' \in \Gamma(k,z)} F(k,k',z) + \beta \mathbf{E}\left[V(k',z') \mid z\right] \qquad (3)$$

where z follows a first order Markov Process : A sequence of random variables $\{z_t\}_{t=0}^{\infty}$ with the property that the conditional expectations depend only on the last realization of the process.

- The only additional complication is merely technical: We cannot be sure that the true value function V^* is integrable so it might not solve the Bellman Equation (3).
- When z is countable again no problem (Bertsekas, 1976).
- In general, we can only state Verification Theorems.
- (Feller Property and Continuity)
- The policy (feed-back) rule describes a Markovian stochastic process: $k' = g(k,z)$ (Stationary distributions).