Absolute and Relative Ambiguity Aversion: A Preferential Approach*

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Abstract

We study absolute and relative attitudes toward ambiguity, determined by wealth effects, from a preferential viewpoint.

1 Introduction

In this paper we study absolute and relative attitudes toward ambiguity from a purely preferential viewpoint, starting from a preferential first principle: a preference is, say, decreasing absolute ambiguity averse if, at a higher wealth level, it becomes comparatively less averse to ambiguity. This first principle implies that a proper analysis of absolute attitudes toward ambiguity requires that the underlying risk preference on lotteries is constant absolute risk averse, so that risk attitudes do not intrude in wealth effects. In turn, this implies that different classes of preferences characterize absolute attitudes toward ambiguity, depending on the risk attitude that the underlying risk preference exhibits. For instance, among uncertainty averse preferences, variational preferences (Maccheroni, Marinacci, and Rustichini [16]) characterize constant absolute ambiguity aversion under risk neutrality, but homothetic preferences (Chateauneuf and Faro [6]) characterize it under risk nonneutrality. Therefore, the

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two quite different properties, both conceptually and mathematically, of constant additivity and positive homogeneity may happen to characterize the preference functionals that are constant absolute ambiguity averse. Our results thus underscore the importance of keeping track of risk attitudes (in terms of aversion or love) that may, otherwise, confound the analysis of ambiguity attitudes. They also underscore the importance of keeping track of the unit of account: Absolute and relative attitudes are, indeed, properly modelled via properties of the monetary certainty equivalents (which are in the same unit of account of wealth).

We consider a standard Anscombe-Aumann set up. This choice is motivated by the fact that we want to study how wealth effects change *ambiguity* attitudes, thus we want to control for the effects due to risk attitudes. We denote by $\mathcal{F}$ the set of all Anscombe-Aumann acts $f : S \to \Delta_0(\mathbb{R})$, where S is a state space and $\Delta_0(\mathbb{R})$ is the set of simple monetary lotteries. As usual, preferences over final wealth levels are modelled by a binary relation $\succsim$. Given a wealth level w and an act f, we define by f^w the act whose final monetary outcomes are the outcomes of f shifted by w (see Section 2.1, for a formal definition). Given this, we define preferences at wealth level w by

$$f \succsim^w g \overset{def}{\iff} f^w \succsim g^w.$$

We say that $\succsim$ is decreasing absolute ambiguity averse if at higher wealth levels, that is $w' > w$, $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$ (in the sense of Ghirardato and Marinacci [9]). This definition is an adaptation to the ambiguity setting of the rather classic definition of decreasing absolute *risk* aversion. In a similar fashion, we also define the notions of increasing and constant absolute ambiguity aversion (see Definition 3).

In the paper, we characterize absolute ambiguity attitudes for the class of rational preferences. This class of preferences is very large and contains several models of choice which are common in the literature (e.g., maxmin, α-maxmin, smooth ambiguity, and variational preferences). These preferences are known to admit a representation of the form $V : \mathcal{F} \to \mathbb{R}$ such that

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F}, \tag{1}$$

where u is a von-Neumann-Morgestern expected utility functional over $\Delta_0(\mathbb{R})$ and I is a normalized and monotone functional that maps utility profiles $s \mapsto u(f(s))$ into the real line. This decomposition of the utility function V dates back to Schmeidler

[18]. There, it played a key role in characterizing Choquet expected utility preferences.[1] From a behavioral point of view, this decomposition is particularly useful since the pair (u, I), other than representing $\succsim$ as in (1), characterizes the attitudes of the decision maker toward risk and ambiguity: Namely, u characterizes the risk attitudes of the decision maker, while I describes the ambiguity attitudes. This specific feature of this decomposition has been emphasized by Ghirardato and Marinacci [8] and exploited several times in the literature.[2] It is no surprise that also in this work the different roles of u and I will play a key role.

As in the risk case, it is not hard to show that absolute attitudes do not provide an exhaustive class of categories with which we can classify rational preferences. In other words, there exist rational preferences that are neither decreasing, nor increasing, nor constant absolute ambiguity averse. When a rational preference relation $\succsim$ exhibits one of these three absolute ambiguity attitudes, we will say that $\succsim$ is *classifiable*. Our first result (Proposition 3) states that if $\succsim$ is a classifiable rational preference, then $\succsim$ must be constant absolute risk averse (henceforth, CARA). Conceptually, this is an important fact because, in this way, risk attitudes do not intrude in wealth effects and all the differences in terms of attitudes toward uncertainty can be then rightfully attributed to attitudes toward ambiguity. With this in mind, we proceed by characterizing absolute ambiguity attitudes using the decomposition (u, I) (Theorem 2 and Corollary 1). The following table provides an informal summary of our characterization for a classifiable $\succsim$:

	Risk averse	Risk loving	Risk neutral
DAAA	I superhomogeneous	I subhomogeneous	I constant superadditive
IAAA	I subhomogeneous	I superhomogeneous	I constant subadditive
CAAA	I homogeneous	I homogeneous	I constant additive

The table should be read as follows: Under the assumption of classifiability, the rows specify the absolute ambiguity attitudes while the columns specify the risk attitudes, be they averse, loving, or neutral;[3] each cell then provides a full characterization in terms of the functional I. For example, consider a preference relation which

[1] In [18], the functional I is the Choquet integral.

[2] For example, it has been useful in characterizing comparative ambiguity attitudes, as in Ghirardato and Marinacci [9], as well as in exploring the relation between ambiguity attitudes and preference for the timing of resolution of uncertainty, as in Strzalecki [19].

[3] Being classifiable, $\succsim$ must be CARA (Proposition 3). Thus, the von-Neumann-Morgenstern

is decreasing absolute ambiguity averse (DAAA) and risk averse. By Theorem 2, I is superhomogeneous. On the other hand, if I is assumed to be superhomogeneous, the table shows that there are only two possibilities for a classifiable preference: either $\succsim$ is risk averse and DAAA or $\succsim$ is risk loving and IAAA.

The table also shows that (Corollary 3) invariant biseparable preferences – so in particular α-maxmin and Choquet expected utility preferences – are classifiable if and only if they are constant absolute ambiguity averse (CAAA). The reason is simple: For this class of preferences, the functional I is both positively homogeneous and constant additive.

The dichotomic properties of the functional I in the risk neutral and nonneutral cases, most evident for CAAA preferences, are the outcome of a unit of account problem. In fact, though wealth effects are in monetary units (as traditional in Economics), for each act f the number $I(u(f))$ is in von-Neumann-Morgenstern utils.[4] In contrast, if v denotes the von-Neumann-Morgenstern utility function on monetary outcomes, then the map $c : \mathcal{F} \to \mathbb{R}$, defined by

$$c(f) = v^{-1}(I(u(f))) \qquad \forall f,$$

is a *monetary* certainty equivalent. Clearly, the unit of account of c is the same of the wealth w. We show that monetary certainty equivalents emerge as the proper representation for absolute attitudes (Proposition 4); for example, $\succsim$ is DAAA if and only if c is wealth superadditive, that is,

$$c(f^w) \geq c(f) + w \qquad \forall w > 0$$

for every act f. To sum up, a consistent use of the unit of account allows for a clear-cut characterization of absolute ambiguity attitudes.

We conclude the section on absolute attitudes toward ambiguity by focusing on the subclass of uncertainty averse preferences. For this class, we provide a characterization of absolute attitudes in terms of their dual representation, that is, in terms of properties of their ambiguity aversion index (Theorem 3). For this particular class, we are able to show how constant absolute ambiguity attitudes are characterized by two radically different models: variational preferences, under risk neutrality, and homothetic preferences under risk nonneutrality (Corollaries 4 and 7).

utility function over monetary outcomes can be normalized to be either $v(c) = -\frac{1}{\alpha}e^{-\alpha c}$ with $\alpha \neq 0$ or $v(c) = c$.

[4]Since I is normalized, if an act f is such that, for some scalar k, $u(f(s)) = k$ for all $s \in S$, then $I(u(f)) = k$.

Finally, we conduct a similar analysis for relative ambiguity aversion. A preference is, say, decreasing relative ambiguity averse if, at a higher proportional wealth level, it becomes comparatively less averse to ambiguity. Similarly to the absolute case, we obtain that a proper analysis of *relative* attitudes toward ambiguity requires that the underlying risk preference on lotteries is constant relative risk averse (CRRA), so that *relative* risk attitudes do not intrude in proportional wealth effects. Our analysis of relative attitudes reinforces our main message: It is fundamental to keep track of risk attitudes in studying ambiguity attitudes, be they absolute or relative.

Related literature Absolute attitudes toward ambiguity have been previously studied in a few insightful papers. On the one hand, Cherbonnier and Gollier [7] propose and characterize a preferential definition of absolute attitudes toward ambiguity within the α-maxmin and the smooth ambiguity models. The key differences with our work are that Cherbonnier and Gollier focus on the portfolio implications of their characterizations and, since they do not operate in an Anscombe and Aumann setup, they are not able to decouple risk and ambiguity attitudes, which is essential to our preferential analysis. On the other hand, Grant and Polak [14] identify the Weak Certainty Independence Axiom (e.g., variational preferences) as modeling constant absolute ambiguity aversion, while Xue [20] and [21] consider general attitudes by suitably weakening such independence axiom and by axiomatizing a constant superadditive version of variational preferences as well as two equivalent representations of uncertainty averse preferences. Independently of Xue, Ghirardato and Siniscalchi [11] studied a similar notion of absolute ambiguity attitudes in a general class of symmetric preferences. Relative to this latter papers, the key difference with our work is that our approach directly addresses the effect of baseline monetary (rather than utility) shifts. As a consequence, our analysis is consistent with their results under risk neutrality. Intuitively, in the latter three papers absolute ambiguity attitudes are defined in terms of utility shifts rather than wealth shifts. A similar intuition is also present in Klibanoff, Marinacci, and Mukerji [15] as well as in Strzalecki [19].

2 Preliminaries

2.1 Setup

We consider a generalized version of the Anscombe and Aumann [1] setup with a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a

convex set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f : S \to X$ that are Σ-measurable and take on finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act that takes value x. Thus, with the usual slight abuse of notation, we identify X with the subset of constant acts in $\mathcal{F}$. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$. Given a binary relation $\succsim$ on $\mathcal{F}$ (a *preference*), for each $f \in \mathcal{F}$ we denote by $x_f \in X$ the certainty equivalent of f, that is, $x_f \sim f$.[5] Given a function $u : X \to \mathbb{R}$, we denote by $\operatorname{Im} u$ the set $u(X)$; in particular, observe that $u \circ f \in B_0(\Sigma)$ when $f \in \mathcal{F}$.[6]

In what follows, we will consider affine maps $^\circ : X \to X$, that is, $(\alpha x + (1-\alpha) y)^\circ = \alpha x^\circ + (1-\alpha) y^\circ$ for all $x, y \in X$ and $\alpha \in [0,1]$. These maps can be naturally extended to $\mathcal{F}$ by defining $f \mapsto f^\circ$ where $f^\circ(s) = f(s)^\circ$ for all $s \in S$.

The paper relies on the following comparative notion of Ghirardato and Marinacci [9].

Definition 1 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$, we say that $\succsim_1$ is more ambiguity averse than $\succsim_2$ if, for each $f \in \mathcal{F}$ and $x \in X$, $f \succsim_1 x$ implies $f \succsim_2 x$.*

An important example of a convex consequence set X is that of all *simple monetary lotteries*:

$$\Delta_0(\mathbb{R}) = \left\{ x \in [0,1]^{\mathbb{R}} : x(c) \neq 0 \text{ for finitely many } c \in \mathbb{R} \text{ and } \sum_{c \in \mathbb{R}} x(c) = 1 \right\}.$$

For our purposes, the most important bijective affine transformation on $\Delta_0(\mathbb{R})$ is the one induced by a scalar w, interpreted as a wealth level: for each x in $\Delta_0(\mathbb{R})$, x^w is the lottery such that $x^w(c) = x(c-w)$ for all $c \in \mathbb{R}$.[7] We thus interpret the outcome of a lottery, $c \in \mathbb{R}$, as a final wealth level. Thus, given x in $\Delta_0(\mathbb{R})$, if the decision maker has wealth w, we interpret x^w as being the distribution on final wealth levels. In fact, lottery x yields a consequence $d \in \mathbb{R}$ (on top of w) with probability $x(d)$ and the probability of having as final wealth $w + d$, that is $x^w(w+d)$, is equal to $x(d)$. This implies that $x^w(w+d) = x(d)$ for all $d \in \mathbb{R}$ which is equivalent to our definition of x^w.

[5] Note that given f, x_f is a lottery that, received with certainty in each state s, is indifferent to f. Thus, x_f is a risky prospect which is independent of the realization on S.

[6] The mathematical notions used in the main text, but not defined there, are collected in Appendix A.

[7] The collection of maps $^w : \Delta_0(\mathbb{R}) \to \Delta_0(\mathbb{R})$ is an Abelian group under the composition operation; in particular, $x^{w+w'} = (x^w)^{w'}$ for all $x \in \Delta_0(\mathbb{R})$ and all $w, w' \in \mathbb{R}$.

2.2 Axioms and representations

We will consider the following classes of preferences $\succsim$ on $\mathcal{F}$: rational preferences (Cerreia-Vioglio et al. [2]), uncertainty averse preferences (Cerreia-Vioglio et al. [3]), invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci [10]), variational preferences (Maccheroni, Marinacci, and Rustichini [16]), and maxmin preferences (Gilboa and Schmeidler [13]). They rely on the following axioms, discussed in the original papers as well as in Gilboa and Marinacci [12].

Axiom A. 1 (Weak Order) *$\succsim$ is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom A. 3 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.*

Axiom A. 4 (Risk Independence) *If $x, y, z \in X$ and $\alpha \in (0,1)$,*

$$x \sim y \Longrightarrow \alpha x + (1-\alpha) z \sim \alpha y + (1-\alpha) z.$$

Axiom A. 5 (Convexity) *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succsim f.$$

Axiom A. 6 (Weak C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x \Longrightarrow \alpha f + (1-\alpha) y \succsim \alpha g + (1-\alpha) y.$$

Axiom A. 7 (C-Independence) *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$,*

$$f \succsim g \iff \alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x.$$

The following omnibus result collects some of the results that the above papers proved for the classes of preferences that they studied.

Theorem 1 (Omnibus) *A preference $\succsim$ on $\mathcal{F}$ satisfies Weak Order, Monotonicity, Continuity, and Risk Independence if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ such that the criterion $V : \mathcal{F} \to \mathbb{R}$, given by*

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F} \tag{2}$$

represents $\succsim$. The function u is cardinally unique and, given u, I is the unique normalized, monotone, and continuous functional satisfying (2). In this case, we say that $\succsim$ is a rational preference. *A rational preference satisfies:*

(i) *C-Independence if and only if I is constant linear; in this case, we say that $\succsim$ is* invariant biseparable.[8]

(ii) *Convexity if and only if I is quasiconcave; in this case, we say that $\succsim$ is an* uncertainty averse preference.

(iii) *Convexity and Weak C-Independence if and only if I is quasiconcave and constant additive; in this case, we say that $\succsim$ is a* variational preference.

(iv) *Convexity and C-Independence if and only if I is quasiconcave and constant linear; in this case, we say that $\succsim$ is a* maxmin preference.

Given u and I as in (2), we call (u, I) a (*canonical*) *representation* of the rational preference $\succsim$.[9]

We say that $\succsim$ on $\mathcal{F}$ is a *homothetic* (*uncertainty averse*) *preference* if there exists a canonical representation (u, I), with $\operatorname{Im} u$ equal to either $(-\infty, 0)$ or $(0, \infty)$, such that

$$I(\varphi) = \min_{p \in \Delta} \int \varphi c(p)^{-\operatorname{sgn} \varphi} dp = \begin{cases} \min_{p \in \Delta} \frac{\int \varphi dp}{c(p)} & \text{if } \operatorname{Im} u = (0, \infty) \\ \min_{p \in \Delta} c(p) \int \varphi dp & \text{if } \operatorname{Im} u = (-\infty, 0) \end{cases}$$

where $c : \Delta \to [0, 1]$ is normalized, upper semicontinuous, and quasiconcave.[10] Note that I is positively homogeneous. These preferences, proposed by Chateauneuf and Faro [6], are a natural counterpart to variational preferences with positive homogeneity in place of constant additivity. As [6] showed, positive homogeneity is implied by a form of homotheticity/independence with respect to a worst consequence, when such a consequence exists (something that in this paper we do not allow for; this is why these preferences are not included in the omnibus theorem).

[8] Invariant biseparable preferences correspond to the general class of $\alpha(f)$-maxmin preferences of Ghirardato, Maccheroni, and Marinacci [10], which includes the Choquet expected utility preferences of Schmeidler [18].

[9] In Appendix B, we discuss more in detail the uniqueness features of canonical representations.

[10] The function c is normalized if and only if $\max_{p \in \Delta} c(p) = 1$.

3 Results

3.1 Induced preferences

A preference $\succsim$ on $\mathcal{F}$ induces, through an affine and bijective transformation $^\circ$ on X, a preference $\succsim^\circ$ on $\mathcal{F}$ given by

$$f \succsim^\circ g \Longleftrightarrow f^\circ \succsim g^\circ.$$

The induced preference inherits some of the properties of the original preference.

Proposition 1 *Let $\succsim$ be a preference on $\mathcal{F}$ and $^\circ : X \to X$ an affine bijection. Then:*

(i) If $\succsim$ is a rational preference, so is $\succsim^\circ$.

(ii) If $\succsim$ is an uncertainty averse preference, so is $\succsim^\circ$.

Next, we compare the ambiguity aversion of different induced preferences.

Proposition 2 *Let $\succsim$ be a rational preference on $\mathcal{F}$ and $^\circ$ and $^\#$ two affine and bijective transformations on X. If $\succsim^\circ$ is more ambiguity averse than $\succsim^\#$, then u° is a positive affine transformation of $u^\#$.*[11]

In the rest of the paper (with the exception of Section 4) we specialize the set of consequences X to be made of monetary lotteries, that is $X = \Delta_0(\mathbb{R})$, and the maps $^\circ$ and $^\#$ to be ^{w} and $^{w'}$. Moreover, note that an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$ takes the form $u(x) = \sum_{c\in\mathbb{R}} v(c)\, x(c)$, where $v : \mathbb{R} \to \mathbb{R}$.

Throughout the paper we make the following assumption.

Assumption The function v is strictly increasing and continuous.

In this monetary setup, we have the following classic notion.

Definition 2 *A preference $\succsim$ on $\mathcal{F}$ is* constant absolute risk averse *(CARA) if, for any two levels w and w' of wealth, the induced preferences $\succsim^w$ and $\succsim^{w'}$ agree on $\Delta_0(\mathbb{R})$.*

This behavioral definition amounts to say that preferences over lotteries are unaffected by the level of wealth w. A routine argument shows that, if $\succsim$ (on lotteries) is

[11] Here, u° and $u^\#$ are part of a canonical representation for, respectively, $\succsim^\circ$ and $\succsim^\#$.

represented by an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$, then $\succsim$ is CARA if and only if there exist $\alpha \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v_\alpha(c) = \begin{cases} -a\frac{1}{\alpha}e^{-\alpha c} + b & \text{if } \alpha \neq 0 \\ ac + b & \text{if } \alpha = 0 \end{cases}, \tag{3}$$

that is, if v_α is either exponential or affine. In the former case, $\succsim$ is a CARA preference which is not risk neutral; in particular, it is (strictly) risk averse if $\alpha > 0$ and (strictly) risk loving if $\alpha < 0$.[12] Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \alpha > 0 \\ (b, +\infty) & \text{if } \alpha < 0 \\ (-\infty, +\infty) & \text{if } \alpha = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\succsim$ is risk averse and $b = \inf \operatorname{Im} u$ when $\succsim$ is risk loving. Momentarily, this extremum role of b will play a key role in Theorem 2.

3.2 Rational preferences

Absolute ambiguity attitudes describe how the decision maker's preferences over uncertain monetary alternatives vary as his wealth changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational and for uncertainty averse preferences.

Definition 3 *A preference $\succsim$ on $\mathcal{F}$ is decreasing (increasing, constant) absolute ambiguity averse if, for any two levels w and w' of wealth, $w' > w$ implies that $\succsim^w$ is more (less, equally) ambiguity averse than $\succsim^{w'}$.*[13]

As this classification is not exhaustive, we say that a preference is (absolutely) *classifiable* in terms of absolute ambiguity aversion if it can be classified according to this definition, that is, if it is either decreasing or increasing or constant absolute ambiguity averse. The next result shows that being CARA is a necessary condition for a preference in order to be classifiable: in fact, in this way risk attitudes do not intrude in wealth effects.

[12] In what follows, we omit "strictly" since a CARA preference is either risk neutral ($\alpha = 0$) or strictly risk averse ($\alpha > 0$) or strictly risk loving ($\alpha < 0$).

[13] Clearly, $\succsim^w$ is less ambiguity averse than $\succsim^{w'}$ if and only if $\succsim^{w'}$ is more ambiguity averse than $\succsim^w$. Similarly, equally ambiguity averse means that $\succsim^w$ is, at the same time, more and less ambiguity averse than $\succsim^{w'}$.

Proposition 3 *A rational preference $\succsim$ is classifiable only if it is CARA.*

We first characterize absolute ambiguity attitudes for rational preferences.

Theorem 2 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with a representation (u, I). The following statements are equivalent:*

(i) $\succsim$ is decreasing absolute ambiguity averse;

(ii) $\succsim$ is CARA and I is:

(a) concave (convex) at b provided $\succsim$ is risk averse (loving);

(b) constant superadditive provided $\succsim$ is risk neutral.

(iii) $\succsim$ is classifiable and I satisfies (a) or (b).

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (a) concavity (convexity) at b reduces to positive superhomogeneity (subhomogeneity).

Dual versions of this theorem are easily seen to hold for increasing and constant absolute ambiguity aversion (for this latter case see Corollary 1). In particular, by keeping the same premises, Theorem 2 takes a similar form with (i), (ii), and (iii) replaced by:

(i)' $\succsim$ is increasing absolute ambiguity averse;

(ii)' $\succsim$ is CARA and I is:

(a) convex (concave) at b provided $\succsim$ is risk averse (loving);

(b) constant subadditive provided $\succsim$ is risk neutral.

(iii)' $\succsim$ is classifiable and I satisfies (a) or (b).

The next result characterizes constant absolute ambiguity aversion for classifiable rational preferences. At the same time, the result still holds if instead of requiring $\succsim$ being classifiable we only require $\succsim$ to be CARA.[14]

Corollary 1 *Let $\succsim$ be a classifiable rational preference on $\mathcal{F}$ with representation (u, I). Then:*

[14] Recall that by Proposition 3, classifiable preferences are CARA.

(i) If $\succsim$ is risk neutral, it is constant absolute ambiguity averse if and only if I is constant additive.[15]

(ii) If $\succsim$ is not risk neutral, it is constant absolute ambiguity averse if and only if I is affine at b.

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (ii) the affinity at b reduces to positive homogeneity, that is, $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\lambda > 0$. Risk neutrality and risk aversion of $\succsim$ may thus translate in, respectively, constant additivity and positive homogeneity of I which are two mathematically and decision theoretically distinct properties.

In behavioral terms, Weak C-Independence underlies the risk neutral case of the previous result.

Corollary 2 *A risk neutral rational preference is constant absolute ambiguity averse if and only if it satisfies Weak C-Independence.*

Along with Corollary 1, the next result shows that invariant biseparable preferences are the rational preferences that, when classifiable, are constant absolute ambiguity averse regardless of their risk attitudes.

Corollary 3 *Let $\succsim$ be an invariant biseparable preference $\succsim$ on $\mathcal{F}$. The following conditions are equivalent:*

(i) $\succsim$ is classifiable;

(ii) $\succsim$ is constant absolute ambiguity averse;

(iii) $\succsim$ is CARA.

As mentioned in the introduction, Corollary 2 (and Corollary 4 below) show that our analysis is consistent, under risk neutrality, with the approach of Grant and Polak [14]. The role of constant superadditivity in Theorem 3 shows that a similar consistency holds with the results of Xue [20] and [21].

Since we are dealing with acts yielding monetary lotteries, it is also possible to discuss *monetary* certainty equivalents. Given a canonical representation (u, I), we can define the functional $c : \mathcal{F} \to \mathbb{R}$ by the rule $c(f) = v^{-1}(I(u(f)))$. Note that,

[15] Recall that $\operatorname{Im} u = \mathbb{R}$ in the risk neutral case.

given $f \in \mathcal{F}$, $c(f)$ is the monetary amount that, received with certainty in each state of the world, makes the decision maker indifferent between f and the constant (risk free) act paying $c(f)$. We will say that c is wealth superadditive (resp., subadditive, additive) if and only if for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) \geq c(f) + w \qquad (\text{resp.}, \ \leq, =).$$

Proposition 4 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). Then:*

(i) $\succsim$ is decreasing absolute ambiguity averse if and only if c is wealth superadditive and $\succsim$ is CARA.

(ii) $\succsim$ is increasing absolute ambiguity averse if and only if c is wealth subadditive and $\succsim$ is CARA.

(iii) $\succsim$ is constant absolute ambiguity averse if and only if c is wealth additive and $\succsim$ is CARA.

3.3 Uncertainty averse preferences

Assume that $\succsim$ is an uncertainty averse preference. By definition, $\succsim$ is also rational. If (u, I) is a (rational) representation of $\succsim$, then there exists a unique (minimal) linearly continuous $G \in \mathcal{G}(\operatorname{Im} u \times \Delta)$ such that $I(\psi) = \inf_{p \in \Delta} G\left(\int \psi dp, p\right)$ for all $\psi \in B_0(\Sigma, \operatorname{Im} u)$. Uncertainty averse preferences are thus characterized by the pair (u, G). In particular, the function G can be seen as an index of ambiguity aversion.[16]

Now we characterize absolute ambiguity attitudes for uncertainty averse preferences in terms of the pair (u, G).

Theorem 3 *Let $\succsim$ be an uncertainty averse preference on $\mathcal{F}$ with a representation (u, G). The following statements are equivalent:*

(i) $\succsim$ is decreasing absolute ambiguity averse;

(ii) $\succsim$ is CARA and G is:

(a) $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t, p) + (1-\lambda) b$ $(\leq)$ for all $(t, p) \in \operatorname{Im} u \times \Delta$ and for all $\lambda \in (0, 1)$ provided $\succsim$ is risk averse (loving);

[16] These facts can be found in [3]. Because of the minimality of G, we have $G(t, p) = \sup_{f \in \mathcal{F}} \left\{ u(x_f) : \int u(f)\, dp \leq t \right\}$.

(b) $G(t+k,p) \geq G(t,p)+k$ *for all* $(t,p) \in \operatorname{Im} u \times \Delta$ *and for all* $k \geq 0$ *provided* $\succsim$ *is risk neutral.*

(iii) $\succsim$ *is classifiable and* G *satisfies (a) or (b).*

Here as well, dual versions of this result hold in the increasing and constant absolute ambiguity averse case (with, respectively, opposite inequalities and equalities).

The next corollary shows that the behavioral characterization established in Corollary 2 leads to variational preferences when preferences are uncertainty averse.

Corollary 4 *A risk neutral uncertainty averse preference is constant absolute ambiguity averse if and only if it is a variational preference.*

The next result reports a noteworthy consequence of the previous theorem for uncertainty averse preferences which feature a concave G (or, equivalently, a concave I).

Corollary 5 *Let* $\succsim$ *be an uncertainty averse preference which is CARA and risk averse. If* G *is concave, then* $\succsim$ *is decreasing absolute ambiguity averse.*

This corollary can be sharpened for variational preferences (which feature a concave G) that are not maxmin, and so are not invariant biseparable.

Corollary 6 *A variational preference, which is not maxmin and not risk neutral, satisfies:*

(i) decreasing absolute ambiguity aversion if and only if it is CARA and risk averse;

(ii) increasing absolute ambiguity aversion if and only if it is CARA and risk loving.

In order to characterize constant absolute ambiguity attitudes when the preference is not risk neutral, we need to consider homothetic preferences.

Corollary 7 *A risk nonneutral uncertainty averse preference is constant absolute ambiguity averse if and only if it is CARA and homothetic.*

To sum up, depending on risk attitudes, homothetic or variational preferences characterize constant absolute ambiguity attitudes for uncertainty averse preferences.

3.4 Smooth ambiguity preferences

Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be a strictly increasing and continuous function, and μ a Borel probability measure over Δ. The preferences represented by a pair (u, I), where

$$I(\varphi) = \phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right) \tag{4}$$

are called *smooth ambiguity preferences* (Klibanoff, Marinacci and, Mukerji [15]). They are uncertainty averse when ϕ is concave.

Proposition 5 *Let $\succsim$ be a CARA smooth ambiguity preference. If $\phi(t) = -e^{-\gamma t}$ with $\gamma > 0$, then*

(i) If $\succsim$ is risk neutral, then it is constant absolute ambiguity averse.

(ii) If $\succsim$ is risk averse, then it is decreasing absolute ambiguity averse.

In our setup an exponential ϕ thus yields constant absolute ambiguity aversion, as argued in [15], as long as $\succsim$ is risk neutral.

Let $c_f(p) \in \mathbb{R}$ be the certainty equivalent of act f under p, that is, $c_f(p) = v^{-1}\left(\int u(f)\, dp\right)$. By setting $w = \phi \circ v : \mathbb{R} \to \mathbb{R}$, the smooth ambiguity representation can be written as

$$\begin{aligned} V(f) &= \left(v \circ w^{-1}\right)\left(\int w(c_f(p))\, d\mu\right) \\ &= \left(v \circ w^{-1}\right)\left(\int \left(w \circ v^{-1}\right)\left(\int u(f)\, dp\right) d\mu\right) \end{aligned}$$

The function w can be interpreted as aversion to epistemic uncertainty.[17] When v is the identity, we have $\phi = w$ and so point (i) of the previous proposition can be interpreted in terms of constant attitude toward such uncertainty. When both $v(c) = -e^{-\alpha c}$ and $w(c) = -e^{-\beta c}$ are risk averse exponentials, with $\beta > \alpha > 0$, then $\phi(t) = -(-t)^{\frac{\beta}{\alpha}}$. The condition $\beta > \alpha$ can be interpreted as higher aversion to epistemic uncertainty than to risk (both being constant absolute averse). The next result shows that in this double exponential case the resulting absolute ambiguity aversion is decreasing.

Proposition 6 *Let $\succsim$ be a CARA smooth ambiguity preference, with $b \leq 0$ in (3), and suppose $\phi(t) = -(-t)^{\gamma}$ for all $t < 0$ with $\gamma > 1$. If $\succsim$ is risk averse, then it is decreasing absolute ambiguity averse.*

[17] See Marinacci [17] for a discussion of this version of the smooth ambiguity model.

4 Relative ambiguity aversion

In this subsection, we briefly explore relative ambiguity aversion.[18] For this reason, we focus on monetary lotteries which yield only strictly positive amounts of money: $X = \Delta_0(\mathbb{R}_{++})$. As before, we consider a group of transformations on $\Delta_0(\mathbb{R}_{++})$, this time, indexed by $\mathbb{R}_{++}$. In particular, given $\nu > 0$, we denote by ${}^{\nu} : \Delta_0(\mathbb{R}_{++}) \to \Delta_0(\mathbb{R}_{++})$ the affine and onto map such that $x^{\nu}(\nu c) = x(c)$ for all $c \in \mathbb{R}_{++}$ and for all $x \in \Delta_0(\mathbb{R}_{++})$.[19] Given $\nu > 0$, the lottery x^{ν} is interpreted as the distribution of final wealth if $\nu\%$ of the decision maker's wealth is invested in x. In this monetary setup, we have the following classic notion.[20]

Definition 4 *A preference $\succsim$ on $\mathcal{F}$ is* constant relative risk averse *(CRRA) if, for any two strictly positive levels ν and η, the induced preferences $\succsim^{\nu}$ and $\succsim^{\eta}$ agree on $\Delta_0(\mathbb{R}_{++})$.*

This behavioral definition amounts to say that preferences over lotteries are unaffected by multiplying factors ν. A routine argument shows that, if $\succsim$ is represented by an affine utility function $u : \Delta_0(\mathbb{R}_{++}) \to \mathbb{R}$, then $\succsim$ is CRRA if and only if there exist $\gamma \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v_{\gamma}(c) = \begin{cases} a\gamma x^{\gamma} + b & \text{if } \gamma \neq 0 \\ a \log c + b & \text{if } \gamma = 0 \end{cases}, \tag{5}$$

that is, if v_{γ} is either a power or the logarithm. Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \gamma < 0 \\ (b, +\infty) & \text{if } \gamma > 0 \\ (-\infty, +\infty) & \text{if } \gamma = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\gamma < 0$ and $b = \inf \operatorname{Im} u$ when $\gamma > 0$. Again, this extremum role of b will play a key role, in particular in Theorem 4. Relative ambiguity attitudes describe how the decision maker's preferences over uncertain monetary alternatives vary as the wealth's proportion invested in them changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational preferences.

[18] Proofs follow very closely the ones carried out for the absolute case.

[19] The collection of maps ${}^{\nu} : \Delta_0(\mathbb{R}_{++}) \to \Delta_0(\mathbb{R}_{++})$ is an Abelian group under the composition operation; in particular, $x^{\nu\eta} = (x^{\nu})^{\eta}$ for all $x \in \Delta_0(\mathbb{R}_{++})$ and all $\nu, \eta \in \mathbb{R}_{++}$.

[20] Even in this section, we maintain the assumption that if $\succsim$ on $\Delta_0(\mathbb{R}_{++})$ is represented by an affine utility function, then its von-Neumann-Morgenstern utility function is strictly increasing and continuous.

Definition 5 *A preference $\succsim$ on $\mathcal{F}$ is decreasing (increasing, constant) relative ambiguity averse if, for any strictly positive ν and η, $\nu > \eta$ implies that $\succsim^{\eta}$ is more (less, equally) ambiguity averse than $\succsim^{\nu}$.*

Since also this classification of preferences is not exhaustive, we say that a preference is *relatively classifiable* (in terms of relative ambiguity aversion) if it can be classified according to this definition, that is, if it is either decreasing or increasing or constant relative ambiguity averse. The next result shows that being CRRA is a necessary condition for a preference in order to be relatively classifiable: in fact, in this way, risk attitudes do not intrude in wealth's proportionality effects.

Proposition 7 *A rational preference $\succsim$ is relatively classifiable only if it is CRRA.*

We next characterize decreasing relative ambiguity attitudes for rational preferences.

Theorem 4 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with a representation (u, I). The following statements are equivalent:*

(i) $\succsim$ is decreasing relative ambiguity averse;

(ii) $\succsim$ is CRRA and I is:

(a) concave (convex) at b provided $\gamma < 0$ ($\gamma > 0$);

(b) constant superadditive provided $\gamma = 0$.

(iii) $\succsim$ is relatively classifiable and I satisfies (a) or (b).

Similar dual characterizations hold for increasing and constant relative ambiguity aversion. Also in this case, it is possible to introduce *monetary* certainty equivalents. Given a canonical representation (u, I), we can again define the functional $c : \mathcal{F} \to \mathbb{R}$ by the rule $c(f) = v^{-1}(I(u(f)))$. We will say that c is wealth superproportional (resp., subproportional, proportional) if and only if for each $f \in \mathcal{F}$ and for each $\nu \geq 1$

$$c(f^{\nu}) \geq \nu c(f) \qquad (\text{resp., } \leq, =).$$

Proposition 8 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). Then:*

(i) $\succsim$ is decreasing relative ambiguity averse if and only if c is wealth superproportional and $\succsim$ is CRRA.

(ii) $\succsim$ *is increasing relative ambiguity averse if and only if* c *is wealth subproportional and* $\succsim$ *is CRRA.*

(iii) $\succsim$ *is constant relative ambiguity averse if and only if* c *is wealth proportional and* $\succsim$ *is CRRA.*

A Appendix: Mathematics

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions. If T is an interval of the real line, set $B_0(\Sigma, T) = \{\psi \in B_0(\Sigma) : \psi(s) \in T \text{ for all } s \in S\}$. We endow both $B_0(\Sigma)$ and $B_0(\Sigma, T)$ with the topology induced by the supnorm.

With a small abuse of notation, we denote by k both the real number and the constant function on S that takes value k. Let $\varphi, \psi \in B_0(\Sigma, T)$. A functional $I : B_0(\Sigma, T) \to \mathbb{R}$ is:

(i) *normalized* if $I(k) = k$ for all $k \in T$;

(ii) *monotone* if $\varphi \geq \psi$ implies $I(\varphi) \geq I(\psi)$;

(iii) *quasiconcave* if $I(\lambda\varphi + (1-\lambda)\psi) \geq \min\{I(\varphi), I(\psi)\}$ for all $\lambda \in (0,1)$;

(iv) *positively superhomogeneous* (*subhomogeneous*) if $I(\lambda\varphi) \geq (\leq) \lambda I(\varphi)$ for all $\lambda \in (0,1)$ such that $\lambda\varphi \in B_0(\Sigma, T)$;

(v) *positively homogeneous* if it is both: positively superhomogeneous and subhomogeneous;[21]

(vi) *concave* (*convex*) at $k \in \mathrm{cl}(T)$ if $I(\lambda\varphi + (1-\lambda)k) \geq (\leq) \lambda I(\varphi) + (1-\lambda)k$ for all $\lambda \in (0,1)$;

(vii) *affine* at $k \in \mathrm{cl}(T)$ if it is both concave and convex at k;

(viii) *constant superadditive* (*subadditive*) if $I(\varphi + k) \geq (\leq) I(\varphi) + k$ for all $k \geq 0$ such that $\varphi + k \in B_0(\Sigma, T)$.

(ix) *constant additive* if I is both constant superadditive and subadditive;[22]

[21] When either $T = (-\infty, 0)$ or $T = (0, \infty)$ or $T = \mathbb{R}$, then I is positively homogeneous if and only if $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\varphi \in B_0(\Sigma, T)$ and for all $\lambda > 0$.

[22] When $T = \mathbb{R}$, then I is constant additive if and only if $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \in \mathbb{R}$.

(x) *constant linear* if I is constant additive and positively homogeneous.

When $k = 0$, concavity (convexity) at k reduces to positive superhomogeneity (subhomogeneity).

As well known, the norm dual space of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ and is a (weak*) compact and convex subset of $ba(\Sigma)$. Elements of Δ are denoted by p or q. We endow Δ and any of its subsets with the weak* topology.

Functions of the form $G : T \times \Delta \to (-\infty, \infty]$, where T is an interval of the real line, will play an important role in the paper. We denote by $\mathcal{G}(T \times \Delta)$ the class of these functions such that:

(i) G is quasiconvex on $T \times \Delta$,

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$,

(iii) $\inf_{p \in \Delta} G(t, p) = t$ for all $t \in T$.

A function $G : T \times \Delta \to (-\infty, \infty]$ is *linearly continuous* if the map

$$\psi \mapsto \inf_{p \in \Delta} G\left(\int \psi dp, p\right)$$

from $B_0(\Sigma, T)$ to $[-\infty, \infty]$ is extended-valued continuous.

B Appendix: Proofs and related material

We begin with a preliminary result that will be used in the appendix.

Lemma 1 *Let $\succsim_1$ and $\succsim_2$ be two rational preferences on $\mathcal{F}$ with representations (u_1, I_1) and (u_2, I_2). The following statements are equivalent:*

(i) $\succsim_1$ is more ambiguity averse than $\succsim_2$;

(ii) There exists $a > 0$ and $b \in \mathbb{R}$ such that $u_1 = au_2 + b$ and $I_1 \leq I_2$ (provided $u_1 = u_2$).

B.1 Generic set of consequences

Proof of Proposition 1. Clearly, $\succsim^\circ$ is well defined. Moreover, we have

$$f \succ^\circ g \iff f \succsim^\circ g \text{ and } g \not\succsim^\circ f \iff f^\circ \succsim g^\circ \text{ and } g^\circ \not\succsim f^\circ \iff f^\circ \succ g^\circ.$$

(i). *Weak Order.* Since $\succsim$ satisfies Weak Order and Monotonicity, it follows that there exist $\bar{x}$ and $\bar{y}$ in X such that $\bar{x} \succ \bar{y}$. Since $^\circ$ is bijective, it follows that there exist $x, y \in X$ such that $\bar{x} = x^\circ$ and $\bar{y} = y^\circ$. By definition of $\succsim^\circ$, we have that

$$\bar{x} \succ \bar{y} \implies x^\circ \succ y^\circ \implies x \succ^\circ y,$$

proving that $\succsim^\circ$ is nontrivial. Consider $f, g \in \mathcal{F}$. Since $f^\circ, g^\circ \in \mathcal{F}$ and $\succsim$ satisfies Weak Order, we have that either $f^\circ \succsim g^\circ$ or $g^\circ \succsim f^\circ$. By definition of $\succsim^\circ$, this implies that either $f \succsim^\circ g$ or $g \succsim^\circ f$ or both, thus proving that $\succsim^\circ$ is complete. Next, consider $f, g, h \in \mathcal{F}$ and assume that $f \succsim^\circ g$ and $g \succsim^\circ h$. By definition of $\succsim^\circ$, we have that $f^\circ \succsim g^\circ$ and $g^\circ \succsim h^\circ$. Since $\succsim$ satisfies Weak Order, we can conclude that $f^\circ \succsim h^\circ$, that is, $f \succsim^\circ h$, proving that $\succsim^\circ$ is transitive. We can conclude that $\succsim^\circ$ satisfies Weak Order.

Monotonicity. Consider $f, g \in \mathcal{F}$ and assume that $f(s) \succsim^\circ g(s)$ for all $s \in S$. By definition of $\succsim^\circ$ and $^\circ$, it follows that $f^\circ(s) = f(s)^\circ \succsim g(s)^\circ = g^\circ(s)$ for all $s \in S$. Since $\succsim$ satisfies Monotonicity, we have that $f^\circ \succsim g^\circ$, that is, $f \succsim^\circ g$.

Continuity. Consider $f, g, h \in \mathcal{F}$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ such that $\alpha_n \to \alpha$ and $\alpha_n f + (1-\alpha_n) g \succsim^\circ h$ for all $n \in \mathbb{N}$. By definition of $\succsim^\circ$ and since $^\circ$ is affine, we have $\alpha_n f^\circ + (1-\alpha_n) g^\circ = (\alpha_n f + (1-\alpha_n) g)^\circ \succsim h^\circ$ for all $n \in \mathbb{N}$. Since $\succsim$ satisfies Mixture Continuity, we have that $(\alpha f + (1-\alpha) g)^\circ = \alpha f^\circ + (1-\alpha) g^\circ \succsim h^\circ$. We can conclude that $\alpha f + (1-\alpha) g \succsim^\circ h$. Thus, the set $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim^\circ h\}$ is closed. A symmetric argument yields the closure of $\{\alpha \in [0,1] : h \succsim^\circ \alpha f + (1-\alpha) g\}$.

Risk Independence. Consider $x, y, z \in X$, $\alpha \in (0,1)$, and assume that $x \sim^\circ y$. It follows that $x^\circ \sim y^\circ$. Since $\succsim$ satisfies Risk Independence and $^\circ$ is affine, we have that

$$(\alpha x + (1-\alpha) z)^\circ = \alpha x^\circ + (1-\alpha) z^\circ \sim \alpha y^\circ + (1-\alpha) z^\circ = (\alpha y + (1-\alpha) z)^\circ,$$

proving that $\alpha x + (1-\alpha) z \sim^\circ \alpha y + (1-\alpha) z$.

(ii). We only need to show that $\succsim^\circ$ also satisfies Convexity.

Convexity. Consider $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$ and assume that $f \sim^\circ g$. It follows that $f^\circ \sim g^\circ$. Since $\succsim$ satisfies Convexity and $^\circ$ is affine, we have that $(\alpha f + (1-\alpha) g)^\circ = \alpha f^\circ + (1-\alpha) g^\circ \succsim f^\circ$, that is, $\alpha f + (1-\alpha) g \succsim^\circ f$. ■

Proof of Proposition 2. By Proposition 1, both preferences $\succsim^{\circ}$ and $\succsim^{\#}$ are rational preferences. By Theorem 1, both preferences have a canonical representation: (u°, I°) and $\left(u^{\#}, I^{\#}\right)$. In particular, u° and $u^{\#}$ are nonconstant and affine. Since $\succsim^{\circ}$ is more ambiguity averse than $\succsim^{\#}$, we have that $y \succsim^{\circ} x$ implies $y \succsim^{\#} x$. Thus, we conclude that $u^{\circ}(y) \geq u^{\circ}(x)$ implies $u^{\#}(y) \geq u^{\#}(x)$. By [10, Corollary B.3], the statement follows. ■

Proposition 9 *Let (u, I) and $\left(\bar{u}, \bar{I}\right)$ be two canonical rational representations. The two representations (u, I) and $\left(\bar{u}, \bar{I}\right)$ represent the same rational preference $\succsim$ if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$\bar{u} = au + b \text{ and } \bar{I}(\cdot) = aI\left(\frac{\cdot - b}{a}\right) + b. \tag{6}$$

Moreover,

(i) I is concave if and only if $\bar{I}$ is concave.

(ii) I is concave (convex, affine) at c if and only if $\bar{I}$ is concave (convex, affine) at $ac + b$.

(iii) I is constant superadditive (subadditive, additive) if and only if $\bar{I}$ is constant superadditive (subadditive, additive), provided $\operatorname{Im} u = \mathbb{R}$.

Proof. The first part of the statement follows from [2, Proposition 1]. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = at + b$ for all $t \in \mathbb{R}$. Define $T : B_0(\Sigma, \operatorname{Im} \bar{u}) \to B_0(\Sigma, \operatorname{Im} u)$ as $T(\varphi) = \frac{\varphi - b}{a}$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} \bar{u})$. Note that both functions are bijective and $\bar{I} = f \circ I \circ T$ as well as $I = f^{-1} \circ \bar{I} \circ T^{-1}$.

(i). "Only if". Assume that I is concave. Since f and T are monotone and affine and $\bar{I} = f \circ I \circ T$, it follows that $\bar{I}$ is concave. "If". Note that $I = f^{-1} \circ \bar{I} \circ T^{-1}$. Assume that $\bar{I}$ is concave. Since f^{-1} and T^{-1} are monotone and affine, it follows that I is concave.

(ii). "Only if". Assume that I is concave (convex, affine) at $c \in \operatorname{cl}(\operatorname{Im} u)$. Note that $\bar{c} = ac + b \in \operatorname{cl}(\operatorname{Im} \bar{u})$. It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} \bar{u})$ and for each

$\lambda \in (0,1)$

$$\begin{aligned}
\bar{I}\left(\lambda\varphi + (1-\lambda)\,\bar{c}\right) &= aI\left(\frac{\lambda\varphi + (1-\lambda)\,\bar{c} - b}{a}\right) + b = aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)\,\frac{\bar{c} - b}{a}\right) + b \\
&= aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)\,\frac{ac + b - b}{a}\right) + b \\
&= aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)\,c\right) + b \\
&\geq (\leq, =)\, a\left(\lambda I\left(\frac{\varphi - b}{a}\right) + (1-\lambda)\,c\right) + b \\
&= \lambda\left(aI\left(\frac{\varphi - b}{a}\right) + b\right) + (1-\lambda)\,(ac + b) = \lambda\bar{I}\left(\varphi\right) + (1-\lambda)\,\bar{c},
\end{aligned}$$

proving that $\bar{I}$ is concave (convex, affine) at $\bar{c}$. "If". Assume that $\bar{I}$ is concave (convex, affine) at $\bar{c} = ac + b$. It follows that for each $\varphi \in B_0\left(\Sigma, \operatorname{Im} u\right)$ and for each $\lambda \in (0,1)$

$$\begin{aligned}
I\left(\lambda\varphi + (1-\lambda)\,c\right) &= \frac{1}{a}\bar{I}\left(a\left(\lambda\varphi + (1-\lambda)\,c\right) + b\right) - \frac{b}{a} \\
&= \frac{1}{a}\bar{I}\left(\lambda\left(a\varphi + b\right) + (1-\lambda)\,(ac + b)\right) - \frac{b}{a} \\
&= \frac{1}{a}\bar{I}\left(\lambda\left(a\varphi + b\right) + (1-\lambda)\,\bar{c}\right) - \frac{b}{a} \\
&\geq (\leq, =)\,\frac{1}{a}\left(\lambda\bar{I}\left(a\varphi + b\right) + (1-\lambda)\,\bar{c}\right) - \frac{b}{a} \\
&= \lambda\left(\frac{1}{a}\bar{I}\left(a\varphi + b\right) - \frac{b}{a}\right) + (1-\lambda)\left(\frac{\bar{c}}{a} - \frac{b}{a}\right) = \lambda I\left(\varphi\right) + (1-\lambda)\,c,
\end{aligned}$$

proving that I is concave (convex, affine) at c.

(iii). "Only if". Assume that I is constant superadditive (subadditive, additive). It follows that for each $\varphi \in B_0\left(\Sigma, \operatorname{Im} \bar{u}\right)$ and for each $k \geq 0$

$$\begin{aligned}
\bar{I}\left(\varphi + k\right) &= aI\left(\frac{\varphi + k - b}{a}\right) + b = aI\left(\frac{\varphi - b}{a} + \frac{k}{a}\right) + b \\
&\geq (\leq, =)\, a\left(I\left(\frac{\varphi - b}{a}\right) + \frac{k}{a}\right) + b = aI\left(\frac{\varphi - b}{a}\right) + b + k = \bar{I}\left(\varphi\right) + k,
\end{aligned}$$

proving that $\bar{I}$ is constant superadditive (subadditive, additive). "If". Assume that $\bar{I}$ is constant superadditive (subadditive, additive). It follows that for each $\varphi \in B_0\left(\Sigma, \operatorname{Im} u\right)$ and for each $k \geq 0$

$$\begin{aligned}
I\left(\varphi + k\right) &= \frac{1}{a}\bar{I}\left(a\left(\varphi + k\right) + b\right) - \frac{b}{a} = \frac{1}{a}\bar{I}\left(a\varphi + b + ak\right) - \frac{b}{a} \\
&\geq (\leq, =)\,\frac{1}{a}\left(\bar{I}\left(a\varphi + b\right) + ak\right) - \frac{b}{a} = \left(\frac{1}{a}\bar{I}\left(a\varphi + b\right) - \frac{b}{a}\right) + k = I\left(\varphi\right) + k,
\end{aligned}$$

proving that I is constant superadditive (subadditive, additive). ■

B.2 Monetary consequences

We next prove a couple of ancillary facts. Moreover, when $\succsim$ (on $\Delta_0(\mathbb{R})$) is represented by an affine u and is CARA, we first assume that v of u corresponds to (3) with $a = 1$ and $b = 0$, that is, we normalize the von Neumann-Morgenstern utility function v to be such that

$$\bar{v}_\alpha(c) = \begin{cases} -\frac{1}{\alpha}e^{-\alpha c} & \text{if } \alpha \neq 0 \\ c & \text{if } \alpha = 0 \end{cases} . \tag{7}$$

In this case, for each $w \in \mathbb{R}$ and for each lottery $x \in \Delta_0(\mathbb{R})$, either $u(x^w) = e^{-\alpha w}u(x)$ or $u(x^w) = u(x) + w$.

Lemma 2 *If $\succsim$ is a CARA rational preference with representation (u, I), then $\succsim^w$ is a rational preference with representation (u, I_w). Moreover, if we choose $v = \bar{v}_\alpha$ as in (7), then I_w is such that*

$$I_w(\varphi) = \begin{cases} I(\varphi + w) - w & \textit{if } \succsim \textit{ is risk neutral} \\ e^{\alpha w} I(e^{-\alpha w}\varphi) & \textit{otherwise} \end{cases} \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

Proof. By Proposition 1, both preferences $\succsim^w$ and $\succsim$ are rational for all $w \in \mathbb{R}$. By assumption, $\succsim$ is CARA. Thus, $\succsim^w$ coincides with $\succsim$ on $\Delta_0(\mathbb{R})$ and it has a canonical representation (u_w, I_w) where v_w of u_w is either exponential or affine as in (7). Wlog, we can thus set $u = u_w$. By [2, Proposition 1], we have that

$$I(\varphi) = u(x_g) \text{ where } x_g \sim g \text{ and } u(g) = \varphi$$

and

$$I_w(\varphi) = u(x_{f,w}) \text{ where } x_{f,w} \sim^w f \text{ and } u(f) = \varphi.$$

a) Assume that $v = \bar{v}_\alpha$ is exponential (risk nonneutral case), that is, $\bar{v}_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$ for all $c \in \mathbb{R}$. This implies that either $\operatorname{Im} u = (0, \infty)$ or $\operatorname{Im} u = (-\infty, 0)$, in particular, for each $w \in \mathbb{R}$ and $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $e^{-\alpha w}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ the certainty equivalent of f for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = e^{-\alpha w}u(f) = e^{-\alpha w}\varphi$ and $u\left(x^w_{f,w}\right) = e^{-\alpha w}u(x_{f,w})$. If we define $g = f^w$, then we also have that x_g can be chosen to be $x^w_{f,w}$, that is,

$$I\left(e^{-\alpha w}\varphi\right) = I(u(g)) = u(x_g) = e^{-\alpha w}u(x_{f,w}) = e^{-\alpha w}I_w(\varphi),$$

and so $I_w(\varphi) = e^{\alpha w}I(e^{-\alpha w}\varphi)$.

b) Assume that $v = \bar{v}_\alpha$ is the identity (risk neutral case). This implies that $\operatorname{Im} u = \mathbb{R}$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ the certainty equivalent of f for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = u(f) + w = \varphi + w$ and $u\left(x^w_{f,w}\right) = u(x_{f,w}) + w$. If we define $g = f^w$, then we also have that x_g can be chosen to be $x^w_{f,w}$, that is,

$$I(\varphi + w) = I(u(g)) = u(x_g) = u(x_{f,w}) + w = I_w(\varphi) + w,$$

and so $I_w(\varphi) = I(\varphi + w) - w$. ■

Proof of Proposition 3. Let $w, w' \in \mathbb{R}$ be such that $w \neq w'$ and $\circ = w$ and $\# = w'$. If $\succsim$ is decreasing or constant absolute ambiguity averse, wlog, we can assume that $w' > w$. If $\succsim$ is increasing absolute ambiguity averse, wlog, we can assume that $w > w'$. By Proposition 2 and since $\succsim$ is classifiable, we have that u_w is a positive affine transformation of $u_{w'}$ and this holds for all $w, w' \in \mathbb{R}$, proving that $\succsim$ is CARA. ■

Proof of Theorem 2. Let $\succsim$ be a rational preference with canonical representation (u, I) where u is such that $u(x) = \sum_{c \in \mathbb{R}} v(c)\, x(c)$ for every $x \in \Delta_0(\mathbb{R})$, with v strictly increasing and continuous. Before starting the proof, we add few extra points.

(iv) $\succsim$ is CARA and $I_w \leq I_{w'}$, provided $w' > w$ and $u_w = u_{w'} = u$ and $v = \bar{v}_\alpha$;

(v) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (7), for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $w, w' \in \mathbb{R}$ such that $w' > w$, either

$$e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \text{ if } \bar{v}_\alpha \text{ is exponential} \tag{8}$$

or

$$I(\varphi + w) - w \leq I(\varphi + w') - w' \text{ if } \bar{v}_\alpha \text{ is the identity.} \tag{9}$$

(vi) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (7), I is:

(a) superhomogeneous (subhomogeneous) provided $\succsim$ is risk averse (loving);

(b) constant superadditive provided $\succsim$ is risk neutral.

(iii) implies (ii). By Proposition 3, we have that $\succsim$ is CARA. The implication trivially follows.

(ii) implies (vi). By assumption, $\succsim$ is CARA. We can thus choose a canonical representation $\left(\bar{u}, \bar{I}\right)$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} =$

$(-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 9, the implication follows.

(vi) implies (v). $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ is as in (7), we have three cases:

a. $\succsim$ is risk averse, that is, $\alpha > 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w-w')} \in (0,1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w-w')}\left(e^{-\alpha w}\varphi\right)\right) \geq e^{\alpha(w-w')} I\left(e^{-\alpha w}\varphi\right) \implies e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \geq e^{\alpha w} I\left(e^{-\alpha w}\varphi\right),$$

since φ was arbitrarily chosen the statement follows.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w'-w)} \in (0,1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w'-w)}\left(e^{-\alpha w'}\varphi\right)\right) \leq e^{\alpha(w'-w)} I\left(e^{-\alpha w'}\varphi\right) \implies e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right),$$

since φ was arbitrarily chosen the statement follows.

c. $\succsim$ is risk neutral, that is, $\alpha = 0$ and $\bar{v}_\alpha$ is the identity. Consider $w' > w$. It follows that $k = (w' - w) > 0$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $\varphi + w, \varphi + w' \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I(\varphi + w + (w' - w)) \geq I(\varphi + w) + (w' - w) \implies I(\varphi + w') - w' \geq I(\varphi + w) - w,$$

since φ was arbitrarily chosen the statement follows.

(v) is equivalent to (iv). By assumption, $\succsim$ is CARA. We consider two cases. For each $w, w' \in \mathbb{R}$:

a. $v = \bar{v}_\alpha$ is exponential. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u). \tag{10}$$

b. $v = \bar{v}_\alpha$ is the identity. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff I(\varphi + w) - w \leq I(\varphi + w') - w' \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

Subpoints a. and b. prove the equivalence between (iv) and (v).

(iv) implies (i). Let $w' > w$. By Lemma 2 and since $\succsim$ is CARA, we have that both preferences, $\succsim^w$ and $\succsim^{w'}$, admit a representation (u_w, I_w) and $(u_{w'}, I_{w'})$. Since $\succsim$ is CARA, we can choose $u_w = u_{w'} = u$ with $v = \bar{v}_\alpha$ for all $w, w' \in \mathbb{R}$. By Lemma 1 and since $I_w \leq I_{w'}$, we can conclude that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$.

(i) implies (iv). By Proposition 3, since $\succsim$ is decreasing absolute ambiguity averse, $\succsim$ is CARA. By Lemma 2, we have that for each $w \in \mathbb{R}$ the preference $\succsim^w$ admits a canonical representation (u_w, I_w). Thus, we can choose $u_w = u$ for all $w \in \mathbb{R}$ with $v = \bar{v}_\alpha$. By Lemma 1 and since $u_w = u_{w'}$ for all $w, w' \in \mathbb{R}$, note that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$ only if $I_w \leq I_{w'}$.

(iv) implies (vi). By the previous part of the proof, we know that (iv) is equivalent to (v). We thus assume (v) and prove (vi). We have three cases.

a. $\succsim$ is risk averse, that is, $\alpha > 0$. In (8) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since α is positive, it follows that $e^{\alpha w'} > 1$ and $\left\{e^{\alpha w'} : w' > 0\right\} = (1, \infty)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma > 1$. In other words, $\lambda I(\varphi) \leq I(\lambda\varphi)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\lambda \in (0,1)$, proving superhomogeneity.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. In (8) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since α is negative, it follows that $\left\{e^{\alpha w'} : w' > 0\right\} = (0,1)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma \in (0,1)$. If $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, then $\lambda\varphi \in B_0(\Sigma, \operatorname{Im} u)$ for all $\lambda \in (0,1)$. We have that

$$I(\lambda\varphi) \leq \lambda I\left(\frac{1}{\lambda}(\lambda\varphi)\right) = \lambda I(\varphi) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall \lambda \in (0,1),$$

proving subhomogeneity.

c. $\succsim$ is risk neutral, that is, $\bar{v}_\alpha$ is the identity. In (9) set $w = 0$ and $k = w'$, so that

$$I(\varphi) \leq I(\varphi + k) - k \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall k > 0.$$

In other words, $I(\varphi) + k \leq I(\varphi + k)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $k > 0$, proving superadditivity.

(vi) implies (ii). By assumption, $\succsim$ is CARA and represented by (u, I). We can thus choose a canonical representation $\left(\bar{u}, \bar{I}\right)$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} = (-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 9, the implication follows.

We thus proved that (iii) implies (ii) and (ii) is equivalent to (i), (iv), (v), and (vi). In particular, it follows that (ii) implies (i), thus $\succsim$ is classifiable and I satisfies condition (a) or (b), that is, (ii) implies (iii). ■

B.3 Other proofs

Proof of Corollary 2. Call (u, I) the rational representation of $\succsim$ on $\mathcal{F}$. Since $\succsim$ is risk neutral, it follows that $\operatorname{Im} u = \mathbb{R}$ and $I : B_0(\Sigma) \to \mathbb{R}$.

"Only if." By point 1 of Corollary 1, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \geq 0$. It is immediate to show that the equality holds for all $k \in \mathbb{R}$. By [16, Lemma 25], it follows that I is a normalized niveloid. By [16, Lemma 28], we can conclude that $\succsim$ satisfies Weak C-Independence.

"If." By [16, Lemma 28], it follows that I is a niveloid. By [16, Lemma 25] and since $\operatorname{Im} u = \mathbb{R}$, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \in \mathbb{R}$. By point 1 of Corollary 1 (recall that it holds by only assuming CARA in place of classifiable), the statement follows. ■

Proof of Corollary 3. Call (u, I) the rational representation of $\succsim$. Wlog, choose v to be such that $a = 1$ and $b = 0$. By [10], there also exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\Sigma) \to \mathbb{R}$ such that

$$\hat{I}(\lambda\varphi + k) = \lambda\hat{I}(\varphi) + k \qquad \forall\lambda > 0, \forall k \in \mathbb{R}$$

and $f \succsim g$ if and only if $\hat{I}(u(f)) \geq \hat{I}(u(g))$. It follows that $\hat{I}$ and I coincide on $B_0(\Sigma, \operatorname{Im} u)$.

(i) implies (iii). By Proposition 3, the implication follows.

(iii) implies (ii). By Corollary 1 (recall that it holds by only assuming CARA in place of classifiable) and since $\hat{I}$ and I coincide on $B_0(\Sigma, \operatorname{Im} u)$, the implication follows.

(ii) implies (i). Trivially, $\succsim$ is classifiable. ■

Proof of Proposition 4. Let (u, I) be the canonical representation of $\succsim$. Wlog, if $\succsim$ is CARA, we choose v to be such that $a = 1$ and $b = 0$ (see equation (3)). In this case, by the definition of c, we have that

$$c(f) = \begin{cases} -\frac{1}{\alpha}\log(-\alpha I(u(f))) & \alpha \neq 0 \\ I(u(f)) & \alpha = 0 \end{cases} \qquad \forall f \in \mathcal{F}.$$

Recall that for each $f \in \mathcal{F}$ and for each $w \in \mathbb{R}$

$$u(f^w) = \begin{cases} e^{-\alpha w}u(f) & \alpha \neq 0 \\ u(f) + w & \alpha = 0 \end{cases}.$$

(i). "Only if". By Proposition 3, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c(f^w) = I(u(f^w)) = I(u(f) + w)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) = I(u(f) + w) \geq I(u(f)) + w = c(f) + w,$$

proving that c is wealth superadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in [0,1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f^w) &= -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \geq -\frac{1}{\alpha}\log\left(-\alpha e^{-\alpha w}I(u(f))\right) \\ &= -\frac{1}{\alpha}\log\left(e^{-\alpha w}\left(-\alpha I(u(f))\right)\right) = -\frac{1}{\alpha}\log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I(u(f))\right) \\ &= c(f) + w, \end{aligned}$$

proving that c is wealth superadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in [0,1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f) &= -\frac{1}{\alpha}\log\left(-\alpha I(u(f))\right) = -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w}u(f)\right)\right)\right) \\ &\leq -\frac{1}{\alpha}\log\left(-\alpha e^{\alpha w}I\left(e^{-\alpha w}u(f)\right)\right) = -\frac{1}{\alpha}\log\left(e^{\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \\ &= -w + c(f^w), \end{aligned}$$

proving that c is wealth superadditive.

"If". First, observe that

$$f \succsim g \iff I(u(f)) \geq I(u(g)) \iff v^{-1}(I(u(f))) \geq v^{-1}(I(u(g))) \iff c(f) \geq c(g).$$

Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and c is wealth superadditive, it follows that

$$c\left(f^{w'}\right) = c\left((f^w)^{w'-w}\right) \geq c(f^w) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \geq c\left(f^{w}\right) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned} f \succsim^{w} x &\implies f^{w} \succsim x^{w} \implies c\left(f^{w}\right) \geq c\left(x^{w}\right) \implies c\left(f^{w}\right) \geq c(x) + w \\ &\implies c\left(f^{w}\right) - w \geq c(x) \implies c\left(f^{w'}\right) - w' \geq c(x) \implies c\left(f^{w'}\right) \geq c(x) + w' \\ &\implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies f^{w'} \succsim x^{w'} \implies f \succsim^{w'} x. \end{aligned}$$

Since f, x, w, and w' were arbitrarily chosen, we have that $\succsim^{w}$ is more ambiguity averse than $\succsim^{w'}$, proving the statement.

(ii). "Only if". By Proposition 3, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c\left(f^{w}\right) = I\left(u\left(f^{w}\right)\right) = I\left(u(f) + w\right)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By what follows right after Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c\left(f^{w}\right) = I\left(u(f) + w\right) \leq I\left(u(f)\right) + w = c(f) + w,$$

proving that c is wealth subadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c\left(f^{w}\right) = v^{-1}\left(I\left(u\left(f^{w}\right)\right)\right) = v^{-1}\left(I\left(e^{-\alpha w} u(f)\right)\right)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in [0,1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c\left(f^{w}\right) &= -\frac{1}{\alpha} \log\left(-\alpha I\left(e^{-\alpha w} u(f)\right)\right) \leq -\frac{1}{\alpha} \log\left(-\alpha e^{-\alpha w} I\left(u(f)\right)\right) \\ &= -\frac{1}{\alpha} \log\left(e^{-\alpha w}\left(-\alpha I\left(u(f)\right)\right)\right) = -\frac{1}{\alpha} \log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha} \log\left(-\alpha I\left(u(f)\right)\right) \\ &= c(f) + w, \end{aligned}$$

proving that c is wealth subadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c\left(f^{w}\right) = v^{-1}\left(I\left(u\left(f^{w}\right)\right)\right) = v^{-1}\left(I\left(e^{-\alpha w} u(f)\right)\right)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in [0,1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f) &= -\frac{1}{\alpha} \log\left(-\alpha I\left(u(f)\right)\right) = -\frac{1}{\alpha} \log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w} u(f)\right)\right)\right) \\ &\geq -\frac{1}{\alpha} \log\left(-\alpha e^{\alpha w} I\left(e^{-\alpha w} u(f)\right)\right) = -\frac{1}{\alpha} \log\left(e^{\alpha w}\right) + -\frac{1}{\alpha} \log\left(-\alpha I\left(e^{-\alpha w} u(f)\right)\right) \\ &= -w + c\left(f^{w}\right), \end{aligned}$$

proving that c is wealth subadditive.

"If". First, recall that $f \succsim g$ if and only if $c(f) \geq c(g)$. Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and c is wealth subadditive, it follows that

$$c\left(f^{w'}\right) = c\left(\left(f^{w}\right)^{w'-w}\right) \leq c\left(f^{w}\right) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \leq c\left(f^{w}\right) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned} f \succsim^{w'} x &\implies f^{w'} \succsim x^{w'} \implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies c\left(f^{w'}\right) \geq c(x) + w' \\ &\implies c\left(f^{w'}\right) - w' \geq c(x) \implies c\left(f^{w}\right) - w \geq c(x) \implies c\left(f^{w}\right) \geq c(x) + w \\ &\implies c\left(f^{w}\right) \geq c\left(x^{w}\right) \implies f^{w} \succsim x^{w} \implies f \succsim^{w} x. \end{aligned}$$

Since f, x, w, and w' were arbitrarily chosen, we have that $\succsim^{w'}$ is more ambiguity averse than $\succsim^{w}$, proving the statement.

(iii). It is an easy consequence of points (i) and (ii). ■

Proof of Theorem 3. Recall that an uncertainty averse preference is a rational preference. In particular, given a canonical representation (u, I), we have that

$$G(t,p) = \sup_{\varphi \in B_0(\Sigma, \operatorname{Im} u)} \left\{ I(\varphi) : \int \varphi dp \leq t \right\} \qquad \forall (t,p) \in \operatorname{Im} u \times \Delta.$$

(i) implies (ii). By Theorem 2, it follows that $\succsim$ is CARA and I is either concave at b, or convex at b, or superadditive, depending on $\succsim$ being, respectively, either risk averse, or risk loving, or risk neutral. We consider the three different cases separately:

- $\succsim$ is risk averse. Thus, $\operatorname{Im} u = (-\infty, b)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\lambda\varphi_n + (1-\lambda) b)\, dp \leq \lambda t + (1-\lambda) b \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since I is concave at b, we have that for each $n \in \mathbb{N}$

$$G(\lambda t + (1-\lambda) b, p) \geq I(\lambda\varphi_n + (1-\lambda) b) \geq \lambda I(\varphi_n) + (1-\lambda) b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t,p) + (1-\lambda) b$.

- $\succsim$ is risk loving. Thus, $\operatorname{Im} u = (b, \infty)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(\lambda t + (1-\lambda) b, p)$ and $\int \varphi_n dp \leq \lambda t + (1-\lambda) b$ for all $n \in \mathbb{N}$. Define $\{\psi_n\}_{n\in\mathbb{N}}$ to be such that

$$\psi_n = \frac{\varphi_n - (1-\lambda) b}{\lambda} \qquad \forall n \in \mathbb{N}.$$

Note also that

$$\psi_n \geq b, \int \psi_n dp \leq t, \text{ and } \varphi_n = \lambda\psi_n + (1-\lambda)\, b \qquad \forall n \in \mathbb{N}.$$

Since I is convex at b, this implies that for each $n \in \mathbb{N}$

$$I(\varphi_n) = I(\lambda\psi_n + (1-\lambda)\, b) \leq \lambda I(\psi_n) + (1-\lambda)\, b \leq \lambda G(t,p) + (1-\lambda)\, b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda)\, b, p) \leq \lambda G(t,p) + (1-\lambda)\, b$.

- $\succsim$ is risk neutral. Thus, $\operatorname{Im} u = \mathbb{R}$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $k \geq 0$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\varphi_n + k)\, dp \leq t + k \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since I is superadditive, we have that for each $n \in \mathbb{N}$

$$G(t+k,p) \geq I(\varphi_n + k) \geq I(\varphi_n) + k.$$

By passing to the limit, it follows that $G(t+k,p) \geq G(t,p) + k$.

(ii) implies (iii) and (i). Recall that

$$I(\psi) = \inf_{p\in\Delta} G\left(\int \psi dp, p\right) \qquad \forall \psi \in B_0(\Sigma, \operatorname{Im} u). \tag{11}$$

Observe also that $\succsim$ is CARA by assumption. As before, we consider three cases:

- $\succsim$ is risk averse. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned} I(\lambda\varphi + (1-\lambda)\, b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda)\, b)\, dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda)\, b, p\right) \\ &\geq \inf_{p\in\Delta}\left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda)\, b\right) \\ &\geq \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda)\, b = \lambda I(\varphi) + (1-\lambda)\, b, \end{aligned}$$

that is, I is concave at b.

- $\succsim$ is risk loving. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned} I(\lambda\varphi + (1-\lambda)\, b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda)\, b)\, dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda)\, b, p\right) \\ &\leq \inf_{p\in\Delta}\left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda)\, b\right) \\ &= \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda)\, b = \lambda I(\varphi) + (1-\lambda)\, b, \end{aligned}$$

that is, I is convex at b.

- $\succsim$ is risk neutral. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $k \geq 0$. We have that

$$I(\varphi + k) = \inf_{p \in \Delta} G\left(\int (\varphi + k)\, dp, p\right) = \inf_{p \in \Delta} G\left(\int \varphi dp + k, p\right)$$
$$\geq \inf_{p \in \Delta} \left(G\left(\int \varphi dp, p\right) + k\right) \geq \inf_{p \in \Delta} G\left(\int \varphi dp, p\right) + k = I(\varphi) + k,$$

that is, I is superadditive.

It follows that $\succsim$ is CARA and I either satisfies (a) or (b) of point (ii) of Theorem 2. By Theorem 2, we can conclude that $\succsim$ is decreasing absolute ambiguity averse and is classifiable.

(iii) implies (ii). By Proposition 3 and since $\succsim$ is classifiable, we have that $\succsim$ is also CARA.

We thus have proved that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), proving the statement. ■

Proof of Corollary 4. Recall that an uncertainty averse preference is a rational preference. By Corollary 2, we can conclude that a risk neutral uncertainty averse preference is constant absolute ambiguity averse if and only if it satisfies Weak C-Independence. At the same time, by definition, uncertainty averse preferences that satisfy Weak C-Independence are exactly variational preferences.

Proof of Corollary 5. Since $\succsim$ is CARA and risk averse, we have that $\operatorname{Im} u = (-\infty, b)$. Recall that $G(t, p) \geq t$ for all $(t, p) \in \operatorname{Im} u \times \Delta$. At the same time, note that for each $(t, p) \in \operatorname{Im} u \times \Delta$ and for each $\lambda \in (0, 1)$

$$G(\lambda t + (1 - \lambda) b, p) \geq G(\lambda t + (1 - \lambda) b_n, p) \geq \lambda G(t, p) + (1 - \lambda) G(b_n, p)$$
$$\geq \lambda G(t, p) + (1 - \lambda) b_n$$

where $b_n = b - \frac{1}{n}$ for all $n \in \mathbb{N}$. By passing to the limit and since (t, p) and λ were arbitrarily chosen, we have that $G(\lambda t + (1 - \lambda) b, p) \geq \lambda G(t, p) + (1 - \lambda) b$. By Theorem 3, the statement follows. ■

Proof of Corollary 6. Observe that a variational preference is a rational preference where the canonical representation (u, I) has the extra property of I being quasiconcave and constant additive. In particular, I is normalized and concave.

(i). By Theorem 2 and since $\succsim$ is not risk neutral, if $\succsim$ is either decreasing absolute ambiguity averse or CARA and risk averse, then v is a positive affine transformation of

$-\frac{1}{\alpha}e^{-\alpha c}$ where $\alpha \neq 0$. Without loss of generality, we assume that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. The first case holds under risk aversion, the second one under risk love. In the first case, since I is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right) \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right)\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(-\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) - (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that I is concave at 0, that is, I is superhomogeneous. In the second case, since I is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\frac{1}{n} \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\frac{1}{n}\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) + (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that I is concave at 0, that is, I is again superhomogeneous.

"If". By Theorem 2 and since I is concave at 0, if $\succsim$ is CARA and risk averse, it follows that $\succsim$ is decreasing absolute ambiguity averse. "Only if". By Theorem 2, if $\succsim$ is decreasing absolute ambiguity averse, then $\succsim$ is CARA. Since $\succsim$ cannot be risk neutral, it can either be risk averse or risk loving. By contradiction, assume it is risk loving. By Theorem 2, it follows that I is convex at 0, that is, I is subhomogeneous. From the previous part of the proof, we can conclude that I is homogeneous. To sum up, we would have that I is normalized, monotone, continuous, concave, constant additive, and homogeneous, that is, $\succsim$ is maxmin, a contradiction.

(ii). It follows from analogous arguments. ■

Proof of Corollary 7. "If". Since $\succsim$ is risk nonneutral, if $\succsim$ is CARA, then either $\succsim$ is risk averse or it is risk loving. If $\succsim$ is homothetic uncertainty averse, then, in both cases, I is positively homogeneous, proving the statement.

"Only if". By Proposition 3 and since $\succsim$ is constant absolute ambiguity averse and uncertainty averse, we have that $\succsim$ is CARA. Since $\succsim$ is uncertainty averse and risk nonneutral, we can consider a canonical representation (u, I) such that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. Since $\succsim$ is constant absolute ambiguity averse, we also have that I is positively homogeneous. Define $\bar{I} : B_0(\Sigma) \to [-\infty, \infty)$ by

$$\bar{I}(\varphi) = \sup\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \qquad \forall \varphi \in B_0(\Sigma).$$

By [3, Theorem 36], it follows that $\bar{I}$ is monotone, lower semicontinuous, quasiconcave, and such that $\bar{I}_{|B_0(\Sigma,\operatorname{Im} u)} = I$. We next show that also $\bar{I}$ is positively homogeneous. Consider $\varphi \in B_0(\Sigma)$ and $\lambda > 0$. We have two cases:

1. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} = \emptyset$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \lambda\varphi\} = \emptyset$, which yields that $\bar{I}(\lambda\varphi) = -\infty = \bar{I}(\varphi) = \lambda\bar{I}(\varphi)$.

2. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \neq \emptyset$. Let $\{\psi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ be such that $\psi_n \leq \varphi$ for all $n \in \mathbb{N}$ and $I(\psi_n) \uparrow \bar{I}(\varphi)$. Let now $\lambda > 0$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, it follows that $\{\lambda\psi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ and it is such that $\lambda\psi_n \leq \lambda\varphi$ for all $n \in \mathbb{N}$. In particular, by the definition of $\bar{I}$, we have that $\bar{I}(\lambda\varphi) \geq I(\lambda\psi_n) = \lambda I(\psi_n) \to \lambda\bar{I}(\varphi)$. We just proved that $\bar{I}(\lambda\varphi) \geq \lambda\bar{I}(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $\lambda > 0$. By choosing $1/\lambda$ with $\lambda > 0$, it follows that

$$\bar{I}(\varphi) = \bar{I}\left(\frac{1}{\lambda}(\lambda\varphi)\right) \geq \frac{1}{\lambda}\bar{I}(\lambda\varphi),$$

that is, $\lambda\bar{I}(\varphi) \geq \bar{I}(\lambda\varphi)$, proving positive homogeneity.

Consider $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ defined by

$$G(t,p) = \sup\left\{\bar{I}(\varphi) : \int \varphi dp \leq t\right\} \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

By [4], we have that G is lower semicontinuous, quasiconvex, and such that

$$\bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma) \tag{12}$$

and $G(\lambda t, p) = \lambda G(t,p)$ for all $\lambda > 0$, for all $t \in \mathbb{R}$, and for all $p \in \Delta$. Define $c_1, c_2 : \Delta \to [0, \infty]$ to be such that

$$c_1(p) = \frac{1}{G(1,p)} \text{ and } c_2(p) = -G(-1,p) \qquad \forall p \in \Delta.$$

We now consider two cases:

Risk averse case. $\operatorname{Im} u = (-\infty, 0)$. Since $\bar{I} \leq 0$ and $\bar{I}(-1) = I(-1) = -1$, observe that $G(-1,p) \leq 0$ and $G(-1,p) \geq -1$, that is, $c_2(p) \geq 0$ and $c_2(p) \leq 1$ for all $p \in \Delta$. Next, we have that for each $\alpha \in \mathbb{R}$

$$\{p \in \Delta : c_2(p) \geq \alpha\} = \{p \in \Delta : -G(-1,p) \geq \alpha\} = \{p \in \Delta : G(-1,p) \leq -\alpha\}.$$

Since G is quasiconvex and lower semicontinuous, the set is convex and closed, proving that c_2 is quasiconcave and upper semicontinuous. By (12), we can conclude that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) = \min_{p\in\Delta}\left(-\int \varphi dp\right) G(-1,p) = \min_{p\in\Delta} c_2(p) \int \varphi dp.$$

Since $-1 = \bar{I}(-1) = \min_{p\in\Delta} -c_2(p)$, we have that c_2 is normalized. The statement follows by setting $c = c_2$.

Risk loving case. $\operatorname{Im} u = (0,\infty)$. Since $\bar{I}(1) = I(1) = 1$, observe that $G(1,p) \geq 1$, that is, $0 \leq c_1(p) \leq 1$. Next, we have that for each $\alpha \in (0,\infty)$

$$\{p \in \Delta : c_1(p) \geq \alpha\} = \left\{p \in \Delta : \frac{1}{G(1,p)} \geq \alpha\right\} = \left\{p \in \Delta : G(1,p) \leq \frac{1}{\alpha}\right\}.$$

Since G is quasiconvex and lower semicontinuous, for each $\alpha \in (0,\infty)$ the set is convex and closed. Since $\{p \in \Delta : c_1(p) \geq \alpha\} = \Delta$ for all $\alpha \leq 0$, it follows that c_1 is quasiconcave and upper semicontinuous. By (12), we can conclude for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p\in\Delta}\left(\int \varphi dp\right) G(1,p) = \min_{p\in\Delta} \frac{\int \varphi dp}{c_1(p)}.$$

Since $1 = \bar{I}(1) = \min_{p\in\Delta} \frac{1}{c_1(p)}$, proving that c_1 is normalized. The statement follows by setting $c = c_1$. ■

Proof of Proposition 5. Since there exists $\gamma > 0$ such that $\phi(t) = -e^{-\gamma t}$ for all $t \in \mathbb{R}$, we have that I, defined as in (4), can be defined over the entire space $B_0(\Sigma)$. Moreover, by [3, Proposition 54], I is normalized, concave and constant additive. In particular, it is concave at b, in case $\succsim$ is either risk averse or risk loving.

(i). By Corollary 1 (recall that it holds by only assuming CARA in place of classifiable) and since I is constant additive, if $\succsim$ is risk neutral, then $\succsim$ is constant absolute ambiguity averse.

(ii). By Corollary 5 and since $\succsim$ is CARA, if $\succsim$ is risk averse, then $\succsim$ is decreasing absolute ambiguity averse. ■

Proof of Proposition 6. Since $\succsim$ is a smooth ambiguity preference, it admits a canonical representation (u, I) where I is as in (4). Since $\succsim$ is CARA and risk averse and $b \leq 0$, we also have that I is defined over $B_0(\Sigma, (-\infty, 0)) \supseteq B_0(\Sigma, \operatorname{Im} u)$. The functional $\hat{I} : B_0(\Sigma, (0,\infty)) \to (0,\infty)$ defined by

$$\hat{I}(\varphi) = \left(\int \left(\int \varphi dp\right)^{\gamma} d\mu\right)^{\frac{1}{\gamma}} \qquad \forall \varphi \in B_0(\Sigma, (0,\infty)).$$

is normalized, monotone, positively homogeneous, and quasiconvex. It follows that $I : B_0(\Sigma, (-\infty, 0)) \to \mathbb{R}$, which is such that $I(\varphi) = -\hat{I}(-\varphi)$, is normalized, monotone, positively homogeneous, and quasiconcave. In particular, by [4, Proposition 7, WP version], it is concave. By Corollary 5 and since $\succsim$ is CARA, the statement easily follows. ■

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