CLASS I: Monopoly, Price Discrimination - SOLUTIONS

October 2, 2009

Exercise 1

Price discrimination

Under price discrimination, the monopolist's maximization problem in market $i = l, h$ is

$$\max_{p_i} (p_i - c)(v_i - p_i)$$

The first-order condition for p_i is

$$v_i - p_i - (p_i - c) = 0$$

from which we get

$$p_i^d = \frac{v_i + c}{2}$$

Total profits are

$$\pi^d = \lambda \left(p_l^d - c\right)\left(v_l - p_l^d\right) + (1-\lambda)\left(p_h^d - c\right)\left(v_h - p_h^d\right) =$$

$$= \lambda \frac{v_l - c}{2}\frac{v_l - c}{2} + (1-\lambda)\frac{v_h - c}{2}\frac{v_h - c}{2} = \lambda \frac{(v_l - c)^2}{4} + (1-\lambda)\frac{(v_h - c)^2}{4}$$

The consumers' surplus in market i can be found as the area of the triangle between the inverse demand function and the market price, i.e.

$$CS_i^d = \frac{1}{2}\left(v_i - p_i^d\right) q_i^d = \frac{1}{2}\left(v_i - p_i^d\right)^2$$

Hence, total consumers' surplus is given by

$$CS^d = CS_l^d + CS_h^d = \frac{1}{2}\lambda \left[v_l - \frac{v_l + c}{2}\right]^2 + \frac{1}{2}(1-\lambda)\left[v_h - \frac{v_h + c}{2}\right]^2 =$$

$$= \frac{1}{8}\lambda (v_l - c)^2 + \frac{1}{8}(1-\lambda)(v_h - c)^2$$

Social welfare is

$$W^d \equiv CS^d + \pi^d = \frac{3}{8}\left[\lambda (v_l - c)^2 + (1-\lambda)(v_h - c)^2\right]$$

Uniform pricing

When price discrimination is banned, the monopolist is obliged to charge a uniform price for both markets and its maximization program becomes

$$\max_p (p-c)\left[\lambda (v_l - p) + (1-\lambda)(v_h - p)\right]$$

The first-order condition for p is

$$\lambda (v_l - p) + (1-\lambda)(v_h - p) + (p-c)\left[-\lambda - (1-\lambda)\right] = 0$$

from which we get

$$p^u = \frac{\lambda v_l + (1-\lambda) v_h + c}{2}$$

Substituting yields

$$\pi^u = (p^u - c)\left[\lambda (v_l - p^u) + (1-\lambda)(v_h - p^u)\right] = \frac{\left[\lambda v_l + (1-\lambda) v_h - c\right]^2}{4}$$

It can be easily shown that, not surprisingly, the uniform price is in between the prices the monopolist would charge if it were allowed to price discriminate, i.e. $p_h^d > p^U > p_l^d$.

Consumers' surplus under uniform pricing is

$$CS^u = \frac{1}{2}\lambda \left[v_l - p^u\right]^2 + (1-\lambda)\left[v_h - p^u\right]^2 =$$

$$= \frac{1}{2}\lambda \left[\frac{2v_l - \lambda v_l - (1-\lambda) v_h - c}{2}\right]^2 + \frac{1}{2}(1-\lambda)\left[\frac{2v_h - \lambda v_l - (1-\lambda) v_h - c}{2}\right]^2.$$

The numerator of the first term in square brackets can be rewritten as $(v_l - v_h) - \lambda (v_l - v_h) + (v_l - c)$, while the numerator of the second term in square brackets boils down to $-\lambda (v_l - v_h) + (v_h - c)$. Therefore, we get

$$CS^u = \frac{1}{8}\lambda \left[(1-\lambda)(v_l - v_h) + (v_l - c)\right]^2 + \frac{1}{8}(1-\lambda)\left[-\lambda (v_l - v_h) + (v_h - c)\right]^2 =$$

$$= \frac{1}{8}\left\{(v_l - v_h)^2 \left[\lambda (1-\lambda)^2 + \lambda^2 (1-\lambda)\right]\right\} +$$

$$+\frac{1}{8}\left\{\lambda\left(v_l-c\right)^2+\left(1-\lambda\right)\left(v_h-c\right)^2+2\lambda\left(1-\lambda\right)\left(v_l-v_h\right)\left(v_l-c\right)-2\lambda\left(1-\lambda\right)\left(v_l-v_h\right)\left(v_h-c\right)\right\}$$

Simplyfying the expression in square brackets and collecting the last two terms in curly braces yields

$$CS^u=\frac{1}{8}\left\{\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2\right\}+\frac{1}{8}\left\{\lambda\left(v_l-c\right)^2+\left(1-\lambda\right)\left(v_h-c\right)^2+2\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2\right\}.$$

Now, we add the first and the last term. Moreover,we sum and subtract $\frac{1}{8}\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2$ so that we get

$$CS^u=\frac{1}{2}\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2+\frac{1}{8}\left\{\lambda\left(v_l-c\right)^2+\left(1-\lambda\right)\left(v_h-c\right)^2-\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2\right\}.$$

Since the expression in curly brackets is equal to $\left[\lambda v_l+\left(1-\lambda\right)v_h-c\right]^2$ we find

$$CS^u=\frac{1}{8}\left[\lambda v_l+\left(1-\lambda\right)v_h-c\right]^2+\frac{1}{2}\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2.$$

Hence, social welfare under uniform pricing is given by

$$W^u=CS^u+\pi^u=\frac{3}{8}\left[\lambda v_l+\left(1-\lambda\right)v_h-c\right]^2+\frac{1}{2}\lambda\left(1-\lambda\right)\left(v_l-v_h\right)^2.$$

Let us show now that, not surprisingly, the firm prefers to price discriminate, because it has two instruments rather than one. This means that $\pi^d>\pi^u$, that is

$$\lambda\frac{\left(v_l-c\right)^2}{4}+\left(1-\lambda\right)\frac{\left(v_h-c\right)^2}{4}>\frac{\left[\lambda v_l+\left(1-\lambda\right)v_h-c\right]^2}{4}.$$

If we sum and subtract v_l the expression in square brackets can be written as $\left[\left(1-\lambda\right)\left(v_h-v_l\right)+\left(v_l-c\right)\right]^2$ and the inequality above becomes

$$\lambda\left(v_l-c\right)^2+\left(1-\lambda\right)\left(v_h-c\right)^2>\left(v_l-c\right)^2+\left(1-\lambda\right)^2\left(v_h-v_l\right)^2+2\left(1-\lambda\right)\left(v_l-c\right)\left(v_h-v_l\right)$$

Simplyfying by $\left(1-\lambda\right)$ and rewriting $2\left(1-\lambda\right)\left(v_l-c\right)\left(v_h-v_l\right)=\left(1-\lambda\right)\left(v_l-c\right)\left(v_h-v_l\right)+\left(1-\lambda\right)\left(v_l-c\right)\left(v_h-v_l\right)$ we get

$$\left(v_h-c\right)^2-\left(v_l-c\right)\left(v_l-c+v_h-v_l\right)>\left(v_h-v_l\right)\left[\left(1-\lambda\right)\left(v_h-v_l\right)+\left(v_l-c\right)\right]$$

from which it is straightforward to see that the inequality holds as long as $\lambda>0$.

Let us consider now welfare comparison

$$\Delta W \equiv W^u - W^d = \frac{3}{8}\left[\lambda v_l + (1-\lambda) v_h - c\right]^2 + \frac{1}{2}\lambda(1-\lambda)(v_l - v_h)^2 - \frac{3}{8}\left[\lambda (v_l - c)^2 + (1-\lambda)(v_h - c)^2\right].$$

Exploiting the fact that $\left\{\lambda (v_l - c)^2 + (1-\lambda)(v_h - c)^2 - \lambda(1-\lambda)(v_l - v_h)^2\right\} = \left[\lambda v_l + (1-\lambda) v_h - c\right]^2$, we find

$$\Delta W = \frac{3}{8}\left[\lambda v_l + (1-\lambda) v_h - c\right]^2 + \frac{1}{2}\lambda(1-\lambda)(v_l - v_h)^2 - \frac{3}{8}\left[\lambda v_l + (1-\lambda) v_h - c\right]^2 - \frac{3}{8}\lambda(1-\lambda)(v_l - v_h)^2$$

which becomes

$$\Delta W = \frac{1}{8}\lambda(1-\lambda)(v_l - v_h)^2 > 0.$$

This means that social welfare is higher under uniform pricing. This is an application of a general result that price discrimination reduces welfare if it does not increase total output. In particular, in our example

$$Q^d = \lambda q_l^d + (1-\lambda) q_h^d = \lambda\left[v_l - \frac{v_l + c}{2}\right] + (1-\lambda)\left[v_h - \frac{v_h + c}{2}\right] =$$

$$= \frac{1}{2}\lambda(v_l - c) + \frac{1}{2}(1-\lambda)(v_h - c) = \frac{1}{2}\left[\lambda v_l + (1-\lambda) v_h - c\right]$$

while

$$Q^u = \lambda q_l^u + (1-\lambda) q_h^u = \lambda\left[v_l - \frac{\lambda v_l + (1-\lambda) v_h + c}{2}\right] + (1-\lambda)\left[v_h - \frac{\lambda v_l + (1-\lambda) v_h + c}{2}\right] =$$

$$= \frac{1}{2}\lambda\left[2v_l - \lambda v_l - (1-\lambda) v_h - c\right] + \frac{1}{2}(1-\lambda)\left[2v_h - \lambda v_l - (1-\lambda) v_h - c\right]$$

Since the term in the first square bracket can be written as $(1-\lambda)(v_l - v_h) + (v_l - c)$ while the term in the second square brackets is equal to $-\lambda(v_l - v_h) + (v_h - c)$ we get

$$Q^u = \frac{1}{2}\lambda(1-\lambda)(v_l - v_h) + \frac{1}{2}\lambda(v_l - c) - \frac{1}{2}(1-\lambda)\lambda(v_l - v_h) + \frac{1}{2}(1-\lambda)(v_h - c)$$

$$= \frac{1}{2}\lambda(v_l - c) + \frac{1}{2}(1-\lambda)(v_h - c) = \frac{1}{2}\left[\lambda v_l + v_h(1-\lambda) - c\right] = Q^d$$

Since price discrimination does not increase the total quantity produced, it is necessarily social welfare detrimental.

Uniform prices (one market not served)

There is a strong assumption behind the analysis just carried out: i.e., the monopolist serves both markets. However, uniform pricing to serve both markets implies reducing prices (and profits) in the high-demand market. The monopolist might prefer just to set the monopoly price $p_h = \frac{v_h + c}{2}$ that maximizes its profits in the high-demand market, even if this entails losing sales in the low demand market. For instance, let us assume that $v_h + c > 2v_l$. In this case, $q_l = v_l - p_h = v_l - \frac{v_h + c}{2} = \frac{1}{2}\left[2v_l - v_h - c\right] < 0$, so the demand in the low-demand market is zero and the monopolist will earn $\pi_h^u = (1 - \lambda)\frac{(v_h - c)^2}{4}$.

Let us see now when the strategy of not serving the low-demand market is profitable provided that price discrimination is banned. In other terms, we want to see for which values of λ we have $\pi_h^u > \pi^u$. This is equal to

$$(1 - \lambda)\frac{(v_h - c)^2}{4} > \frac{\left[\lambda v_l + (1 - \lambda) v_h - c\right]^2}{4}$$

Since the expression in square brackets may be rewritten as $(1 - \lambda)(v_h - v_l) + (v_l - c)$ we find after taking the square root of both sides of the inequality

$$\sqrt{1 - \lambda}\,(v_h - c) > (1 - \lambda)(v_h - v_l) + (v_l - c)$$

If we define $\sqrt{1 - \lambda} \equiv y$, we get a second-degree equation, whose roots are $y_1 = 1$ and $y_2 = \frac{v_l - c}{v_h - v_l}$, so we get

$$\pi_h^u > \pi^u \text{ if } 0 < \lambda < \frac{(v_h - c)(v_h - 2v_l - c)}{(v_h - v_l)^2}.$$

This means that the lower the share of the low-demand market in total demand the more likely it will not be served is price discrimination is banned.

Let us check that $\lambda^* \equiv \frac{(v_h - c)(v_h - 2v_l - c)}{(v_h - v_l)^2}$ increases with v$_h$ and decreases with v_l. Indeed,

$$\frac{d\lambda^*}{dv_h} = 2\frac{v_h - v_l}{(v_h - v_l)^4}(v_l - c)^2 > 0$$

and

$$\frac{d\lambda^*}{dv_l} = -2(v_h - c)(v_l - c)\frac{v_h - v_l}{(v_h - v_l)^4} < 0$$

The higher the gap between demands the more likely only one market will be served. As a matter of fact,

if $v_h \uparrow \Leftrightarrow$ *gap* $\uparrow \Rightarrow \lambda^* \uparrow \Rightarrow$ *condition slackier*

if $v_l \uparrow \Leftrightarrow$ *gap* $\downarrow \Rightarrow \lambda^* \downarrow \Rightarrow$ *condition more binding.*

Finally, it is straightforward to check that if only one market is served total welfare is higher under price discrimination, because consumers' suplus and profits in the high-demand market are the same under both regimes, while low-demand market "disappears" only under uniform pricing.

Exercise 2

(I) If it does not price discriminate, the monopolist faces the following aggregate demand function

$$q^U = q_D + q_E = 240 - 20p$$

and its maximization program is the following

$$\max_p pq^U - cq^U = (240 - 20p)\, p - 3\,(240 - 20p)$$

The first-order condition for p is

$$-20p + 240 - 20p + 60 = 0$$

from which we get

$$p^U = \frac{300}{40} = 7.50$$

Hence, daytime attendance is

$$q_D^U = 100 - 10p^U = 25.$$

Evening attendance is

$$q_E^U = 140 - 10p^U = 65.$$

Total profit per day is

$$\pi^U = \left(240 - 20p^U\right) p^U - 3\left(240 - 20q^U\right) =$$

$$= (240 - 20 \cdot 7.50) \cdot 7.50 - 3\,(240 - 20 \cdot 7.50) = 675 - 270 = 405.$$

(II) A monopolist which price discriminates among consumers solves two different maximization problems.

As regards early moviegoers, its problem is

$$\max_{p_D} (100 - 10p_D)\, p_D - 3\,(100 - 10p_D)$$

The first-order condition for p_D is

$$-10p_D + 100 - 10p_D + 30 = 0$$

from which we get

$$p_D^D = \frac{130}{20} = 6.5.$$

As regards evening moviegoers, the monopolist's maximization problem is

$$\max_{p_E} (140 - 10p_E)\, p_E - 3\,(140 - 10p_E)$$

The first-order condition for p_E is

$$-10p_E + 140 - 10p_E + 30 = 0$$

from which

$$p_E^D = \frac{170}{20} = 8.5.$$

Now, daily attendance is

$$q_D^D = 100 - 10p_D^D = 100 - 10 \cdot 6.5 = 35,$$

while evening attendance is

$$q_E^D = 140 - 10p_E^D = 140 - 10 \cdot 8.5 = 55.$$

Notice that total attendance per day is $q^D = q_D^D + q_E^D = 35 + 55 = 90 = q^U$.
Total profit per day is

$$\pi^D = \pi_D^D + \pi_E^D = \left(100 - 10p_D^D\right) p_D^D - 3\left(100 - 10p_D^D\right) + \left(140 - 10p_E^D\right) p_E^D - 3\left(140 - 10p_E^D\right) =$$

$$= (100 - 10 \cdot 6.5) \cdot 6.5 - 3\,(100 - 10 \cdot 6.5) + (140 - 10 \cdot 8.5) \cdot 8.5 - 3\,(140 - 10 \cdot 8.5) =$$

$$= 122.5 + 302.5 = 425 > \pi^U = 405.$$

(III) Without discriminatory pricing consumers' surplus for daytime consumers is derived as follows

$$q_D = 100 - 10p \Rightarrow p = 10 - \frac{1}{10} q_D$$

Since consumers' surplus is given by the area between the inverse demand function and the equilibrium market price, we get

$$CS_D^U = \frac{1}{2}\,(10 - 7.50) \cdot 25 = 31.25.$$

Analogously, consumers' surplus for evening consumers is found as follows

$$q_E = 140 - 10p \Rightarrow p = 14 - \frac{1}{10} q_E$$

from which we get

$$CS_D^U = \frac{1}{2}\,(14 - 7.50) \cdot 65 = 211.25.$$

Since producer surplus, that is firm's profit, is $\pi^U = 405$, total surplus under uniform pricing is

$$W^U = \pi^U + CS_D^U + CS_E^U = 405 + 31.25 + 211.25 = 647.50$$

With discriminatory pricing consumers' surplus for daytime consumers is

$$CS_D^D = \frac{1}{2}(10 - 6.50) \cdot 35 = 61.25.$$

Consumers' surplus for evening consumers is

$$CS_E^D = \frac{1}{2}(14 - 8.50) \cdot 55 = 151.25.$$

Since producer surplus is $\pi^D = 425$ total surplus under price discrimination is

$$W^D = \pi^D + CS_D^D + CS_E^D = 425 + 61.25 + 151.25 = 637.50.$$

Discriminatory pricing has lowered total surplus by 10. This result confirms Motta's conclusion, according to which ‹‹economic theory shows that price discrimination unambiguously reduces welfare only when it does not raise total output›› [Motta (2004), p. 496].

Exercise 3

(I) Under second-degree price discrimination the monopolist typically offers a two-part-tariff $T + pq$ and thus it has two choice variables to be determined.

The monopolist maximization program is

$$\max_{p,T} \pi = \lambda (p - c)(v_l - p) + (1 - \lambda)(p - c)(v_h - p) + T$$

$$s.t.$$

$$CS_l = \lambda \left[\frac{(v_l - p)^2}{2} - T \right] \geq 0$$

$$CS_h = (1 - \lambda) \left[\frac{(v_h - p)^2}{2} - T \right] \geq 0$$

Since the profit function is inceasing in T the monopolist finds it optimal to extract all l-type consumers' surplus through the fixed amount T, so

$$T^D = \frac{(v_l - p)^2}{2}$$

Since the other constraint is slack at the optimum, the maximization problem boils down to

$$\max_p \lambda \left(p-c\right)\left(v_l-p\right)+\left(1-\lambda\right)\left(p-c\right)\left(v_h-p\right)+\frac{\left(v_l-p\right)^2}{2}$$

The first-order condition for p is

$$\lambda\left[v_l-p-\left(p-c\right)\right]+\left(1-\lambda\right)\left[v_h-p-\left(p-c\right)\right]-\left(v_l-p\right)=0$$

from which we get

$$p^D=c+\left(1-\lambda\right)\left(v_h-v_l\right).$$

Moreover,

$$q_l^D=v_l-p^D=v_l-c-\left(1-\lambda\right)\left(v_h-v_l\right)$$

$$q_h^D=v_h-p^D=v_h-c-\left(1-\lambda\right)\left(v_h-v_l\right)$$

and

$$T^D=\frac{1}{2}\left[v_l-c-\left(1-\lambda\right)\left(v_h-v_l\right)\right]^2$$

The consumers' surplus is given by

$$CS^D=CS_l^D+CS_h^D=0+\left(1-\lambda\right)\left\{\frac{1}{2}\left[v_h-c-\left(1-\lambda\right)\left(v_h-v_l\right)\right]^2-\frac{1}{2}\left[v_l-c-\left(1-\lambda\right)\left(v_h-v_l\right)\right]^2\right\}$$

The monopolist's profit is given by

$$\pi^D=\left(p^D-c\right)\left[\lambda q_l^D+\left(1-\lambda\right)q_h^D\right]+T^D=$$

$$=\left(1-\lambda\right)\left(v_h-v_l\right)\left\{\lambda v_l-\lambda c-\lambda\left(1-\lambda\right)\left(v_h-v_l\right)+\left(1-\lambda\right)v_h-\left(1-\lambda\right)c-\left(1-\lambda\right)^2\left(v_h-v_l\right)\right\}+$$

$$+\frac{1}{2}\left[v_l-c-\left(1-\lambda\right)\left(v_h-v_l\right)\right]^2$$

After some calculations we get

$$\pi^D=\frac{1}{2}\left(v_l-c\right)^2+\frac{1}{2}\left(1-\lambda\right)^2\left(v_h-v_l\right)^2$$

(II) Banning price discrimination implies that the monopolist charges a uniform price to all consumers. The monopolist's maximization problem is the following

$$\max_p \lambda\left(p-c\right)\left(v_l-p\right)+\left(1-\lambda\right)\left(p-c\right)\left(v_h-p\right)$$

The first-order condition for p is

$$\lambda [v_l - p - (p-c)] + (1-\lambda)[v_h - p - (p-c)] = 0$$

from which we get

$$p^U = \frac{c + \lambda v_l + (1-\lambda) v_h}{2}.$$

Equilibrium quantities are

$$q_l^U = v_l - p^U = v_l - \frac{c + \lambda v_l + (1-\lambda) v_h}{2} = \frac{(2-\lambda) v_l - (1-\lambda) v_h - c}{2}$$

and

$$q_h^U = v_h - p^U = v_h - \frac{c + \lambda v_l + (1-\lambda) v_h}{2} = \frac{(1+\lambda) v_h - \lambda v_l - c}{2}.$$

The monopolist's profit is equal to

$$\pi^U = \left(p^U - c\right)\left[\lambda q_l^U + (1-\lambda) q_h^U\right] =$$

$$= \frac{\lambda v_l + (1-\lambda) v_h - c}{4}\left[\lambda (2-\lambda) v_l - \lambda (1-\lambda) v_h - \lambda c + \left(1-\lambda^2\right) v_h - \lambda (1-\lambda) v_l - (1-\lambda) c\right] =$$

$$= \frac{\left[\lambda v_l + (1-\lambda) v_h - c\right]^2}{4}.$$

Let us compare now π^D and π^U. If we sum and subtract v_l in the expression in square brackets of π^U, we may rewrite the latter as

$$\pi^U = \frac{\left[v_l - c + (1-\lambda)(v_h - v_l)\right]^2}{4}$$

Hence, we get

$$\Delta\pi \equiv \pi^D - \pi^U =$$

$$= \frac{1}{2}(v_l - c)^2 + \frac{1}{2}(1-\lambda)^2 (v_h - v_l)^2 - \frac{1}{4}(v_l - c)^2 - \frac{1}{4}(1-\lambda)^2 (v_h - v_l)^2 - \frac{1}{2}(v_l - c)(1-\lambda)(v_h - v_l)$$

Standard calculations imply

$$\Delta\pi = \frac{1}{4}(v_l - c)^2 + \frac{1}{4}(1-\lambda)^2 (v_h - v_l)^2 - \frac{1}{2}(1-\lambda)(v_l - c)(v_h - v_l) =$$

$$= \frac{1}{4}\left[(v_l - c) - (1-\lambda)(v_h - v_l)\right]^2 \geq 0$$

so price discrimination is (weakly) profit improving.