CLASS II: Bertrand, Cournot, Hotelling SOLUTIONS

October 9, 2009

Exercise 1

(I) Since they compete à la Bertrand and have the same marginal (and average) costs, all n firms set a price equal to marginal costs, i.e. $p_i = c$ for any $i = 1, ...n$.

Substituting into the demand function, we derive equilibrium quantities

$$p = c = a - bq \Rightarrow q^B = \frac{a-c}{b}$$

Being symmetric, each firm will produce

$$q_i^B = \frac{a-c}{bn}$$

(II) Firm i's profit is zero, i.e. $\pi_i^B = 0$.

(III) If the n firms merge, they act as a monopoly. Therefore, the equilibrium price and quantity can be derived from the following maximization program

$$\max_p (p-c) \frac{a-p}{b}$$

The first-order condition for p is

$$\frac{a-p}{b} - \frac{1}{b}(p-c) = 0$$

from which we get

$$p^M = \frac{a+c}{2}$$

and

$$q^M = \frac{1}{b}\left(a - \frac{a+c}{2}\right) = \frac{a-c}{2b}$$

(IV) To see whether there are incentives to merge, we need to compare firm i's profits under Betrand competition and monopoly. The monopoly profit is

$$\pi^M = \left[\frac{a+c}{2} - c\right]\frac{a-c}{2b} = \frac{(a-c)^2}{4b}$$

so each firm's profit when merger occurs is

$$\pi_i^M = \frac{(a-c)^2}{4bn} > \pi_i^B = 0$$

so each firm has an incentive to merge.

Exercise 2

(I) To derive the demand functions for the two varieties, we first find the position on the line (i.e. the ideal variety) of the consumer who is indifferent between variety 0 and variety 1. Let us call it t_{01}. Then we get

$$8 - p_0 - [0 - t_{01}]^2 = 8 - p_1 - [1 - t_{01}]^2$$

from which we get

$$t_{01} = \frac{p_1 - p_0 + 1}{2}$$

The firms' demands functions are

$$D_0(p_0, p_1) = t_{01} = \frac{p_1 - p_0 + 1}{2}$$

while

$$D_1(p_0, p_1) = 1 - t_{01} = \frac{p_0 - p_1 + 1}{2}$$

Firm 0's maximization program is

$$\max_{p_0} (p_0 - 0)\frac{p_1 - p_0 + 1}{2}$$

The first-order condition for p_0 is

$$\frac{p_1 - p_0 + 1}{2} - \frac{1}{2}p_0 = 0$$

from which we get

$$p_0(p_1) \equiv R_0(p_1) = \frac{1 + p_1}{2}$$

Firm 1's maximization program is

$$\max_{p_1} (p_1 - 0)\frac{p_0 - p_1 + 1}{2}$$

The first-order condition for p_1 is

$$\frac{p_0 - p_1 + 1}{2} - \frac{1}{2} p_1 = 0$$

from which we get

$$p_1 (p_0) \equiv R_1 (p_0) = \frac{1 + p_0}{2}$$

Substituting implies

$$p_0 = \frac{1}{2} \left[1 + \frac{1 + p_0}{2} \right]$$

which finally yields

$$p_0 = p_1 = 1$$

Moreover,

$$D_0 (p_0, p_1) = D_1 (p_0, p_1) = \frac{1}{2}$$

and

$$\pi_0 (p_0, p_1) = \pi_1 (p_0, p_1) = \frac{1}{2}$$

(II) If the entrant (firm 2) offers the same variety as firm 0, they compete à la Bertrand and set prices $p_0^{BN'} = p_2^{BN'} = 0$, earning zero profits $[\pi_0^{BN'} = \pi_2^{BN'} = 0]$.

Now let us compute the new demand functions. In particular, t'_{01} is such that

$$8 - 0 - \left[0 - t'_{01} \right]^2 = 8 - p_1 - \left[1 - t'_{01} \right]^2$$

from which we get

$$t'_{01} = \frac{1 + p_1}{2}$$

Hence,

$$D_0 + D_2 = t'_{01} = \frac{1 + p_1}{2}$$

and

$$D_1 = 1 - t'_{01} = \frac{1 - p_1}{2}$$

Now, firm 1's maximization program is

$$\max_{p_1} (p_1 - 0) \frac{1 - p_1}{2}$$

The first-order condition for p_1 is

$$\frac{1-p_1}{2} - \frac{1}{2}p_1 = 0$$

from which we get

$$p_1^{'} = \frac{1}{2}$$

and

$$D_1^{'} = \frac{1}{4}$$

Finally,

$$\pi_1^{'} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

(III) Since the monopolist is not allowed to price discriminate, it charges all consumers a uniform price. In order to maximize its profit, the monopolist sets the highest possible price which is compatible with the participation constraint of the most distant consumer, namely the one located at $\frac{1}{2}$. So, let us impose

$$8 - p - \left(\frac{1}{2}\right)^2 = 0$$

from which we get

$$p^M = \frac{31}{4}$$

while $D^M = 1$ since the three firms have merged.
Finally,

$$\pi^M = \left[\frac{31}{4} - 0\right] \cdot 1 = \frac{31}{4}$$

Each firm gets after merger

$$\pi_1^M = \pi_2^M = \pi_3^M = \frac{31}{12} > \frac{1}{8} > 0$$

so each firm has an incentive to merge.

Exercise 3

(I) Firm 1's and 2's unit costs are respectively

$$c_1 = w + r$$

and

$$c_2 = 2w + r$$

Firm 1's maximization program is

$$\max_{q_1} q_1 [1 - q_1 - q_2] - [w + r] q_1, \text{ for } q_2 \text{ given}$$

The first-order condition for q_1 is

$$1 - q_1 - q_2 - q_1 - w - r = 0$$

from which we get

$$q_1^C (q_2) \equiv R_1 (q_2) = \frac{1 - q_2 - w - r}{2}$$

Firm 2's maximization program is

$$\max_{q_2} q_2 [1 - q_1 - q_2] - [2w + r] q_2, \text{ for } q_1 \text{ given.}$$

The first-order condition for q_2 is

$$1 - q_1 - q_2 - q_2 - 2w - r = 0$$

from which we get

$$q_2 \equiv R_2 (q_1) = \frac{1 - q_1 - 2w - r}{2}$$

Since the equilibrium is given by the intersection between the two best response functions we have

$$q_1 = \frac{1}{2}\left[1 - \frac{1 - q_1 - 2w - r}{2} - w - r\right] = \frac{1}{4}(1 + q_1 - r)$$

which finally yields

$$q_1^C = \frac{1 - r}{3}$$

Substituting into the expression for q_2 implies

$$q_2^C = \frac{1}{2}\left[1 - \frac{1 - r}{3} - 2w - r\right] = \frac{1}{3}[1 - r - 3w]$$

(II) Firm 1's profit is given by

$$\pi_1 = \max_{q_1} \left\{ q_1 \left[1 - q_1 - q_2^C - (r + w)\right] \right\}$$

From the *Envelope theorem* we know that

$$\frac{d\pi_1}{dw} = \left.\frac{\partial \pi_1}{\partial w}\right|_{q_1 = q_1^C}$$

which implies

$$\frac{d\pi_1}{dw} = q_1^C \left[-\frac{\partial q_2^C}{\partial w} - 1 \right] = \frac{1-r}{3} [+1-1] = 0.$$

A change in w has two effects, which in this case cancel out. It raises firm 1's costs and it weakens firm 2's strategic position (because it becomes more inefficient as well). Since firm 2 is highly labour intensive, it must reduce its output considerably. In general, either effect may dominate.

Exercise 4

Without any loss of generality let us suppose that firm 1 is located at 0 while firm 2 is located at 1. A consumer who is indifferent between the two firms is located at $x = D_1(p_1, p_2)$, where x is given by equating generalized costs, i.e.

$$p_1 + t[x - 0] = p_2 + t[1 - x]$$

from which we get

$$D_1(p_1, p_2) = x = \frac{p_2 - p_1 + t}{2t}$$

and

$$D_2(p_1, p_2) = 1 - x = \frac{p_1 - p_2 + t}{2t}$$

Hence the demand for firm i can be written as

$$D_i(p_i, p_j) = \frac{p_j - p_i + t}{2t}$$

Firm i chooses p_i so as to maximize its profit given the price p_j charged by its rival, i.e.

$$\pi_i = \max_{p_i} \pi_i(p_i, p_j) = \max_{p_i} (p_i - c_i) \frac{p_j - p_i + t}{2t}$$

The first-order condition for p_i is

$$\frac{p_j - p_i + t}{2t} - \frac{1}{2t}(p_i - c_i) = 0$$

$$p_i(p_j) \equiv R_i(p_j) = \frac{p_j + c_i + t}{2}$$

Reversing the roles of i and j yields

$$p_j(p_i) \equiv R_j(p_i) = \frac{p_i + c_j + t}{2}$$

We know that in a Nash equilibrium we must have

$$p_i\left(p_j\right) \equiv R_i\left(p_j\right) = R_i\left[R_j\left(p_i\right)\right]$$

which means

$$p_i\left(p_j\right) = \frac{1}{2}\left[\frac{p_i + c_j + t}{2} + c_i + t\right] = \frac{1}{4}\left[p_i + c_j + 2c_i + 3t\right]$$

Finally, we get

$$p_i\left(p_i, p_j\right) = t + \frac{c_j + 2c_i}{3}$$

and consequently

$$p_j\left(p_i, p_j\right) = t + \frac{c_i + 2c_j}{3}$$

The reduced-form profits are

$$\pi_i\left(c_i, c_j\right) = \left[p_i - c_i\right] D_i\left(p_i, p_j\right) = \left[t + \frac{c_j + 2c_i}{3} - c_i\right] \cdot \frac{1}{2t}\left[t + \frac{c_i + 2c_j}{3} - t - \frac{c_j + 2c_i}{3} + t\right] =$$

$$= \frac{1}{2t}\left[t + \frac{c_j - c_i}{3}\right] \cdot \left[t + \frac{c_j - c_i}{3}\right] = \frac{1}{2t}\left[t + \frac{c_j - c_i}{3}\right]^2.$$

(II) To show that $\frac{\partial^2 \pi_i(c_i, c_j)}{\partial c_i \partial c_j} < 0$, let us compute first

$$\frac{\partial \pi_i}{\partial c_i} = -\frac{1}{3t}\left[t + \frac{c_j - c_i}{3}\right]$$

which entails

$$\frac{\partial^2 \pi_i}{\partial c_i \partial c_j} = -\frac{1}{3t} \cdot \frac{1}{3} = -\frac{1}{9t} < 0$$

This means that the higher c_j the lower the impact of c_i on π_i. We may also say that the goods produced by the two firms are strategic substitutes in costs.