Class IV: Collusion, Review
SOLUTIONS

October 19, 2009

Exercise 1

Let us consider first market α. The critical discount factor is given by

$$\overline{\delta}_i^{\alpha} \equiv \frac{\pi_i^{d\alpha} - \pi_i^{c\alpha}}{\pi_i^{d\alpha} - \pi_i^{p\alpha}}$$

We start by deriving firm i' s $(i = 1, 2)$ profits under collusion, punishment and deviation.

Under collusion, the two firms share out the monopoly profits given by

$$\max_q q\left[1-q\right]$$

The first-order condition for q is

$$1 - q - q = 0$$

from which we get

$$q^{c\alpha} = \frac{1}{2}$$

$$p^{c\alpha} = 1 - q^{c\alpha} = \frac{1}{2}$$

and

$$\pi^{c\alpha} = \frac{1}{4}$$

Therefore, firm i's profit under collusion is

$$\pi_i^{c\alpha} = \frac{1}{8}$$

Under punishment, firm i's profits come from Cournot competition, i.e.

$$\max_{q_i} q_i\left[1 - q_i - q_j\right], \text{ for } q_j \text{ given}$$

The first-order condition for q_i is

$$1 - q_i - q_j - q_i = 0$$

from which we get

$$q_i\left(q_j\right) = \frac{1 - q_j}{2}$$

Symmetry implies

$$q_i = \frac{1 - q_i}{2}$$

We get

$$q_i^{p\alpha} = \frac{1}{3}$$

and

$$p^{p\alpha} = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

Hence, firm i's profit under punishment is equal to

$$\pi_i^{p\alpha} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

Firm i which deviates competes à la Cournot, while the other one sticks on its collusive quantity. This implies

$$\max_{q_i} q_i \left[1 - q_i - \frac{1}{4}\right]$$

The first-order condition for q_i is

$$1 - q_i - \frac{1}{4} - q_i = 0$$

from which we get

$$q_i^{d\alpha} = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

and

$$p^{d\alpha} = 1 - \frac{3}{8} - \frac{1}{4} = \frac{3}{8}$$

Hence, firm i's profits from deviation are

$$\pi_i^{d\alpha} = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}$$

Therefore, in market α, the critical discount factor is equal to

$$\overline{\delta}_i^{\alpha} \equiv \overline{\delta}^{\alpha} \equiv \frac{\pi_i^{d\alpha} - \pi_i^{c\alpha}}{\pi_i^{d\alpha} - \pi_i^{p\alpha}} = \frac{\frac{9}{64} - \frac{1}{8}}{\frac{9}{64} - \frac{1}{9}} = \frac{9}{17}$$

Let us consider now market β. The critical discount factor is given by

$$\overline{\delta}_i^{\beta} \equiv \frac{\pi_i^{d\beta} - \pi_i^{c\beta}}{\pi_i^{d\beta} - \pi_i^{p\beta}}$$

We compute now firm i's profits under collusion, punishment and deviation. Firm i's profits under collusion are the same as before, i.e. $\pi_i^{c\beta} = \frac{1}{8}$
Under punishment, firm i's profits arise from Betrand competition, i.e.

$$\pi_i^{p\beta} = 0$$

Deviating firm i's profit when the other sticks on its collusive price is

$$\pi_i^{d\beta} \approx \frac{1}{4}$$

Finally, the critical discount factor in market β is given by

$$\overline{\delta}_i^{\beta} \equiv \overline{\delta}^{\beta} \equiv \frac{\pi_i^{d\beta} - \pi_i^{c\beta}}{\pi_i^{d\beta} - \pi_i^{p\beta}} = \frac{\frac{1}{4} - \frac{1}{8}}{\frac{1}{4} - 0} = \frac{1}{2}$$

Notice that $\overline{\delta}^{\beta} = 0.5 < \overline{\delta}^{\alpha} \approx 0.52$. The harsher the punishment (Bertrand competition is tougher than Cournot competition) the easier collusion.

Exercise 2

We know that the condition for the sustainability of collusion may be written as

$$\delta \geq \overline{\delta}_i \equiv \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^p}$$

Let us compute the two firms' profits in case of punishment, collusion and deviation to derive the critical discount factors $\overline{\delta}_1$ and $\overline{\delta}_2$ for firms 1 and 2 respectively.

Under punishment each firm's prifit arises from Bertrand competition, which implies

$$\pi_i^p = 0$$

Under collusion firms 1 and 2, which choose a collusive price $\widehat{p}$, share out the profit $\pi(\widehat{p})$ with weights λ and $(1 - \lambda)$ respectively. Hence,

$$\pi_1^c = \lambda\pi(\widehat{p})$$

and

$$\pi_2^c = (1-\lambda)\,\pi(\widehat{p})$$

If firm i deviates, choosing a price slightly lower than $\widehat{p}$, while firm $j \neq i$ still sticks on the collusive behaviour, we find

$$\pi_i^d \approx \pi(\widehat{p})$$

and

$$\pi_j^d = 0$$

Hence, firm 1 will find it profitable to sustain collusion if and only if

$$\delta \geq \overline{\delta}_1 \equiv \frac{\pi_1^d - \pi_1^c}{\pi_1^d - \pi_1^p} = \frac{\pi(\widehat{p}) - \lambda\pi(\widehat{p})}{\pi(\widehat{p}) - 0} = 1-\lambda$$

Firm 2 finds it profitable to sustain collusion if and only if

$$\delta \geq \overline{\delta}_2 \equiv \frac{\pi_2^d - \pi_2^c}{\pi_2^d - \pi_2^p} = \frac{\pi(\widehat{p}) - (1-\lambda)\,\pi(\widehat{p})}{\pi(\widehat{p}) - 0} = \lambda$$

(II) To have collusion in equilibrium, both conditions above must hold. Since $\lambda \in \left(\frac{1}{2}, 1\right)$, we know that $\lambda > 1-\lambda$ and so the condition for the sustainability of collusion in equilibrium is

$$\delta > \lambda$$

It is immediate to see that the higher λ the more binding the condition for sustainability of collusion. This is not surprisingly, since it is a well-founded result in the economic theory that asymmetries between firms, for instance in the market shares, deter collusion.

(III) To prove this, just notice that the condition above is satisfied for any $\widehat{p} \in \left[c, p^M\right]$. Values for $\widehat{p}$ outside this interval are not sensible, because if $\widehat{p}$ were less than c no firm would be viable. On the other hand, no firm would choose $\widehat{p}$ higher than p^M, independently of its rival's decisions.

Exercise 3

(I) With Cournot competition, firm i's maximization program is the following

$$\max_{qi} q_i\left[1 - q_i - q_j\right], \text{ given } q_j$$

The first-order condition for q_i is

$$1 - q_i - q_j - q_i = 0$$

from which we get

$$q_i\left(q_j\right)=R_i\left(q_j\right)=\frac{1-q_j}{2}$$

Symmetry implies

$$q_i=\frac{1-q_i}{2}$$

Finally we get

$$q_i^C=\frac{1}{3}$$

$$p_i^C=\frac{1}{3}$$

and

$$\pi_i^C=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$$

In a plane (q_1,q_2) the best response function for firm 1 can be represented by $q_1=\frac{1-q_2}{2}\Rightarrow q_2=1-2q_1$, while the best reply for firm 2 is simply $q_2=\frac{1-q_1}{2}=\frac{1}{2}-\frac{1}{2}q_1$.

The graphical representation of the best reply functions is the following

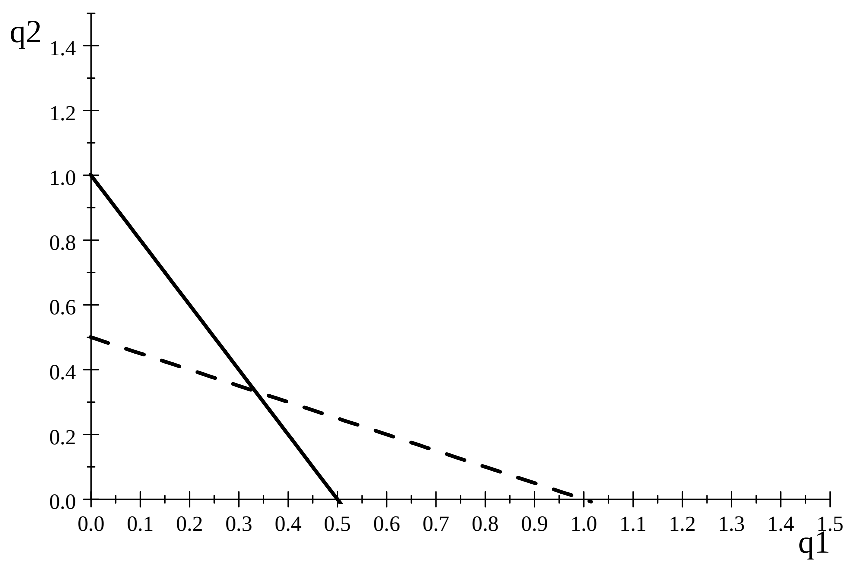

Fig. 2

In Figure 2 the solid line represents the best reply for firm 1, while the brocken line captures the best reply for firm 2.

(II.a) Profit function of Mr A is

$$\pi_A = \pi_1 + \alpha\pi_2 = pq_1 + \alpha p q_2 = p\left[q_1 + \alpha q_2\right]$$

while profit function of Mr B is

$$\pi_B = (1-\alpha)\, q_2$$

(II.b) Notice that Mr A, even if he has a share of firm 2, does not decide q_2. This means that Mr A and Mr B respectively choose q_1 and q_2 simultaneously and noncooperatively. So, Mr A's maximization problem is the following

$$\max_{q_1} \pi_A = \left[1 - q_1 - q_2\right]\left[q_1 + \alpha q_2\right]$$

The first-order condition for q_1 is

$$-\left(q_1 + \alpha q_2\right) + \left(1 - q_1 - q_2\right) = 0$$

which yields

$$q_1\left(q_2, \alpha\right) = R_1\left(q_2, \alpha\right) = \frac{1 - q_2\left(1+\alpha\right)}{2}.$$

Let us draw now on a plane (q_1, q_2) firm 1's best response function in (I) $q_1 = \frac{1-q_2}{2} \Rightarrow q_2 = 1 - 2q_1$ and Mr A's best response function in (II.b) $q_1\left(q_2, \alpha\right) = \frac{1-q_2(1+\alpha)}{2} \Rightarrow q_2 = \frac{1}{1+\alpha} - \frac{2}{1+\alpha} q_1$.

$y = 1 - 2x$

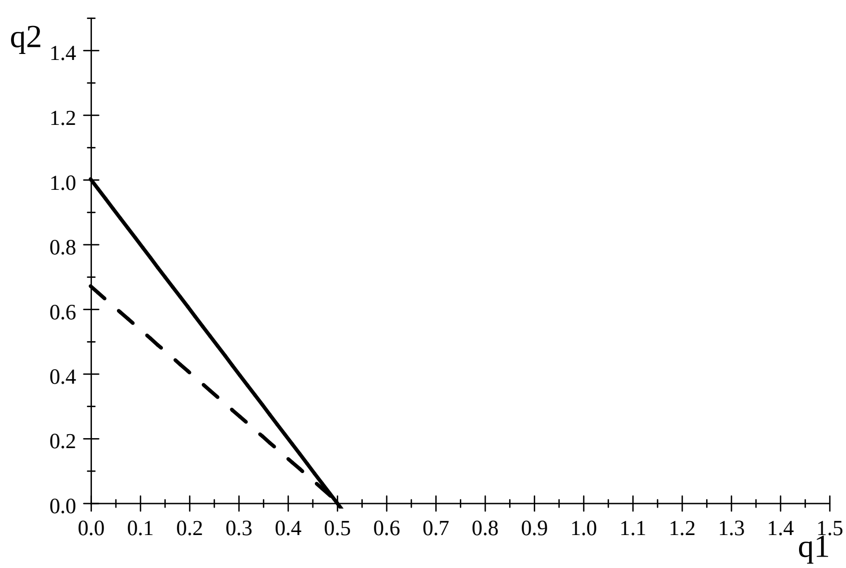

Fig. 3 ($\alpha = \frac{1}{2}$)

In Figure 3, the solid line represent the best reply for firm 1 in (I), while the brocken line captures Mr A's best response in (II.b), computed for $\alpha = \frac{1}{2}$. Notice that $q_1(q_2, \alpha)$ is rotated inwards around the horizontal intercept.

Mr B's maximization problem is

$$\max_{q_2} \pi_B = [1 - q_1 - q_2](1 - \alpha) q_2$$

The first order condition for q_2 is

$$-(1 - \alpha) q_2 + (1 - \alpha)[1 - q_1 - q_2] = 0$$

from which we get

$$q_2(q_1) = R_2(q_1) = \frac{1 - q_1}{2}$$

Notice that this is the same best reply as that derived in (I).

(II.c) Solving the system of the best replies we compute the equilibrium quantities

$$q_1 = \frac{1 - q_2(1 + \alpha)}{2} = \frac{1}{2}\left[1 - (1 + \alpha)\frac{1 - q_1}{2}\right]$$

from which we get

$$q_1^C(\alpha) = \frac{1 - \alpha}{3 - \alpha}$$

Moreover we get

$$q_2^C(\alpha) = \frac{1}{2}\left[1 - q_1^C(\alpha)\right] = \frac{1}{2}\left[1 - \frac{1 - \alpha}{3 - \alpha}\right] = \frac{1}{3 - \alpha}$$

The equilibrium price is

$$p^C(\alpha) = 1 - q_1^C - q_2^C = 1 - \frac{1 - \alpha}{3 - \alpha} - \frac{1}{3 - \alpha} = \frac{1}{3 - \alpha}$$

Firm 1's profits are

$$\pi_1^C(\alpha) = p^C(\alpha) q_1^C(\alpha) = \frac{1}{3 - \alpha} \cdot \frac{1 - \alpha}{3 - \alpha} = \frac{1 - \alpha}{(3 - \alpha)^2}$$

Firm 2' profits are

$$\pi_2^C(\alpha) = p^C(\alpha) q_2^C(\alpha) = \frac{1}{3 - \alpha} \cdot \frac{1}{3 - \alpha} = \frac{1}{(3 - \alpha)^2}$$

Finally, Mr A's profits are

$$\pi_A^C(\alpha) = p^C(\alpha)\left[q_1^C(\alpha) + \alpha q_2^C(\alpha)\right] = \frac{1}{3 - \alpha}\left[\frac{1 - \alpha}{3 - \alpha} + \alpha\frac{1}{3 - \alpha}\right] = \frac{1}{(3 - \alpha)^2} = \pi_2^C(\alpha)$$

and Mr B's profits are

$$\pi_B^C(\alpha) = p^C(\alpha)(1-\alpha) q_2^C(\alpha) = \frac{1}{3-\alpha}(1-\alpha)\frac{1}{3-\alpha} = \frac{1-\alpha}{(3-\alpha)^2} = \pi_1^C(\alpha)$$

(IV.d) Let us consider

a) $\frac{dq_2^C(\alpha)}{d\alpha} = \frac{d\left[\frac{1}{3-\alpha}\right]}{d\alpha} = -(3-\alpha)^{-2}(-1) > 0 \Leftrightarrow q_2^C(\alpha)$ increases in α

b) $\frac{dq_1^C(\alpha)}{d\alpha} = \frac{d\left[\frac{1-\alpha}{3-\alpha}\right]}{d\alpha} = \frac{-(3-\alpha)-(1-\alpha)(-1)}{(3-\alpha)^2} = -\frac{2}{(3-\alpha)^2} < 0 \Leftrightarrow q_1^C(\alpha)$ decreases in α

c) $\frac{dp^C(\alpha)}{d\alpha} = \frac{d\left[\frac{1}{3-\alpha}\right]}{d\alpha} = -(3-\alpha)^{-2}(-1) > 0 \Leftrightarrow p^C(\alpha)$ increases in α

d) $\frac{d\pi_1^C(\alpha)}{d\alpha} = \frac{d\pi_B^C(\alpha)}{d\alpha} = \frac{d\left[\frac{1-\alpha}{(3-\alpha)^2}\right]}{d\alpha} = \frac{-(3-\alpha)^2-2(1-\alpha)(3-\alpha)(-1)}{(3-\alpha)^4} = -\frac{1+\alpha}{(3-\alpha)^3} < 0 \Leftrightarrow \pi_1^C(\alpha) = \pi_B^C(\alpha)$ decreases in α

e) $\frac{d\pi_2^C(\alpha)}{d\alpha} = \frac{d\pi_A^C(\alpha)}{d\alpha} = \frac{d\left[\frac{1}{(3-\alpha)^2}\right]}{d\alpha} = -2(3-\alpha)^{-3}(-1) > 0 \Leftrightarrow \pi_2^C(\alpha) = \pi_A^C(\alpha)$ increases in α.