Dark Pool Trading Strategies*

Sabrina Buti[†] Barbara Rindi[‡]

Ingrid M. Werner[§]

June, 2011

*We thank Albert Menkveld and Dan Li for useful comments as well as participants at 6th Early Career Women in Finance Conference (2010), 4th IIROC DeGroote Conference on Market Structure and Market Integrity (2010), Northern Finance Association Meeting (2010), and seminar participants at Bocconi University and Toronto University. The usual disclaimer applies. We acknowledge financial support from IGIER, Bocconi University.

[†]University of Toronto, Rotman School of Management

[‡]Bocconi University and IGIER

[§]Fisher College of Business, Ohio State University.

Dark Pool Trading Strategies

Abstract

We model a dynamic financial market where traders submit orders either to a limit order book (LOB) or to a Dark Pool (DP). We show that there is a positive liquidity externality in the DP, that orders migrate from the LOB to the DP, but that overall trading volume increases when a DP is introduced. We also demonstrate that DP market share is higher when LOB depth is high, when LOB spread is narrow, when the tick size is large and when traders seek protection from price impact. Further, while inside quoted depth in the LOB always decreases when a DP is introduced, quoted spreads can narrow for liquid stocks and widen for illiquid ones. We also show that traders' interaction with both LOB and DP generates interesting systematic patterns in order flow: differently from Parlour (1998), the probability of a continuation is greater than that of a reversal only for liquid stocks. In addition, when depth decreases on one side of LOB, liquidity is drained from DP. When a DP is added to a LOB, total welfare as well as institutional traders' welfare increase but only for liquid stocks; retail traders' welfare instead always decreases. Finally, when flash orders provide select traders with information about the state of the DP, we show that more orders migrate from the LOB to the DP, and DP welfare effects are enhanced.

1 Introduction

According to the U.S. Securities and Exchange Commission (SEC), Dark Pools (DP) are Alternative Trading Systems (ATS) that do not provide their best-priced orders for inclusion in the consolidated quotation data. DP offer trading services to institutional investors that try to trade in size while minimizing adverse price impact. While undisplayed liquidity has always been a feature of U.S. equity markets, it is only recently that DP have been singled out for regulatory scrutiny. In 2009, the SEC proposed DP-related rule changes ranging from a ban of flash orders to increased pre- and post-trade transparency for DP venues. Moreover, the recent *SEC 2010 Concept Release on Equity Market Structure* shows concerns on the effect of undisplayed liquidity on market quality as well as on fair access to sources of undisplayed liquidity.

Unfortunately, there is to date very limited academic research that sheds light on these issues. Existing models focus on the comparison between a dealer market and a crossing network (e.g. Degryse, Van Achter and Wuyts, 2009), thus overlooking the features that drive the strategic interaction of DP with limit order books (LOB). We extend this literature by building a theoretical model of a dynamic limit order market where traders can choose to submit orders either to the fully transparent LOB or to a DP. We derive the optimal dynamic trading strategies and characterize the resulting market equilibrium. Specifically, we show how stock liquidity, volatility, tick size and price pressure affect DP market share. We also demonstrate how the introduction of a DP affects overall trading volume and LOB measures of market quality. Finally, in an extension of our model, we show how flash orders affect DP market share.

There are over thirty active DP in U.S. equity markets according to the SEC. A growing number of DP also operates in European equity markets. DP are characterized by limited or no pre-trade transparency, anonymity and derivative (almost exclusively mid-quote) pricing. However, they differ in terms of whether or not they attract order flow through Indications Of Interest (IOI)[1] and whether or not they allow interaction with proprietary and black box order flow. DP report their executed trades in the consolidated trade data, but the trade reports were until very recently not required to identify the ATS that executed the trade. As a result, it is difficult to accurately measure DP trading activity. Recent estimates suggest that DP represent over 10% of matched volume (*Rosenblatt Securities*, December 2009). As illustrated in Figure 1, there are four broad categories of DP, namely Public Crossing Networks, Internalization Pools, Exchange-Based Pools and Consortium-Based Pools.[2]

[**Insert Figure 1 here**]

[1] IOI are sales messages reflecting an indication of interest to either buy or sell securities. They can contain security names, prices and order size.

[2] Exchanges offer dark liquidity facilities that represent another 4% of matched volume (*Rosenblatt Securities*, December 2009).

As mentioned above, there are several concerns associated with DP growth. A main concern of the SEC relates to the possible migration of volume from transparent to dark markets, and hence to the effect of DP trading on the execution quality of those retail and institutional investors who display their orders in the lit markets. Another relevant concern of regulatory authorities is the fair access to DP liquidity. While there are several aspects of the fair access issue, the problem that the SEC has focused on in their rule making is IOI messages. Indeed "actionable IOI" messages[3] work as public quotes with implicit pricing and, by creating a leakage of privileged information to select investors, they can unfairly discriminate against public investors. Hence on October 21, 2009 SEC Chairman Schapiro noted that "DP now represent a significant source of liquidity in U.S. stocks", creating a "two-tiered market", and announced a SEC proposal for DP regulatory change.[4]

We use our model of a dynamic limit order market with a DP to shed light on these concerns raised by the SEC. We first determine factors that drive DP market share. We find that there is a positive liquidity externality in the DP so that DP orders beget more DP orders. In our model, orders migrate from the LOB to the DP, but overall trading volume can actually increase when a DP is introduced. We also demonstrate that DP market share is higher for stocks with higher inside order book depth and for stocks with narrow order book spreads. Intuitively, this can be explained as follows. Traders optimally trade off the execution uncertainty and the midquote price in the DP against the trading opportunities in the LOB. For stocks with larger depth at the inside or narrower spread, an order submitted to the LOB has to be more aggressive to gain priority over existing orders in the order book. As a result, the alternative of a midquote execution in the DP becomes relatively more attractive. We also demonstrate that DP market share is higher when the tick size is larger. This follows since a trader would have to make a larger price concession to gain priority in the order book, which makes the DP a more attractive alternative. Our model shows that traders use DP orders to reduce the price impact of their large orders. In particular we prove that when large orders generate price pressure, traders either reduce their order size or they resort to DP orders.

We also use our model to gain insights on the effect of DP trading activity on LOB market quality. We find that inside quoted depth and volume in the LOB always decreases

[3] According to the SEC (2009), IOI messages are "actionable" if they explicitly or implicitly convey information on: the security's name, the side of the order, a price that is equal or better than the NBBO, and a size that is at least equal to one round lot. IOI are typically targeted to specific institutional customers and not broadcasted more widely.

[4] More precisely, the SEC proposal addresses 3 issues: 1) actionable IOI: amendment of the definition of "bid" or "offer" in Rule 600(b)(8) of Regulation NMS to apply explicitly to actionable IOI, and exclusion of "size-discovery IOI", i.e. actionable who are reasonably believed to represent current contra-side trading interest of at least $200,000; 2) lower substantially the trading volume threshold (from the current 5% to 0.25%) in Rule 301(b) of Regulation ATS that triggers the obligation for ATS to display their best-priced orders in the consolidated quotation data; 3) require real-time disclosure of the identity of ATS on the reports of their executed trades.

when a DP is introduced because orders migrate from the LOB to the DP. However, total volume in the LOB and DP combined actually increases. The results also show that the introduction of a DP is associated with tighter quoted spreads for liquid stocks but wider quoted spreads for illiquid stocks. The explanation is subtle and takes into account both the migration of orders to the DP and the switch between limit and market orders in the LOB. When the initial LOB depth is high, both market and limit orders switch to the DP, leaving spreads tight in the LOB. By contrast, when the initial LOB depth is low, competition from DP decreases limit orders execution probability and hence increases the use of market orders, thus widening the spread. Furthermore, we analyze the dynamic pattern in order flow: differently from Parlour (1998), only liquid stocks exhibit a probability of continuation which is higher than that of a reversal, and the opposite holds for illiquid stocks. Also, we find an externality originating from the interaction of a LOB with a DP, whereby the latter acts as a liquidity buffer. In terms of traders' welfare we show that, when a DP is added to a LOB, total welfare and institutional traders' welfare increase for liquid stocks, and decrease for illiquid ones. Retail traders' welfare instead always decreases.

Finally, we use our model to understand how introducing IOI messages such as flash orders affects the equilibrium. We model flash orders as a mechanism that provides select traders with information about the state of the DP before they submit orders. In this setting, we show that more orders migrate from the LOB to the DP: the reason is that everyone knows that informed institutions will use the DP. This means that the execution probability of DP orders increases, which reinforces the already existing liquidity externality for the DP. As a consequence, flash orders have the overall effect of enhancing DP effects on market quality and traders' welfare. Indeed, compared to the market without asymmetric information, private information on the state of the DP reduces the execution risk of DP trading, thus making market orders less competitive than DP orders. The result is an improvement of both order book spread and depth, a reduction of LOB volumes but a further increase of total trading volume. Noticeably, we find that institutional traders benefit of flash orders whereas retail traders bear extra losses.

The paper is organized as follows. Section 2 reviews the related literature. Section 3 presents and discusses the general framework of the model, the benchmark cases with both a Limit Order Book (LOB) and a Dealer Market (DM), as well as the protocol with a DP. The equilibrium is derived in Section 4. Section 5 reports results on market quality, on the dynamic pattern in order flow and on traders' welfare. In Section 6, we extend the model to include asymmetric information on the state of the DP. Section 7 discusses the model's empirical implications and Section 8 summarizes the results. All proofs are in the Appendix.

2 Literature on Dark Pools

Most of the existing theory on undisplayed liquidity focuses on the interaction between crossing networks (CN) and dealer markets. Hendershott and Mendelson (2000) model the interaction between a CN and a DM and show costs and benefits of order flow fragmentation. Donges and Heinemann (2004) model intermarket competition as a coordination game among traders and investigate when a DM and a CN can coexist; Foster, Gervais and Ramaswamy (2007) show that a volume-conditional order-crossing mechanism next to a DM market Pareto improves the welfare of additional traders. The model we propose differs from these as it considers the interaction between a LOB and a DP rather than a DM and a CN; furthermore, it focuses on the dynamic, rather than static, order submission strategies of traders. More recently, Ye (2009) uses Kyle's model to find the insider's optimal strategic use of a DP and to show that DP harm price discovery especially for stock with high volatility; however Ye assumes that only the insider can strategically opt to trade in the DP and he models uninformed traders as noise traders. An opposite result on price discovery is obtained by Zhu (2011) who models the interaction among insiders, constrained and unconstrained noise traders by using a Glosten and Milgrom type framework. Kratz and Schoeneborn (2009) prove existence and uniqueness of optimal trading strategies for a traders who can split orders between an exchange and a DP, but assume that the price impact and the DP's liquidity are exogenously given.

The paper which is closest to ours is that by Degryse, Van Achter, and Wuyts (DVW, 2009), who investigate the interaction of a CN and a DM and show that the composition and dynamics of the order flow on both systems depends on the level of transparency. Our paper differs from DVW (2009) in that it considers the interaction between a LOB -rather than a DM- with the DP: this means that in our model traders can use both market orders and limit orders, and it is precisely the effect of competition from limit orders that drives the results we obtain compared to those of DVW. Modelling competition between a LOB and a DP entails considering a price grid where traders can choose to place their orders. We also extend our model to include asymmetric information on the state of the DP.

Other strands of the academic literature are relevant for understanding the role of DP in today's markets. DP are characterized by limited or no pre-trade transparency, and issues of anonymity and transparency are therefore important.[5] DP also coexist with more transparent venues, which suggest a link with the literature on multimarket trading.[6] Finally, DP are

[5] See for example the theoretical works by Admati and Pfleiderer (1991), Baruch (2005), Fishman and Longstaff (1992), Forster and George (1992), Madhavan (1995), Pagano and Röell (1996), Röell (1991), and Theissen (2001). Several empirical papers have recently explored the significance of anonymity and transparency in experimental settings and real data: Bloomfield and O'Hara (1999, 2000), Boehmer, Saar, and Yu (2005), Flood, Huisman, Koedijk and Mahieu (1999) and Foucault, Moinas and Theissen (2007).

[6] See among the others: Barclay, Hendershott, and McCormick (2003), Baruch, Karolyi and Lemmon (2007), Bennett and Wei (2006), Bessembinder and Kaufman (1997), Boehmer and Boehmer (2003), Easley,

currently competing with other dark options offered by exchanges to market participants, which builds a connection with the recent literature on hidden orders.[7]

Empirical work on crossing networks is relatively limited. Gresse (2006) finds that POSIT's crossing network (CN) has a market share of one to two percent of share volume and, by investigating the relation between the CN trading and the liquidity of the SEAQ quote-driven segment of the LSE, finds no negative effect of the CN on the dealership market. Gresse (2006) results show also that there isn't any significant increase in adverse selection or inventory risk, but rather a spread decrease due to increased competition and risk sharing. Conrad, Johnson, and Wahal (2003) find that realized execution costs are generally lower on alternative trading systems and that institutional orders sent to traditional brokers have higher execution costs than those executed in the CN. Naes and Odegaard (2006) provide evidence that orders from large institutional investors have lower realized execution costs for the component of the orders sent to the CN, but higher costs of delay if one considers the entire orders and includes the component sent to standard exchanges. Fong, Madhavan and Swan (2004) find no evidence that competition from the upstairs market and the CN has an adverse effect on the limit order book of the Australian Stock Exchange. To our knowledge, there is still limited empirical analysis of DP in the academic literature. Ready (2009) studies monthly volume by stock in three DP: Liquidnet, POSIT, and Pipeline during the period June 2005 to September 2007. The data suggests that these three DP execute roughly 2.5 percent of consolidated volume (third quarter 2007) in stocks where they were active during a month, but only 1 percent of market consolidated volume. Moreover, he finds that these three DP execute roughly 20 percent of "potential institutional volume" defined as the minimum of quarterly buying and selling activity by institutions estimated using 13F filings. While his results are preliminary, he finds that DP execute most of their volume in liquid stocks (low spreads, high share volume), but they execute the smallest fraction of share of volume in those same stocks. Buti, Rindi and Werner (2010) examine a unique data on dark pool activity for a large cross section of US securities and find that liquid stocks are characterized by more dark pool activity. They also find that dark pool volume increases for stocks with narrow quoted spreads and high inside bid depth suggesting that a higher degree of competition in the limit order book enhances dark pools activity.

Kiefer, and O'Hara (1996), Goldstein, Shkilko, Van Ness and Van Ness (2008), Karolyi (2006), Lee (1993), Nguyen, Van Ness, and Van Ness (2007), Pagano (1989), Reiss and Werner (2004) and Subrahmanyam (1997).

[7] There is little theoretical work on hidden orders: Buti and Rindi (2008), Esser and Mönch (2007) and Moinas (2006). The empirical literature is instead rather extensive: Bessembinder, Panayides and Venkataraman (2009), De Winne and D'Hondt (2007), Frey and Sandas (2008), Hasbrouck and Saar (2004), Pardo and Pascual (2006) and Tuttle (2006).

3 The Model

In this Section we present a model for three different market organizations. We start with a limit order book with both retail and institutional traders and use it as a benchmark model. We then add a Dark Pool which allows us to consider a market organization where traders can choose between the two platforms. Finally, we study the market structure formed by both a dealership market and a DP, that has extensively been modelled by previous literature (e.g. Degryse, Van Achter, and Wuyts, 2009). We find remarkable differences when we compare this market structure with the LOB plus DP mechanism.

Market Structure

We consider a discrete time protocol that, as in Parlour (1998), features a limit order book for a security, which pays v at each period and is assumed constant through the trading periods. Trading occurs during a day that is divided into T periods: $t = 1, ..., T$. In each period t a new risk neutral trader arrives who can be with equal probability either a large institutional trader or a small retail trader. Large traders can trade $j = [0, 2]$ shares, whereas small traders can only trade up to 1 share at a time. Upon arrival at the market the trader selects both a trading venue and an order type, and his optimal trading strategy cannot be modified thereafter: small traders can only trade in the LOB, while large traders can choose to trade either in the LOB or in the DP. Traders' personal evaluation of the asset, β_t, is drawn from a uniform distribution with support $[0, 2]$: traders with a high value of β are impatient to buy the asset, while traders who arrive at the market with a low β value the asset very little and therefore are impatient to sell it; traders with a β next to 1 are patient as their evaluation of the asset is close to the common value.

The LOB is characterized by a set of four prices and associated quantities, denoted as $\{p_i^B \& q_i^B, p_i^A \& q_i^A\}$, where A (B) indicates the ask (bid) side of the market and $i = \{1, 2\}$ the level on the price grid. Hence, prices are defined relative to the common value of the asset, v:

$$\begin{aligned} p_i^A &= v + i\ \tau \\ p_i^B &= v - i\ \tau \end{aligned}$$

where τ is the minimum price increment that traders are allowed to quote over the existing price, and hence it is the minimum spread that can prevail on the LOB. The associated quantities denote the number of shares that are available at that price. Following Parlour (1998) and Seppi (1997), we assume that a trading crowd absorbs whatever amount of the risky asset is demanded or offered at p_2^A and p_2^B. Hence at the second level of the book depth is unlimited and traders can only demand liquidity, whereas the number of shares available at p_1^A (p_1^B) forms the state of the book that characterizes time t and is defined as $b_t = [q_1^A q_1^B]$.

The DP operates next to the LOB; it allows market participants to enter unpriced orders to buy or sell the asset, and it is organized as a crossing network. In this trading venue orders are crossed at the end of time T at the spread midquote prevailing on the LOB in that period, p_{Mid}. The novelty of the DP compared to the standard crossing networks, however, is that traders have no access to any information regarding the orders previously submitted by the other market participants to the DP. It follows that they can only infer its depth by monitoring the LOB. If a trader submits an order to the DP, he will have this order executed provided that there will be sufficient depth to match it. As will be discussed more in detail below, we consider the last three periods of the trading game and we assume that at $T-2$ agents assign equal probabilities to the following three states of the DP's depth:

$$DP_{T-2} = \begin{cases} +6 \quad with\ prob = \frac{1}{3} \\ 0 \quad with\ prob = \frac{1}{3} \\ -6 \quad with\ prob = \frac{1}{3} \end{cases} \tag{1}$$

This means that at time $T-2$ traders believe that either the DP is empty, or that it is full on one or the other side of the market.[8] We also assume that traders strictly monitor the book and that when they do not observe any market or limit orders, they Bayesian update their expectations on the state of the DP. So traders at T can face a double uncertainty as they have to make inference on the state of the DP at both $T-2$ and $T-1$.

It is straightforward to extend the model discussed so far to include a dealership market (DM) that competes with a DP. Technically, this is the case when the trading crowd is moved to the first level of the LOB and is precisely the market structure discussed in DVW (2009).

Order Submission Strategies

Upon arrival at the market each trader decides his optimal trading venue as well as the optimal order type. To this end he compares the expected profits from the different order types he can choose. The feasibility and profitability of these orders depend on the traders' type (β_t) as well as on both the state of the LOB (b_t) and the state of the DP ($\widetilde{DP}_t$) at the time of the order submission.

For example, if an impatient large seller arriving at time t opts for the LOB, he will submit a market order of size j, $\varphi(j, \overline{p}_i^B)$, that gives expected profits equal to $\pi_t^e[\varphi(j, \overline{p}_i^B)] = j(\overline{p}_i^B - \beta_t v)$ if it is completely executed at the best price available, $\overline{p}_i^B$. If instead j exceeds the depth associated with the best opposite price, the order $\varphi(j, \overline{p}^B)$ will walk down the LOB and in this case the trader's expected payoff will be equal to: $\pi_t^e[\varphi(2, \overline{p}^B)] = (\overline{p}_1^B + \overline{p}_2^B) - 2\beta_t v$.[9] A more patient large seller can instead choose to submit a limit sell order of size j to p_1^A on the LOB, $\varphi(j, p_1^A)$, so that his expected payoff is:

[8] As at $T-2$ there are only three periods left in the trading game, if for example six shares to sell are already standing on the ask side of the DP (-6), the execution probability of any other share posted to the ask side is zero, the reason being that only two shares can be executed at each trading round.

[9] Clearly in this case $j = 2$, and, as the order hits different prices, we do not use an index i for the level of the book as we do for the other order types.

$$\pi_t^e[\varphi(j,p_1^A)] = E\left\{(p_1^A-\beta_t v)\sum_{w_{t+1}=1}^{j} w_{t+1}\Pr_{w_{t+1}}(p_1^A|\Omega_{t+1})+\right.$$
$$\left.I_t\left[\sum_{l=t+2}^{T}(p_1^A-\beta_t v)\sum_{W=0}^{1}\sum_{w_l=1}^{j-W} w_l\Pr_{w_l}(p_1^A|\Omega_l)\Pr(\sum_{m=t+1}^{l-1} w_m = W|\Omega_{l-1})\right]\right\}$$

where $\Omega_t = \{b_t, v, \widetilde{DP}_t\}$, $\Pr_{w_l}(p_1^A|\Omega_l)$ is the probability that w_l shares will be executed at $t=l$, W is the number of shares executed up to $t=l-1$, and I_t is an indicator function equal to 0 for $t=T-1$ and 1 otherwise. Notice that when submitting a limit order at time t, the trader will have to compute the probability that each unit will be executed from time $t+1$ to T.

The large seller can also decide to submit a j-order to sell to the DP that will be executed at the end of the trading game at the spread midpoint. This strategy, $\varphi(-j,p_{Mid})$, has the following expected payoff:

$$\pi_t^e[\varphi(-j,p_{Mid})] = E[j(p_{Mid} - \beta_t v)\Pr_{-j}(p_{Mid}|\Omega_T)]$$

where $\Pr_{-j}(p_{Mid}|\Omega_T)$ is the probability that j shares to sell will be executed in the DP. Finally a large seller can also decide not to trade so that his expected payoff will be equal to zero, $\pi_t^e[\varphi(0)] = 0$. Specular strategies are available to a large buyer. Notice also that a small trader has access to analogous strategies, with the exception of DP orders.

4 Market for Liquidity

The model is solved under three specifications that correspond to three different market structures. First, we present a benchmark model that describes the working of a pure LOB; then we focus on the protocol with an LOB competing with a DP (LOB&DP), and we compare the results obtained to the case where a DM, rather than a LOB, competes with the DP (DM&DP). This analysis allows us to discuss the driving factors of DP trading, and finally the effects that the price impact generated by blocks can have on traders' choice between a LOB and a DP.

4.1 Benchmark Model

We focus on a three-period trading game that ends at T. Figure 2 shows an example of the extensive form of the trading game: the market opens at $T-2$ with two units on the best bid and offer, $b_{T-2} = [22]$; then nature selects a small or a large trader with equal probability, who, in turn, chooses his order among the available strategies. Given the opening state of the LOB assumed in Figure 2, the equilibrium strategies for large traders include a limit order at either the best ask or the best bid, or a market order that hits the limit order standing at the first level of the book. Still referring to the example presented in Figure 2, suppose that nature selects at $T-2$ a small trader who is rather patient and decides to submit a limit order at p_1^A, $\varphi(1, p_1^A)$; then at $T-1$ the book will open with 3 units on p_1^A, and if at $T-1$ nature still selects another small trader who decides not to trade, then the book will open at T unchanged. It follows that at T incoming traders will choose among market buy, market sell and no trade. The reason why traders do not submit limit orders at time T is that their execution probability is zero as the market closes. At time $T-1$ and $T-2$, instead, traders can submit limit orders as their execution probability can be positive.

[**Insert Figure 2 here**]

At each trading round, the risk-neutral large trader will choose the optimal order submission strategy, φ_{LT,β_t,b_t}, which maximizes his expected profits conditional on the state of the LOB, b_t, and his type, β_t. A large trader (LT) thus chooses:

$$\max_{\varphi_{LT,\beta_t,b_t,j}} \pi_t^e[\varphi(j, \overline{p}_i^B), \varphi(j, \overline{p}^B), \varphi(j, p_1^A), \varphi(j, \overline{p}_i^A), \varphi(j, \overline{p}^A), \varphi(j, p_1^B), \varphi(0)] \tag{2}$$

and the small trader (ST) chooses:

$$\max_{\varphi_{ST,\beta_t,b_t}} \pi_t^e[\varphi(1, \overline{p}_i^B), \varphi(1, p_1^A), \varphi(1, \overline{p}_i^A), \varphi(1, p_1^B), \varphi(0)] \tag{3}$$

We find the solution of this game by backward induction and by assuming that the tick size, τ, is equal to 0.1. We start from the end-nodes at time T and compare trading profits for both large and small traders. This allows us to determine the probability of the equilibrium trading strategies at T that can be market orders, as well as no trading. We can hence calculate the execution probabilities of limit orders placed at $T-1$ that allow us to compute the equilibrium order submission strategies in that period. Given the probability of market orders at $T-1$, we can finally compute the equilibrium order submission strategies at $T-2$.

The framework described so far can also be simplified to analyze a pure dealership market; this can be accomplished by moving the trading crowd to the first level of the book, thus allowing traders to only submit market orders or not to trade as in DVW (2009).

4.2 Intermarket Competition: LOB&DP vs. DM&DP

Once a DP is added to the LOB, large traders have the option to submit an order to buy or to sell to the DP, and, provided that there will be enough depth to match it, the order will be executed at the end of time T. All else equal, the optimization problem now adds this new order type to the strategies of large traders.

[**Insert Figure 3 here**]

At each trading round the risk-neutral large trader chooses the optimal order submission strategy, $\varphi_{LT,\beta_t,b_t,\widetilde{DP}_t}$, which maximizes his expected profits conditional on the state of the LOB, b_t, his type, β_t,and the state of the DP, $\widetilde{DP}_t$. A large trader thus chooses the order that leads to the largest profits:

$$\max_{\varphi_{LT,\beta_t,b_t,j,\widetilde{DP}_t}} \pi_t^e[\varphi(j,\overline{p}_i^B),\varphi(j,\overline{p}^B),\varphi(j,p_1^A),\varphi(j,\overline{p}_i^A),\varphi(j,\overline{p}^A),\varphi(j,p_1^B),\varphi(\pm j,p_{Mid}),\varphi(0)] \quad (4)$$

Small traders still solve problem (3), however they will now condition their strategies not only on their type and on the state of the LOB, but also on the state of the DP.
If instead a DP is added to a DM, the optimization problem for large traders simplifies to:

$$\max_{\varphi_{LT,\beta_t,j,\widetilde{DP}_t}} \pi_t^e[\varphi(j,\overline{p}_1^B),\varphi(j,\overline{p}_1^A),\varphi(\pm j,p_{Mid}),\varphi(0)] \quad (5)$$

as in the DM&DP protocol traders cannot submit limit orders.

[**Insert Figure 4 here**]

A relevant issue in market design is to establish whether by adding a new trading opportunity to a limit order book more volume is created, or whether volume is simply diverted to the new trading venue. The results from this model show that when a DP is added to a LOB, volumes shift to the DP and there is no trade creation. Conversely, when a DP is added to a DM, it indeed induces some traders to enter the market. The latter result replicates the case studied by DVW (2009). The intuition is rather simple: in the dealership market some patient traders, who do not have the possibility to compete for the provision of liquidity by using limit orders, refrain from trading and do not enter the market to avoid paying the spread; however, when they are offered the opportunity to submit orders to the DP, they take that option: these orders can gain positive profits as their possible execution takes place at the midquote rather than at the best bid-offer. In the LOB instead there is not such effect as patient traders are allowed to submit limit orders. The following Proposition summarizes the results obtained by comparing the two protocols.

Proposition 1 .

When a Dark Pool is added to a limit order book, it induces order migration to the dark market. When a Dark Pool is added to a dealership market, it produces trade creation.

Order migration is more intense for highly liquid stocks where competition for the provision of liquidity is strong;

Dark Pools generate a liquidity-externality effect: as existing dark liquidity begets future liquidity, it increases the execution probability of dark orders.

Table 1 reports results on equilibrium trading strategies of large traders for $b_{(T-2,T-1,T)} = [22]$. Notice that at $t = T$ traders cannot submit limit orders and in this case, where the book opens with two shares at the inside spread, the LOB&DP framework converges to the DM&DP one. As traders cannot submit market orders for a size greater than two shares, the LOB is full and works like a DM where dealers offer unlimited liquidity at the BBO (as in DVW, 2009). In the previous periods instead traders can compete for the provision of liquidity by submitting limit orders to the LOB, and the role of these orders is crucial to understand the differences between the LOB&DP and the DM&DP frameworks. Clearly, the longer the time to the end of the trading game, the more relevant the role of limit orders, as their execution probability increases; hence the comparison between the equilibrium trading strategies at $T-2$ and those at T is the most appropriate to capture the differences between the two models.

Results show that at $T-2$ by moving from the LOB to the LOB&DP, the probability that traders submit limit orders decreases from .0314 to .0109 and large traders opt for DP orders with probability .0279 (Table 1); this means that there is no trade creation but only order migration to the DP. The same comparative static exercise performed at time T, when traders cannot submit limit orders, results in trade creation exactly as in DVW (2009). Actually at T, when a DP option is offered to market participants, those traders who were not willing to enter the market move to the DP with probability equal to .0375. However, as discussed above, at T the LOB converges to the DM and hence trade creation takes place only because in a DM traders cannot submit limit orders.

The overall effect of intermarket competition also depends on the state of the LOB. Table 2 reports results obtained by assuming that at $T-2$ the LOB opens empty. Clearly, when the LOB is empty, there is more room for limit orders as traders can post at the top of the queue and have their limit orders executed more quickly. Hence, when the LOB is empty at $T-2$, competition from limit orders is so intense that crowds out the DP. In this case, by allowing traders to choose between a LOB and a DP, they opt for the former. However the relative probability of market to limit orders increases as, due to competition from the DP in the following periods, limit orders' execution probability decreases and hence market orders become more profitable. As a result, the overall effect of DP competition is less intense for less liquid stocks where limit orders represent a more profitable trading strategy than for a deep LOB. Hence for less liquid stocks trade migration to DP is less severe, whereas trade creation from DM to DM&DP is more intense.

Finally Table 1 shows that traders' perception of DP liquidity influences the execution probability of DP orders and hence their use. When at $T-2$ traders do not observe any change in the LOB, they assume that either no trade occurred or that an order was submitted to the DP. As they perceive that liquidity is building in the DP, they update their estimate of the DP depth and assign a higher probability of execution to their DP orders; the result is that they opt for DP more frequently. As an example, this effect can be observed by comparing the results for $T-1$ presented in Table 1 for the case of "vis_{T-2}", where traders observe a change in the LOB at $T-2$, and "inv_{T-2}", where instead traders observe no change. In the latter case, they submit orders to the DP more intensively (.050) than when they have no uncertainty (.0379). Analogous results are shown for the DM&DP market where the probability of DP orders increases from .0379 to .0444 when traders do not observe any change in the LOB at $T-2$. We can therefore conclude that the positive liquidity-externality effect produced by a DP intensifies when traders perceive that DP volume is growing.

4.3 Dark Pool Drivers

We have shown that the state of the LOB affects traders' choice between disclosed and DP trading. We now extend this analysis by discussing more in depth the main factors related to the state of the LOB, i.e. depth, spread and tick size, that affect traders' choice to submit orders to the dark market. The following Proposition summarizes the results.

Proposition 2 *The probability that traders submit orders to the DP:*

- *increases with market depth and the tick size, and*
- *decreases when the inside spread widens.*

Opposite results relative to depth and inside spread are obtained for a DM&DP.

To investigate the effects of depth, spread and tick size on traders' choice, we compare, as we did before, the equilibrium strategies at $T-2$ and at T. Once again, the longer the time to the end of the game, the higher the probability of limit order execution and the stronger the effect of limit order competition. Table 3 (Panel A) shows that at $T-2$ an increase in depth on the top of the book from [11] to [22] reduces competition from limit orders and increases the probability that traders opt for the DP: $\varphi(2, p_1^A)$ and $\varphi(2, p_1^B)$ decrease (from .0832 to .0109) and, even though market orders increase from .4168 to .4612, traders now use the DP. If instead the same comparative static exercise is performed at time T, where there is no competition from limit orders and the LOB resembles a dealership market, we obtain the same result as in DVW (2009). Indeed, Table 4 shows that when depth on the ask side increases from [11] to [21], market orders to buy increase from .4250 to .4625 and crowd out DP buy orders, which decrease from .0750 to .0375. The same results are obtained when depth increases only on the bid side and on both sides of the market (from [11] to [12] and to [22]). We conclude that when market participants can compete for the provision of liquidity by using limit orders and can also opt for DP orders, the deeper the limit order book, the longer the queue for their limit orders -due to time priority-, and the greater the probability that they opt for DP orders. In dealership markets competition from limit orders is absent and greater depth instead fosters traders' aggressiveness thus increasing market orders to the detriment of DP orders.

Results obtained from comparative statics on the inside spread confirm those from market depth. For the LOB&DP framework, the more liquid the market, the more intense competition in limit orders, and the higher the probability that traders opt for the DP. To isolate the effect of a spread variation, one has to control for market depth: this can be achieved by comparing two states of the book that have enough liquidity at the BBO to absorb large orders. Table 3 and 5 show results for both time $T-2$ and T. Starting again from period $T-2$, when the inside spread increases, i.e. the state of the book changes from [22] to [00], the increased competition for liquidity provision crowds out DP orders, even though the probability of market orders decreases. We thus find that the wider the inside spread, the more convenient are limit orders submitted at the top of the book, and the greater is the probability that traders choose limit instead of DP orders. Opposite conclusions can be drawn from the same simulation performed at time T for the DM&DP framework. Table 4 shows that when the spread increases (from [22] to e.g. [00]) competition from market orders decreases and, because at T there is no competition from limit orders, the probability that traders opt for DP increases from .0375 to .1125.

Proposition 2 also informs us about the effects of a change in the tick size on the probability of DP orders: when the tick size increases traders become more willing to supply liquidity. An example for the book [11] is shown in Table 3 where, following an increase in the tick size, market orders decrease, while limit and DP orders increase. The intuition for this result is that an increase in the tick size produces two effects: it widens the inside spread, and hence makes market orders more expensive, and it increases the minimum price change, thus making it more convenient for traders to supply liquidity. The outcome is that more patient traders will opt for limit orders whereas less patient (but not so impatient) traders will choose to trade in the DP.

4.4 Price Impact, Price Pressure and Dark Pool Trading

A widespread view shared by market participant in the financial community is that large institutional traders submit orders to a Dark Pool to reduce price impact. A price impact can arise both when impatient traders submit a large order to the top of a LOB that is not deep enough to absorb the order, and when a patient trader submits a limit order that produces a price pressure, thus temporarily moving the asset value against the trader's order. Price impact resulting from trades has been extensively investigated: for example, Engle and Patton (2004) analyze the price impact of 100 NYSE stocks stratified by trade frequency and find strong evidence of short-run price impact for trades initiated by both buyers and sellers.[10] Price pressure[11] arising from passive order placement through limit orders has been recently explored by Hendershott and Menkveld (2010), who estimate the price impact

[10] See also Hasbrouck (1991) and Dufour and Engle (2000).

[11] See Gabaix et al. (2006), Brunnermeier and Pedersen (2009) and Parlour and Seppi (2008).

arising from liquidity supply. They find a large daily transitory volatility in returns for stocks listed at the NYSE due to price pressure.

In our model what drives large institutional traders to operate in a DP is their wish to buy or sell large blocks with the lowest price impact. Consider first impatient traders who are concerned about the price impact that can be generated by a market order. These traders face the standard trade-off between price risk (i.e. bearing a price impact) and execution risk. If they choose a market order, they will obtain immediate execution but will pay a greater price impact, which is increasing in the lack of depth available on the opposite side of the market. If instead traders opt for the DP, their order will be executed with a lower price impact at the spread midquote; however, the order execution will be uncertain and it will depend on the state of the DP. Hence the trader will choose the DP only if the price impact of his order is large. Our model shows this effect in period T when traders are naturally impatient due to the proximity of the end of the game. Table 5 shows that the impatient trader will more probably opt for a DP order when depth on the other side of the market is shallow and therefore his price impact is large: for example, when the book opens with only 1 share on top of the ask side, $b_{T=}[12]$, instead of 2 shares, $b_{T=}[22]$, large traders use DP orders more intensively (with probability .0750 instead of .0375) as in the former case their order will move the price up to obtain execution.

We now extend our model to embed the temporary price impact that can be generated by passive traders submitting limit orders. To this end we assume that large limit orders produce a short term price pressure that lasts for one period, as shown in Figure 5.

[**Insert Figure 5 here**]

Suppose that a large seller arrives at the market at $T-2$ and submits a large limit order at p_1^A; following this submission, the asset value jumps down by 1 tick and the next period the market opens with $v_{T-1} = v - \tau$. Clearly at time T the temporary price effect vanishes and the asset value jumps back to v. We consider two different specifications: a benchmark model with no DP (LOB&PP) and a model that allows for DP trading (LOB&DP&PP). The results are summarized in the following Proposition.

Proposition 3 *When large orders generate price pressure, traders either reduce the size of their order, or, if available, switch to DP orders.*

Notice that, when price pressure is introduced, the execution probability of a large limit order decreases for two reasons. First, the initial order is now on the second level of the book and hence further away from the asset value; second, it can be easily front-run in the following period by an incoming trader posting a limit order at the now empty first level of the book. The result is that traders switch to those order types that protect from price impact either because they are small in size, or because they are undisclosed. In Table 5 one can indeed notice that moving from the standard LOB protocol to the specification with price pressure, traders reduce their price impact by switching from large to small limit orders. When instead we introduce price pressure in the model with a DP, traders actually minimize their price impact by submitting DP buy and sell orders with larger probability (from .0279 to .0316).

5 Market Quality, Systematic Pattern in Order Flow and Welfare

So far we have shown how traders react when a DP is added to a LOB and concluded that DP produces order migration. The next relevant issue is to investigate how the introduction of a DP affects the quality of the LOB, and how it impacts the dynamic pattern in order flow and traders' welfare.

5.1 Market Quality

To evaluate the effect of DP trading on market quality we consider inside spread (S_t) market depth (D_t) and volume (V_t). We compute expected spread and depth in period $t+1$ by weighing the value that characterizes a particular state of the book with the corresponding order submission probabilities in the previous period t:

$$y_{t+1}^{e} = \sum_{a=ST,LT} \Pr(a)\ E_{b_t}\left[\int_0^2 y_{t+1}(\varphi^{*}_{a,b_t,\widetilde{DP}_t}|\beta_t) \times f\left(\beta_t\right) d\beta_t\right] \tag{6}$$

where $y_{t+1} = \{S_{t+1}, D_{t+1}\}$ and $\varphi^{*}_{a,b_t,\widetilde{DP}_t}$ is the optimal trading strategy of agent a, conditional on b_t and $\widetilde{DP}_t$.

We calculate the expected LOB volume in each period t in a similar way and adequately weigh the market orders submitted to the LOB by both retail and institutional traders by their size:

$$V_t^{e} = \sum_{a=ST,LT} \Pr(a)\ E_{b_t}\left[\int_0^2 q_t(\varphi^{*}_{a,b_t,\widetilde{DP}_t}|\beta_t) \times f\left(\beta_t\right) d\beta_t\right] \tag{7}$$

where $q_t(\varphi^{*}_{a,b_t,\widetilde{DP}_t})$ is the traded quantity which is a function of the agent's type a, the state of both the LOB and the DP. Proposition 4 summarizes the results.

Proposition 4 *When a Dark Pool is added to a Limit Order Book, market quality changes as follows:*

- *market depth at the best bid-offer decreases;*

- *the inside spread decreases when the LOB opens deep, the opposite holding when it opens empty;*
- *LOB volume decreases, whereas total volume increases.*

We have previously proved that the DP can attract orders from the LOB; more precisely, Table 1 shows that when the book opens deep ($b_{T-2} = [22]$) by moving from the LOB to the LOB&DP protocol, the probability of both limit and market orders decrease, as traders switch to the DP. Clearly the effect of the order migration on liquidity and volume depends on the proportion of limit vs market orders that leave the book. A reduction of the probability that traders post limit orders to the LOB decreases the provision of liquidity and hence worsens market depth and inside spread; a reduction of the demand for liquidity, i.e. the probability that traders submit market orders, instead, certainly decreases volume but can have positive effects on both depth and inside spread, as market orders subtract liquidity from the book.

Table 6 shows that when the book opens with two shares on both sides, $b_{T-2} = [22]$, the introduction of the DP decreases average depth by 1.30% and volume by 2.39%, but it improves the inside spread by .41%: this means that the positive effect on spread of the reduction of market orders more than outweighs the negative effect of the reduction of limit orders. The two opposite effects on liquidity can also be explained by the fact that if the opening book is already very deep at the inside spread, then the proportion of orders that move to the DP leaves the best bid-offer very tight.[12] When instead the book opens empty at $T-2$ or with only one share at the best bid-offer, all the three measures of liquidity worsen on average. For the case with an empty book, for example, Table 2 shows that the introduction of a DP makes limit orders less attractive so that traders opt for market orders, and as a result the inside spread increases.

Overall Proposition 4 shows that by moving from the LOB to the LOB&DP, depth and volume decrease, whereas the effect on the inside spread depends on the depth initially available at the top of the book. When the book is empty or has only 1 share available, then the migration makes the inside spread wider and the whole market quality deteriorates. When instead the book opens with 2 shares at the inside, then the effect on the average inside spread is positive and the overall effect on liquidity is mixed. Finally, Table 6 shows that the overall effect of the introduction of a DP on total volume is positive: the sum of the LOB&DP and the DP volume is in fact systematically greater than the amount of volume traded in the benchmark LOB.

[12] Technically, when the book is deep and tight and market orders move to the DP, the probability that the spread remains small increases; the opposit happens when the initial spread is wide.

5.2 Systematic Pattern in Order Flow

Traders' strategic interaction with the two sides of the LOB and with the DP allows us to draw conclusions on the systematic pattern of the order flow, which are summarized in the following Proposition.

Proposition 5 *The following systematic pattern typifies order flows in the LOB and in the LOB&DP market:*

- *when the book is deep, the probability of a continuation is greater than that of a reversal, whereas when the book is shallow the opposite holds;*
- *the DP has a positive externality on the limit order book: if depth decreases on one side, competition for limit orders increases and liquidity gets drained from the DP to the LOB. Hence, volumes show a smaller decline in the LOB&DP protocol.*

Parlour (1998) shows that the interaction of traders with the two sides of the book entails a probability of a continuation greater than that of a reversal, and this is consistent with Biais, Hillion and Spatt (1999). We find that this effect only holds when the book is deep, whereas it is not supported by the model when the top of the book is shallow and traders have to walk up (or down) the book in search of execution. The difference in the results originates from the fact that in Parlour's LOB the trading crowd is positioned at the top of the book, whereas in our model there is a two-level price grid and the trading crowd is not posted at the first level, but rather at the highest (second) level. This means that in our model traders have to walk up (or down) the book when there is not enough liquidity at the top. An example will help understanding why the need to walk up the book entails a probability of a continuation smaller than that of a reversal.

Consider Table 7 where the equilibrium strategies at $T-2$ are reported for two different states of both the LOB and the LOB&DP, $b_{T-2} = [20]$ and $b_{T-2} = [10]$. Comparison between these two books allows us to compute the equilibrium trading strategies of a buyer arriving at the market at $T-2$ and facing a book either with 2 shares at the best ask, or with only 1 share; the latter state of the book can occur if, for example, at time $T-3$ the book opens with 2 units [20] and a market buy order arrives leaving the book with only 1 share on the ask side. Consider first a small buyer who has to decide whether to submit a limit or a market order. The observed reduction of the depth on the opposite side of the book informs him that future sellers will rather post a limit order to sell than a market order to sell:[13] Table 7 shows for example that moving from [20] to [10] the probability to observe limit sell orders increases in percentage by 1.950 and by .8120, respectively for the two cases with and without a DP, and that the probability to observe market sell orders decreases

[13] Here we refer to the average probability of limit orders and market orders submitted by both large and small traders.

by .0859 and .1207. This shift from market to limit sell orders implies that the probability of execution of any eventual limit order to buy decreases, thus inducing the small trader to submit more market than limit buy orders. As a result, the continuation probability of a small buy order becomes greater than that of a reversal. Indeed, Table 7 shows that, after observing a reduction of the depth at the best ask, the probability that a small trader at $T-2$ submits $\varphi(1, \overline{p}_1^A)$ increases by.0128 and .0151 respectively in the market with and without a DP.

Notice that for the small trader in both cases the top of the ask side of the book is deep enough to have a buy order executed without walking up the book. If instead we consider the choice of a large buyer arriving at the market at $T-2$, we observe that in a book [10], despite the lower execution probability of a limit buy order, he will submit fewer rather than more market buy orders. The reason is that when the book changes from [20] to [10], the large trader will have to walk up the book to have his order executed, thus paying a higher price. As the reversal effect for large traders is stronger than the continuation effect for small traders, the average probability of a continuation is smaller than that of a reversal for both the LOB and the LOB&DP markets: $-.0739$ and $-.0702$. Clearly, if after the arrival of a market buy order the final state of the book at $T-2$ were always deep enough even for a large trader (e.g. moving from [30] to [20]), then the 'Parlour effect' would still hold and the probability of a continuation would be higher than that of a reversal.[14] Conversely, if the final state were [00], thus forcing even small traders to walk up the book in search of liquidity, then the probability of market orders to buy would decrease for a small trader too: this is evident by comparing Table 7 with Table 3 and noticing that for the LOB&DP case the probability of a market buy order decreases to .2142.

Table 7 also shows that when depth decreases on one side of the book, e.g. from [20] to [10], trading volume decreases as both market orders to sell and market orders to buy decrease. However, one should notice that the decrease in volume is more contained for the LOB&DP framework: orders decrease in total by .1946 in the LOB, whereas they diminish by .1561 in the presence of a DP. When depth decreases on the ask side of the book, competition for $\varphi(2, p_1^A)$ increases by 8.2881 in the LOB&DP as large traders move from the DP to the book: this means that when the book needs liquidity to attract market orders, this is drained from the DP, which functions like a liquidity buffer.

Proposition 5 offers at least two empirical implications for the dynamic pattern of the order flow: first, the model predicts that liquid stocks should exhibit a probability of a continuation which is higher than that of a reversal, whereas for illiquid stocks the opposite should hold; second, the model foresees an externality originating from the coexistence of a limit order book with a DP. When market depth on the former decreases (increases), it creates a liquidity injection (drain) from the DP to the limit order book.

[14] For brevity we do not report these results here.

5.3 Welfare

In this Section we present results for welfare, measured by the gains from trade of market participants. Formally, welfare for a small ($W_{t,ST}$) and a large trader ($W_{t,LT}$) arriving at the market at time t is defined as:

$$W_{t,a} = E_{b_t}\left[\int_0^2 \pi_t(\varphi^*_{a,\beta_t,b_t,\widetilde{DP}_t}) f(\beta_t)\, d\beta_t\right] \tag{8}$$

The expected value of welfare at time t is computed over all the possible equilibrium states of the book. We measure total expected welfare as the sum of all agents expected gains from trade, which includes welfare of both institutional and retail traders:

$$W_t = \tfrac{1}{2}W_{t,LT} + \tfrac{1}{2}W_{t,ST} \tag{9}$$

Results are summarized in the next Proposition.

Proposition 6 *The introduction of a DP on a limit order book affects traders' welfare. In particular, when a DP is added to a LOB we find that:*

- *total welfare and institutional traders' welfare increases in liquid stocks whereas it decreases in illiquid stocks;*
- *retail traders' welfare always decreases.*

We compute welfare for both the benchmark LOB and the LOB&DP market and report the results obtained from comparisons across different books and over time (Table 8). Results for time T, where limit orders play no role, refer to a dealership market, in the spirit of DVW (2009). As expected, total welfare increases with the liquidity of the opening book for both the benchmark and the LOB&DP market. All traders clearly benefit from a more liquid market as, when the book opens deeper, trading volume and market depth are higher and inside spread is narrower (Table 6).

Liquidity also drives the change in traders' welfare after the introduction of a DP: when a DP is added to a liquid stock (b_{T-2} = [22]) total welfare increases (.0295%), whereas when it is added to an illiquid stock (b_{T-2} = [00]), it decreases (−.4735%). As stated in Proposition 1, DP trading generates trade migration which harms the market when the stock is illiquid by further reducing the already thin liquidity provision. Trade migration is instead beneficial when the stock is liquid as it decreases the competitive pressure in the LOB and hence enhances orders submission As a result, institutional and especially retail traders are harmed by DP when the stock is illiquid, whereas in the case of liquid stocks, the increase of institutional traders' welfare outweighs the losses of retail traders. Indeed large investors benefit from a DP added to a liquid stock as they move their orders to the DP to avoid competition in the LOB. Retail traders, instead, are harmed by DP trading: even when the stock is liquid and DP narrows the inside spread, they cannot access the DP market and hence cannot fully benefit of the decrease in competition. We can therefore conclude that for a LOB, when a DP is added to a liquid stock, it creates an improvement in total welfare, whereas when it is added to an illiquid stock it harms both large and small traders.

6 Asymmetric Information on the State of the DP

The Security and Exchange Commission has recently proposed various changes in the regulation of non-public trading interest that have been grouped under the SEC release No. 34-60997. This proposal takes its move from the widespread use of IOI messages by DP managers that creates a leakage of privileged information to only some select investors. IOI and Alert messages risk creating a two-tiered market that can deprive the public of information about stock prices and liquidity. In this Section we extend the model to include asymmetric information on the state of the DP to show the effects on liquidity of the resulting two-tiered market.

Assume that, all else equal, one group of large traders receives IOI or Alert messages, such as flash orders, about the state of the DP. This feature can be embedded in the model by assuming that at each trading round nature selects with probability 1/2 a small trader, with probability 1/4 a large uninformed trader and with probability 1/4 a large informed one. If a trader arrives at the market and is informed, then he knows the state of the DP and trades accordingly. The following Proposition summarizes the results obtained for this two-tiered market.

Proposition 7 *When some large traders receive private information on the state of the Dark Pool,*

- *the probability that large traders, whether informed or uninformed, choose to trade in the Dark Pool increases and hence orders move from the LOB to the Dark Pool;*
- *the quality of the LOB measured by depth and best bid-offer improves, trading volume in the LOB decreases, whereas total trading volume increases;*
- *in terms of welfare, institutional traders benefit from asymmetric information, whereas retail traders bear extra losses.*

Panels A and B of Table 9 summarize the results obtained in this extended version of the model with two types of large traders: Panel A reports the equilibrium trading strategies of large informed and uninformed traders, and Panel B reports those of small traders. The model has been solved by starting at $T-2$ with 2 shares on both sides of the LOB: this is the regime with greater access to the DP and hence it is more interesting to discuss the role of informed messages. By comparing the results reported in this Table for $T-2$ with those from the model with only one type of large trader (Table 1), it can be noticed that when information on the state of the DP is asymmetric, large traders use the DP rather intensively (.1079). When a large trader knows that the DP is full on one side, he submits an order to the other side with probability greater that 1/2 (.5034), whereas when he notices that the DP is empty, he submits an order with a tiny probability (.0123). If instead a large uninformed trader arrives, he submits an order to the DP with probability equal to .0437, which means that, compared to the case with only one type of large traders (.0279), he uses the DP more intensively. This is due to the fact that at $T-2$ he anticipates that large informed traders will submit their orders to the DP more frequently, and that, for this reason, the DP volume will be enhanced (externality effect), with the result that the execution probability of the orders submitted to the DP will increase. And if at $T-2$ he does not observe any trade (Table 9, Panel A.2), the probability that at $T-1$ he submits to the DP increases even further (.052) than in the case without informed messages as he knows that the probability of DP trading is higher under asymmetric information.

It follows that if IOI and Alert messages create a two-tiered market, with some large traders holding precise information about the state of the DP, then liquidity moves from the LOB to the DP. In fact all traders anticipate that the informed will use the DP more intensively and this increases the probability of execution of DP orders thus reinforcing the DP externality effect.

Table 10 shows that when traders move to the DP more frequently, spread and depth in the LOB improve. Compared to the protocol without asymmetric information, here not only the spread improves but also market depth increases; the reason is that Alert and IOI messages have the overall effect of reducing the execution risk of DP trading, thus making market orders less attractive than DP orders. Considering again the probability of order submission under asymmetric information, Table 9 (Panel A.1) shows that with IOI and Alert messages the probability to observe market orders decreases by 18.1%, whereas without asymmetric information the reduction is tiny (1.6%). Further, limit order submissions decrease less with asymmetric information, even though the difference in difference is much smaller. The result of this change in order submission probabilities is that volume in the LOB decreases even more than in the case without information leakage; however, due to the heavier use of the DP, total volume executed in both the limit order book and the DP increases to 4.1549 (Table 10).

In terms of welfare the overall consequence of asymmetric information is to amplify the effects of the introduction of the DP. When the DP is added to a LOB and DP managers can use flash messages to disseminate information on the state of the DP the welfare of all large traders increases: indeed, not only large informed traders can profit from their privileged information, but also uninformed traders can take advantage of their more precise inference on the state of the DP and use the DP more intensively. Table 11 shows that large uninformed traders' welfare increases with the addition of a DP, and it further increases when IOI and Alert messages are disseminated. Quite the opposite occurs to retail traders whose welfare at $T-2$ decreases by an extra .04%: as all large traders move to the DP more frequently, the execution probability of small traders' limit orders posted to the LOB decreases thus generating extra losses.

In conclusion, the main upshot of asymmetric information is to reduce execution risk from DP trading which enhances the use of DP orders at the expense of market orders that are usually submitted by traders sensitive to this type of risk. With less market orders, LOB volume decreases, but LOB depth as well as total volume increase. This is the reason why, when the DP is added to the LOB, large traders benefit from asymmetric information, whereas small traders bear higher losses.

7 Empirical Implications

A great deal of empirical implications can be derived from our model, that we summarize in this Section. First of all, our results show that when a DP is added to a LOB, volume migrate to the DP so that volume in the LOB decreases; yet, the sum of the volume traded on both the LOB and the DP increases. Second, we show that the overall effect of intermarket competition crucially depends on how deep and tight the LOB is. We expect that trade migration is more intense in liquid than in illiquid stocks as in the latter competition from limit orders crowds out DP orders. Following the volume migration, depth at the top of the book deteriorates, whereas the effect on inside spread depends on the liquidity of the stock. For liquid stocks where the book is very deep, the relative proportion of market to limit orders that move to the DP leaves the inside spread very tight, whereas for illiquid stocks it widens the spread. Our results also show that when traders believe that liquidity is growing in the dark pool, dark trading intensifies so that we expect DP volume in liquid stocks to increase more intensively than in illiquid stocks.

Beside depth and spread, our model has also suggestions for a third determinant of DP trading, as it shows that DP volume increases with the tick size. An increase in the tick size on the one hand increases the inside spread, thus making market orders more expensive, on the other hand it makes limit orders more convenient with the result that patient traders, who are not so patient to submit limit orders, will trade more in the DP. This is an interesting and intriguing empirical prediction of our model that, however, should be tested with caution. The type of dark pools that our model features are different from those internalization pools that are used by broker-dealers to internalize trades. Also in the latter an increase of the tick size raises dark volume, but the effect is driven by broker-dealers' profits from sub-penny trading. Consequently, to separate the effect that a tick size change can have on different dark venues, empiricists should control for the average order size that in internalization pools is much smaller than in traditional dark pools (*Rosenblatt Securities*, October 2010).

Our benchmark model also qualifies standard results on systematic pattern of LOB order flows, as it shows that Parlour's (1998) main finding that the probability of a continuation is larger than that of a reversal only characterizes liquid stocks, the opposite holding for illiquid ones. Furthermore, the model predicts an externality originating from the coexistence of a limit order book and a DP, as it shows that when market depth on the former decreases, it creates a liquidity injection from the DP to the limit order book. Finally, the model shows that IOI messages tend to move volume from the LOB to the DP and to reinforce the liquidity-externality effect. However, it also shows that spread and depth in the LOB improve following the introduction of IOI messages, as they reduce execution risk from DP trading and hence crowd out market orders that are submitted by traders who are sensitive to this type of risk. Hence even though volume in the LOB decreases, depth and spread improve.

8 Conclusions

The dynamic microstructure model presented in this paper solves for the equilibrium trading strategies of different agents who can choose to trade either in a Limit Order Book (LOB) or in a Dark Pool (DP). A DP is an Alternative Trading System that does not provide its best-priced orders for inclusion in the consolidated quotation data. The existing theory shows that dark crossing networks increase liquidity. Our model shows that this is true only when a DP is added to a dealership market where traders cannot compete for the provision of liquidity by submitting limit orders. When a DP is added to a LOB, orders migrate away from the LOB to the dark market. The model thus demonstrates that the dark option offered to market participants produces order migration rather than order creation as in Degryse, Van Achter and Wuyts (2009).

We also show that current DP orders stimulate the arrival of future DP orders thus increasing their execution probability (liquidity-externality effect). Traders' choice between LOB and DP depends on the state of the LOB as well as on agents' expectations on the state of the DP: the model shows that high depth and small spread increase traders' use of DP, and that a reduction in the tick size reduces the profitability of liquidity provision and hence the use of DP orders. According to the model, when large visible limit orders produce price pressure in the LOB, traders resort to DP orders even more extensively to reduce price impact.

In terms of market quality, when a DP is added to an LOB we find that depth and volume deteriorate on the LOB, whilst total volume increases. The effect of the introduction of a DP on the inside spread of the LOB instead depends on the state of the book, improving when it is deep and worsening when it is shallow.

The model also offers new insights on the systematic patterns of order flow that can arise from traders' interaction with the LOB: for liquid stocks the probability of a continuation is greater than that of a reversal, the opposite being true for illiquid stocks. Furthermore, DP act as liquidity buffers by supplying liquidity after a reduction of market depth on the LOB. In terms of welfare, we show that total welfare and institutional traders' welfare increase only when a DP is added to a liquid stock, and that a DP always harms retail traders.

Finally, we show how asymmetric information on the state of the DP creates a two-tiered market that moves liquidity from the LOB to the DP. When traders know that other traders are informed on the state of the DP, they anticipate that the informed will use the DP more intensively and that this will increase the probability of execution of DP orders. Hence, consistently with the recent SEC Proposal on Regulation of Non-Public Trading Interest (SEC Release No. 34-60997), the model shows that IOI and Alert messages, that inform some traders on the state of the DP, can draw orders away from the transparent market. However, compared to the protocol with a DP and without asymmetric information, we show that the use of flash orders can improve not only order book spread but also market depth when they are allowed to be used for liquid stocks.

Appendix

Proof of Proposition 1

Consider first the benchmark case. The model is solved by backward induction, starting from $t = T$. The T-trader solves a simplified version of program (2), if large, or (3), if small:

$$\max_{\varphi_{LT,\beta_T,b_T,j}} \pi_T^e \left\{ \varphi(j, \overline{p}_i^B), \varphi(j, \overline{p}^B), \varphi(0), \varphi(j, \overline{p}_i^A), \varphi(j, \overline{p}^A) \right\} \quad (2')$$

$$\max_{\varphi_{ST,\beta_T,b_T}} \pi_T^e \left\{ \varphi(1, \overline{p}_i^B), \varphi(0), \varphi(1, \overline{p}_1^A) \right\} \quad (3')$$

Without loss of generality, assume that depending on β_T the trader selects among $n \in N_T$ possible equilibrium strategies. These strategies are ordered so that $\beta_{T,a}^{\varphi^{*,n-1},\varphi^{*,n}} < \beta_{T,a}^{\varphi^{*,n},\varphi^{*,n+1}}$, where $a = \{ST, LT\}$ and the β-thresholds are given by:

$$\beta_{T,a}^{\varphi^{*,n-1},\varphi^{*,n}} : \pi_T^e(\varphi_{a,\beta_T,j}^{*,n-1} \mid \Omega_T) - \pi_T^e(\varphi_{a,\beta_T,j}^{*,n} \mid \Omega_T) = 0$$

The ex-ante probability that a trader submits a certain order type at T is determined as follows:

$$\Pr_T(\varphi_{a,\beta_T,j}^{*,n} \mid \Omega_T) = F(\beta_{T,a}^{\varphi^{*,n},\varphi^{*,n+1}}) - F(\beta_{T,a}^{\varphi^{*,n-1},\varphi^{*,n}})$$

Consider now period $t = T-1$. The incoming trader solves program (2) or (3) if large or small respectively, and uses $\Pr_T(\varphi_{a,\beta_T,j}^{*,n} \mid \Omega_T)$, $\forall n \in N_T$, to compute the execution probabilities of his limit orders. Given the optimal strategies $\varphi_{a,\beta_{T-1},b_{T-1},j}^{*}$, the β-thresholds and the order type probabilities at $T-1$ are derived as shown for period T. This procedure is then reiterated for period $T-2$. The solution of the LOB&DP model follows the same methodology, but now the large trader solves program (4).

We provide examples for the LOB&DP protocol for the three trading periods analyzed, that belong to the case where the book opens as $b_{T-2} = [22]$. From now onwards we assume that in program (4) $j^* = \max_j [\varphi_{LT,\beta_T,j} \mid \Omega_T]$, since $\partial \pi_t(\varphi_{LT,\beta_T,j})/\partial j \geq 0$ due to agents' risk neutrality.

Consider the following books at T: (a) $b_T = [20]$, $vis_{T-2}\ vis_{T-1}$, (b) $b_T = [20]$, $inv_{T-2}\ vis_{T-1}$. In the first case traders observe a change in the LOB in both periods, while in the second one only at $T-1$. We focus on the large trader's profits that for (a) are:

$$\pi_T^e[\varphi(2, \overline{p}_2^B) \mid \Omega_T^{[20,vv]}] = 2(p_2^B - \beta_T v) = 2(1 - \tfrac{3\tau}{2} - \beta_T)$$

$$\pi_T^e[\varphi(2, \overline{p}_1^A) \mid \Omega_T^{[20,vv]}] = 2(\beta_T - v p_1^A) = 2(\beta_T - 1 - \tfrac{\tau}{2})$$

$$\pi_T^e[\varphi(-2, p_{Mid}) \mid \Omega_T^{[20,vv]}] = 2(\tfrac{p_1^A + p_2^B}{2} - \beta_T v)\Pr_{-2}(\tfrac{p_1^A + p_2^B}{2} | \Omega_T) = 2(1 - \tfrac{\tau}{2} - \beta_T) \times \tfrac{1}{3}$$

$$\pi_T^e[\varphi(+2, p_{Mid}) \mid \Omega_T^{[20,vv]}] = 2(\beta_T v - \tfrac{p_1^A + p_2^B}{2})\Pr_{+2}(\tfrac{p_1^A + p_2^B}{2} | \Omega_T) = 2(\beta_T - 1 + \tfrac{\tau}{2}) \times \tfrac{1}{3}$$

where $y_t \in \{v, i\}$, with $v = vis_t$ and $i = inv_t$, and $\Omega_T^{[xx,yy]} : [xx, yy] = \{b_T, y_{T-2}y_{T-1}\}$. By solving program (2') for this case it is straightforward to show that all strategies are optimal in equilibrium ($N_T = 4$) and that for the LT $\varphi^{*,1}_{LT,[20,vv]} = \varphi(2, \overline{p}_2^B)$, $\varphi^{*,2}_{LT,[20,vv]} = \varphi(-2, p_{Mid})$, $\varphi^{*,3}_{LT,[20,vv]} = \varphi(+2, p_{Mid})$ and $\varphi^{*,4}_{LT,[20,vv]} = \varphi(2, \overline{p}_1^A)$. As an example we compute the probability of $\varphi^{*,1}_{LT,[20,vv]}$:

$$\begin{aligned}
\beta_{T,LT}^{\varphi^{*,1}_{[20,vv]},\varphi^{*,2}_{[20,vv]}} &: \pi_T^e[\varphi^{*,1}_{LT,[20,vv]}] - \pi_T^e[\varphi^{*,2}_{LT,[20,vv]}] = 0 \rightarrow \beta_{T,LT}^{\varphi^{*,1}_{[20,vv]},\varphi^{*,2}_{[20,vv]}} = 1 - 2\tau \\
\Pr_T(\varphi^{*,1}_{LT,[20,vv]}) &= F(\beta_{T,LT}^{\varphi^{*,1}_{[20,vv]},\varphi^{*,2}_{[20,vv]}}) = \tfrac{1}{2}(1 - 2\tau)
\end{aligned}$$

In case (b), profits for DP orders differ:

$$\pi_T^e[\varphi(-2, p_{Mid}) \mid \Omega_T^{[20,iv]}] = 2\left(\frac{p_1^A + p_2^B}{2} - \beta_T v\right)\left(\frac{1}{3}\times 1 + \frac{1}{3}\frac{\Pr_{T-2}(\varphi^*(+2,p_{Mid}))}{\Pr_{T-2}(\varphi^*(+2,p_{Mid})) + \Pr_{T-2}(\varphi^*(-2,p_{Mid})) + \Pr_{T-2}(\varphi^*(0))}\right)$$

$$\pi_T^e[\varphi(+2, p_{Mid}) \mid \Omega_T^{[20,iv]}] = 2\left(\beta_T v - \frac{p_1^A + p_2^B}{2}\right)\left(\frac{1}{3}\times 1 + \frac{1}{3}\frac{\Pr_{T-2}(\varphi^*(-2,p_{Mid}))}{\Pr_{T-2}(\varphi^*(+2,p_{Mid})) + \Pr_{T-2}(\varphi^*(-2,p_{Mid})) + \Pr_{T-2}(\varphi^*(0))}\right)$$

where we omit that all probabilities at $T-2$ are conditional to Ω_{T-2}. In this case both the β-thresholds and order probabilities are a function of equilibrium order strategies at $T-2$, $\varphi^*_{a,\beta_{T-2},b_{T-2},j}$, that are rationally computed by the T-trader. For example, if we define $\varphi^1_{LT,[20,iv]} = \varphi(2, \overline{p}_2^B)$ and $\varphi^2_{LT,[20,iv]} = \varphi(-2, p_{Mid})$:

$$\begin{aligned}
\beta_{T,LT}^{\varphi^1_{[20,iv]},\varphi^2_{[20,iv]}} &: \pi_T^e[\varphi^1_{LT,[20,iv]}] - \pi_T^e[\varphi^2_{LT,[20,iv]}] = 0 \\
\beta_{T,LT}^{\varphi^1_{[20,iv]},\varphi^2_{[20,iv]}} &= \frac{(2-7\tau)\Pr_{T-2}\varphi^*(+2,p_{Mid}) + 4(1-2\tau)[\Pr_{T-2}\varphi^*(-2,p_{Mid}) + \Pr_{T-2}(\varphi^*(0)]}{2\Pr_{T-2}\varphi^*(+2,p_{Mid}) + 4(\Pr_{T-2}\varphi^*(-2,p_{Mid}) + \Pr_{T-2}(\varphi^*(0))} \\
\Pr_T\left(\varphi^1_{LT,[20,iv]}\right) &= F(\beta_{T,LT}^{\varphi^1_{[20,iv]},\varphi^2_{[20,iv]}}) = \tfrac{1}{2}\beta_{T,LT}^{\varphi^1_{[20,iv]},\varphi^2_{[20,iv]}}
\end{aligned}$$

To determine the equilibrium strategies $\varphi^{*,n}_{LT,[20,iv]}$ for $n \in N_T$, the model has to be solved up to period $T-2$. We anticipate that $\varphi(2, \overline{p}_2^B)$ is indeed an equilibrium strategy, and that the corresponding probability is: $\Pr_T(\varphi^{*,1}_{LT,[20,iv]}) = \frac{(2-5\tau)}{4}$.

For $T-1$ and $T-2$ we only specify the profit formulas, as the derivation of the β-thresholds and order probabilities follow the same steps presented for period T. Consider the case $b_{T-1} = [20]$, vis_{T-2}. Small trader's profits are as follows:

$$\begin{aligned}
\pi_{T-1}^e[\varphi(1, \overline{p}_2^B) &\mid \Omega_{T-1}^{[20,v]}] = (p_2^B - \beta_{T-1} v) \\
\pi_{T-1}^e[\varphi(1, p_1^A) &\mid \Omega_{T-1}^{[20,v]}] = \pi_{T-1}^e[\varphi(0)] = 0 \\
\pi_{T-1}^e[\varphi(1, p_1^B) &\mid \Omega_{T-1}^{[20,v]}] = (\beta_{T-1} v - p_1^B)\tfrac{1}{2}[\Pr_T(\varphi(1, \overline{p}_1^B) \mid \Omega_T^{[21,vv]}) + \Pr_T(\varphi(2, \overline{p}_1^B) \mid \Omega_T^{[21,vv]})] \\
\pi_{T-1}^e[\varphi(1, \overline{p}_1^A) &\mid \Omega_{T-1}^{[20,v]}] = (\beta_{T-1} v - p_1^A)
\end{aligned}$$

where $\Omega_{T-1}^{[xx,y]}: [xx,y] = \{b_{T-1}, y_{T-2}\}$. Large trader's strategies are similar, the only difference being that $j=2$, with the exception of DP orders:

$$\begin{aligned}\pi^e_{T-1}[\varphi(-2,p_{Mid})] &= E[(p_{Mid}-\beta_{T-1}v)\Pr_{-2}(p_{Mid}|\Omega_T)] \\ \pi^e_{T-1}[\varphi(+2,p_{Mid})] &= E[(\beta_{T-1}v-p_{Mid})\Pr_{+2}(p_{Mid}|\Omega_T)]\end{aligned}$$

We specify the first one:

$$\begin{aligned}\pi^e_{T-1}[\varphi(-2,p_{Mid})] = {} & \tfrac{1}{3}\times 2\times\tfrac{1}{2}(\tfrac{p_1^A+p_2^B}{2}-\beta_{T-1}v)\Pr_T(\varphi(+2,p_{Mid})\mid\Omega_T^{[20,vi,0]}) \\ & +\tfrac{1}{3}\times 2\times\tfrac{1}{2}\{(\tfrac{p_2^A+p_2^B}{2}-\beta_{T-1}v)\Pr_T(\varphi(2,\overline{p}_1^A)\mid\Omega_T^{[20,vi,+6]})+(\tfrac{p_1^A+p_2^B}{2}-\beta_{T-1}v) \\ & [1+\Pr_T(\varphi(+2,p_{Mid})\mid\Omega_T^{[20,vi,+6]})+\Pr_T(\varphi(-2,p_{Mid})\mid\Omega_T^{[20,vi,+6]})+\Pr_T(\varphi(2,\overline{p}_2^B)\mid\Omega_T^{[20,vi,+6]})]\}\end{aligned}$$

where $\Omega_T^{[xx,yy,z]}: [xx,yy,z] = \{b_T, y_{T-2}y_{T-1}, DP_T\}$. At $T-2$ we consider the book $b_{T-2}=[22]$ and present profit formulas only for the sell side of the market, the buy side being symmetric:

$$\begin{aligned}\pi^e_{T-2}[\varphi(2,\overline{p}_1^B)] &= 2(p_1^B-\beta_{T-2}v) \\ \pi^e_{T-2}[\varphi(0)] &= 0 \\ \pi^e_{T-2}[\varphi(2,p_1^A)] &= (p_1^A-\beta_{T-2}v)\{\tfrac{1}{2}\Pr_{T-1}(\varphi(1,\overline{p}_1^A)\mid\Omega_{T-1}^{[42,v]})[\tfrac{1}{2}\Pr_T(\varphi(2,\overline{p}_1^A)\mid\Omega_T^{[32,vv]})] \\ &\quad +\tfrac{1}{2}\Pr_{T-1}(\varphi(2,\overline{p}_1^A)\mid\Omega_{T-1}^{[42,v]})[\tfrac{1}{2}\Pr_T(\varphi(1,\overline{p}_1^A)\mid\Omega_T^{[22,vv]})+\tfrac{1}{2}2\Pr_T(\varphi(2,\overline{p}_1^A)\mid\Omega_T^{[22,vv]})]\} \\ \pi^e_{T-2}[\varphi(-2,p_{Mid})] &= E\{(p_{Mid}-\beta_{T-2}v)[\Pr_{-2}(p_{Mid}|\Omega_{T-1})+(1-\Pr_{-2}(p_{Mid}|\Omega_{T-1}))\Pr_{-2}(p_{Mid}|\Omega_T)]\}\end{aligned}$$

where to economize space we do not specify $\pi^e_{T-2}[\varphi(-2,p_{Mid})]$. Results from Proposition 1 are derived by comparing equilibrium strategies for the LOB&DP case and for the pure LOB case presented in Table 1 and Table 2. In Figures A1-A4 we provide graphical plots for the large trader's profits at $T-2$ as a function of β that relate to the four main points of the Proposition. We consider only selling strategies, the picture being symmetric for the buy side.

Proof of Proposition 2

Results from Proposition 2 are obtained by straightforward comparison of the equilibrium strategies derived in the proof of Proposition 1, for different states of the LOB. Also for this proof we provide graphical plots. Consider first the DP&LOB: compare Figures A1 and A5 for the effect of market depth, A1 and A3 for the effect of spread, and A5 and A6 for the tick size. For DM&DP, consider Figure A7.

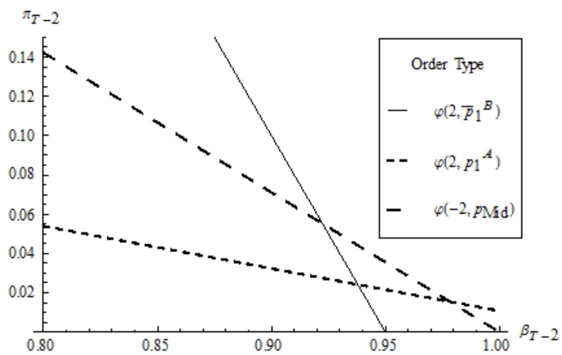

Figure A1. Trade Migration on the LOB&DP – $b_{T\text{-}2}$=[22]

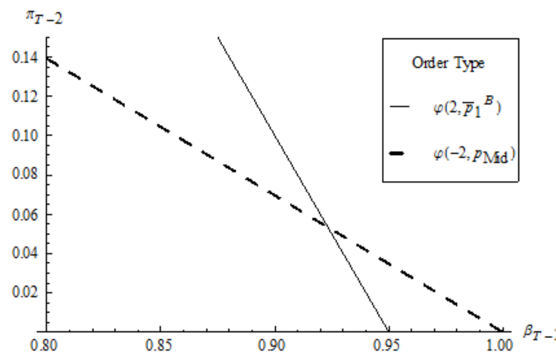

Figure A2. Trade Creation on the DM&DP

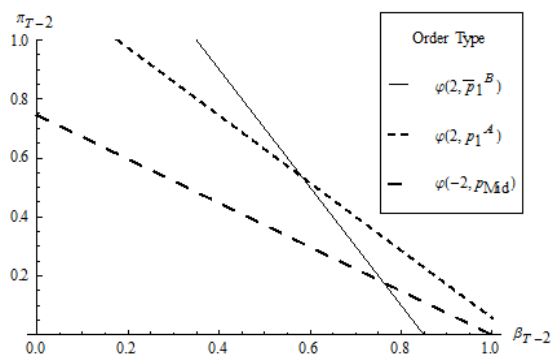

Figure A3. Trade Migration on the LOB&DP – $b_{T\text{-}2}$=[00]

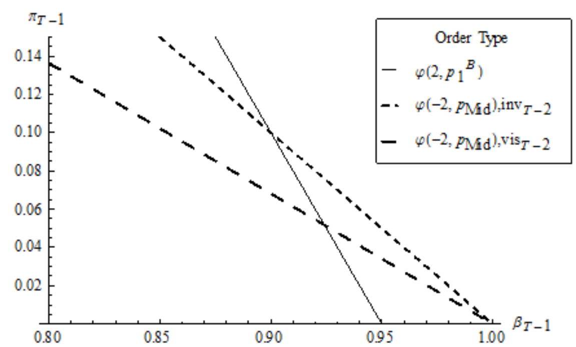

Figure A4. Liquidity Externality Effect of DP – $b_{T\text{-}1}$=[22]

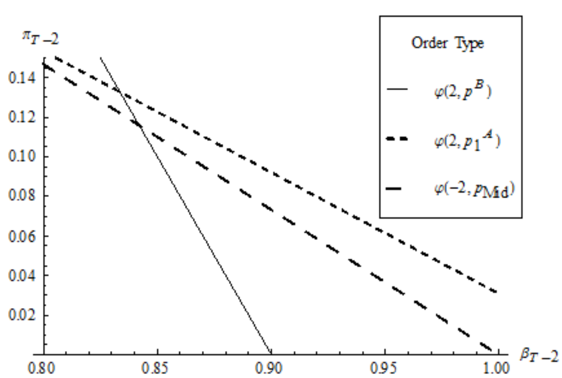

Figure A5. Dark Pool Drivers – LOB&DP – $b_{T\text{-}2}$=[11]

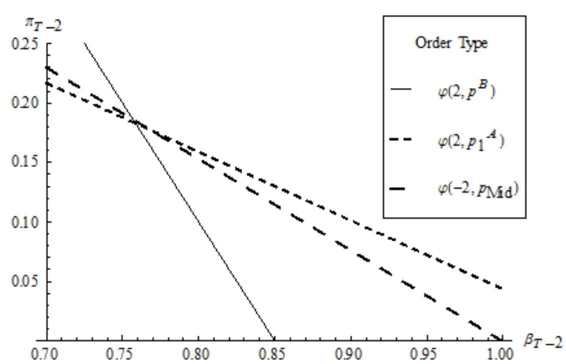

Figure A6. Dark Pool Drivers - $b_{T\text{-}2}$=[11] – τ=0.15

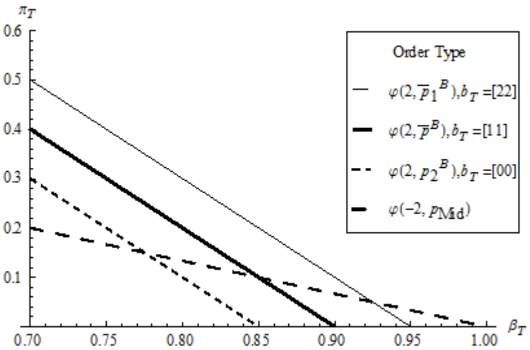

Figure A7. Dark Pool Drivers – DM&DP

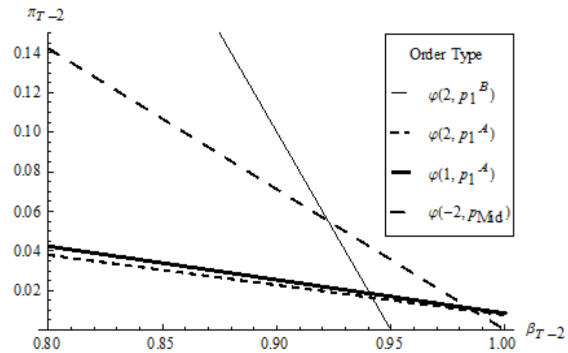

Figure A8. Price Pressure – b_{T-2}=[22]

Proof of Proposition 3

The benchmark and the LOB&DP model are solved following the procedure already illustrated in Proposition 1. To clarify the difference between the case with and without price pressure, consider an opening book $b_{T-2} = [22]$ where a large limit sell order, $\varphi_{T-2}(2, p_1^A)$, is submitted. With no price pressure, at $T-1$ the book is full, $b_{T-1} = [42]$, so the feasible strategies are only market orders, dark pool orders or no trade. With price pressure, instead, the new asset value becomes $v_{T-1} = v - \tau$, and, as shown in Figure 5, the price grid shifts accordingly so that the book turns empty, $b_{T-1} = [00]$. This implies that limit orders, $\varphi_{T-1}(j, p_1^A)$ and $\varphi_{T-1}(j, p_1^B)$, are now included within the available strategies. Large traders arriving at $T-2$ will rationally anticipate the profit reduction in 2-units limit orders due to price pressure, so $j = 2$ is not necessarily optimal anymore. We hence specify as an example the profit formulas for both $\pi^e_{T-2}[\varphi(2, p_1^A)]$ and $\pi^e_{T-2}[\varphi(1, p_1^A)]$:

$$\pi^e_{T-2}[\varphi(2,p_1^A)] = (p_1^A - \beta_{T-2}v_T)\{\tfrac{1}{2}\Pr_{T-1}(\varphi(1,\overline{p}_2^{A_d}) \mid \Omega_{T-1}^{[00,v]})[\tfrac{1}{2}\Pr_T(\varphi(2,\overline{p}_1^A) \mid \Omega_T^{[20,vv]})]$$
$$+ \tfrac{1}{2}\Pr_{T-1}(\varphi(2,\overline{p}_2^{A_d}) \mid \Omega_{T-1}^{[00,v]})[\tfrac{1}{2}\Pr_T(\varphi(1,\overline{p}_1^A) \mid \Omega_T^{[22,vv]}) + \tfrac{1}{2}2\Pr_T(\varphi(2,\overline{p}_1^A) \mid \Omega_T^{[20,vv]})]\}$$

$$\pi^e_{T-2}[\varphi(1,p_1^A)] = (p_1^A - \beta_{T-2}v_T)\{\tfrac{1}{2}\Pr_{T-1}(\varphi(1,\overline{p}_1^A) \mid \Omega_{T-1}^{[32,v]})[\tfrac{1}{2}\Pr_T(\varphi(2,\overline{p}_1^A) \mid \Omega_T^{[22,vv]})]$$
$$+ \tfrac{1}{2}\Pr_{T-1}(\varphi(2,\overline{p}_1^A) \mid \Omega_{T-1}^{[32,v]})[\tfrac{1}{2}\Pr_T(\varphi(1,\overline{p}_1^A) \mid \Omega_T^{[12,vv]}) + \tfrac{1}{2}\Pr_T(\varphi(2,\overline{p}^A) \mid \Omega_T^{[12,vv]})]\}$$

In Figure A8 we plot large trader's profits for the model with price pressures, that can be compared to Figure A1. Numerical values are reported in Table 5.

Proof of Proposition 4

Results, presented in Table 6, are obtained by comparing the three market quality measures for both the benchmark and the LOB&DP protocol. As an example, we consider the LOB&DP model with an opening book equal to $b_{T-2} = [22]$ and specify formulas for the estimated spread and depth at $T-1$ and for the executed volume at $T-2$. Similar computations make it possible to derive the market quality measures for all the other cases. We define equilibrium strategies at $T-2$ for a LT as follows: $\varphi^{*,1}_{LT,[22]} = \varphi(2,\overline{p}^B_2)$, $\varphi^{*,2}_{LT,[22]} = \varphi(-2, p_{Mid})$, $\varphi^{*,3}_{LT,[22]} = \varphi(2, p^A_1)$, $\varphi^{*,4}_{LT,[22]} = \varphi(2, p^B_1)$, $\varphi^{*,5}_{LT,[22]} = \varphi(+2, p_{Mid})$ and $\varphi^{*,6}_{LT,[22]} = \varphi(2,\overline{p}^A_1)$. The ones for a ST are: $\varphi^{*,1}_{ST,[22]} = \varphi(1,\overline{p}^B_2)$, $\varphi^{*,2}_{ST,[22]} = \varphi(1, p^A_1)$, $\varphi^{*,3}_{ST,[22]} = \varphi(1, p^B_1)$ and $\varphi^{*,4}_{ST,[22]} = \varphi(2,\overline{p}^A_1)$.

$$\begin{aligned}
S^e_{T-1} = {} & \tfrac{1}{2}\{\Pr_{T-2}(\varphi^{*,1}_{LT,[22]})\left(p^A_1 - p^B_2\right) + \Pr_{T-2}(\varphi^{*,6}_{LT,[22]})\left(p^A_2 - p^B_1\right) \\
& + [\Pr_{T-2}(\varphi^{*,2}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,3}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,4}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,5}_{LT,[22]})]\left(p^A_1 - p^B_1\right)\} \\
& + \tfrac{1}{2}[\Pr_{T-2}(\varphi^{*,1}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,2}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,3}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,4}_{ST,[22]})]\left(p^A_1 - p^B_1\right)
\end{aligned}$$

$$\begin{aligned}
D^e_{T-1} = {} & \tfrac{1}{2}\{2 \times \Pr_{T-2}(\varphi^{*,1}_{LT,[22]})\left(p^A_1 - p^B_2\right) + \Pr_{T-2}(\varphi^{*,6}_{LT,[22]})\left(p^A_2 - p^B_1\right) \\
& + 6 \times [\Pr_{T-2}(\varphi^{*,3}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,4}_{LT,[22]})] + 4 \times [\Pr_{T-2}(\varphi^{*,2}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,5}_{LT,[22]})]\} \\
& + \tfrac{1}{2}\{3 \times [\Pr_{T-2}(\varphi^{*,1}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,4}_{ST,[22]})] + 5 \times [\Pr_{T-2}(\varphi^{*,2}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,3}_{ST,[22]})]\}
\end{aligned}$$

$$V^e_{T-2} = \tfrac{1}{2}[\Pr_{T-2}(\varphi^{*,1}_{ST,[22]}) + \Pr_{T-2}(\varphi^{*,4}_{ST,[22]})] + \tfrac{1}{2} \times 2[\Pr_{T-2}(\varphi^{*,1}_{LT,[22]}) + \Pr_{T-2}(\varphi^{*,6}_{LT,[22]})]$$

Proof of Proposition 5

Results are derived by comparing equilibrium trading strategies for two new starting books: $b_{T-2} = [20]$ and $b_{T-2} = [10]$. As the solutions of these two cases follow the same steps as the one presented in the proof of Proposition 1 for the book $b_{T-2} = [22]$, they are omitted and available upon the authors at request. In Figure A9-A10 we plot profits for large and small traders respectively. Numerical values are provided in Table 7.

Proof of Proposition 6

Results presented in Table 8 are obtained by comparing welfare levels for both the benchmark and the LOB&DP protocol. As an example we consider again the LOB&DP case opening with $b_{T-2} = [22]$. Notice that it is sufficient to compute the welfare for the ask side, as traders' strategies on the other side are perfectly symmetric. We start with period $T-2$, equilibrium strategies for both a LT and a ST are defined in the proof of Proposition 4.

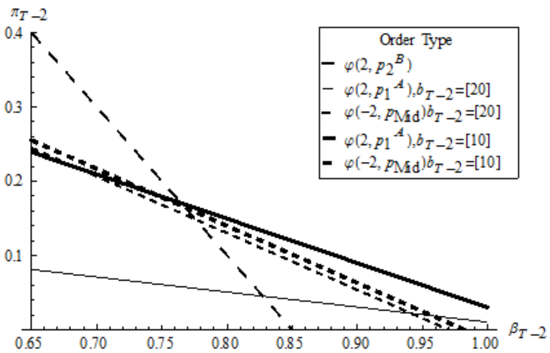

Figure A9. Pattern in Order Flow – LT

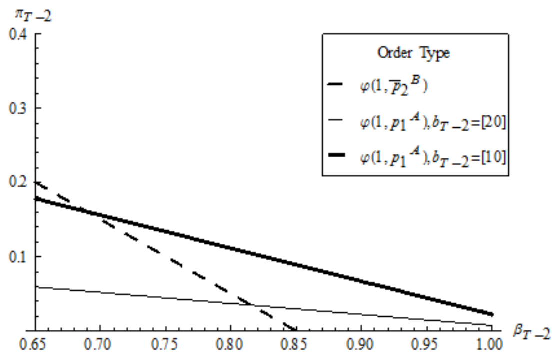

Figure A10. Pattern in Order Flow – ST

$$
\begin{aligned}
W_{T-2,LT} = {} & 2\{\underset{T-2}{\Pr}(\varphi^{*,1}_{LT,[22]})(2p_1^B - \frac{\beta_{T,LT}^{\varphi^{*,1}_{[22]},\varphi^{*,2}_{[22]}}}{2}v) + \underset{T-2}{\Pr}(\varphi^{*,2}_{LT,[22]})(2p_{Mid} - \frac{\beta_{T,LT}^{\varphi^{*,2}_{[22]},\varphi^{*,3}_{[22]}} + \beta_{T,LT}^{\varphi^{*,1}_{[22]},\varphi^{*,2}_{[22]}}}{2}v) \\
& + \underset{T-2}{\Pr}(\varphi^{*,3}_{LT,[22]})(2p_1^A - \frac{\beta_{T,LT}^{\varphi^{*,3}_{[22]},\varphi^{*,4}_{[22]}} + \beta_{T,LT}^{\varphi^{*,2}_{[22]},\varphi^{*,3}_{[22]}}}{2}v)\{2\underset{2}{\Pr}(p_1^A|\Omega_{T-1}) \\
& + \underset{1}{\Pr}(p_1^A|\Omega_{T-1})[\underset{0}{\Pr}(p_1^A|\Omega_T) + 2\underset{1}{\Pr}(p_1^A|\Omega_T)] + \underset{0}{\Pr}(p_1^A|\Omega_{T-1})[\underset{1}{\Pr}(p_1^A|\Omega_T) + 2\underset{2}{\Pr}(p_1^A|\Omega_T)]\}\}
\end{aligned}
$$

$$
\begin{aligned}
W_{T-2,ST} = {} & 2\{\underset{T-2}{\Pr}(\varphi^{*,1}_{ST,[22]})(p_1^B - \frac{\beta_{T,ST}^{\varphi^{*,1}_{[22]},\varphi^{*,2}_{[22]}}}{2}v) + \underset{T-2}{\Pr}(\varphi^{*,2}_{ST,[22]})(p_1^A - \frac{\beta_{T,ST}^{\varphi^{*,2}_{[22]},\varphi^{*,3}_{[22]}} + \beta_{T,ST}^{\varphi^{*,1}_{[22]},\varphi^{*,2}_{[22]}}}{2}v)[\underset{1}{\Pr}(p_1^A|\Omega_{T-1}) \\
& + \underset{0}{\Pr}(p_1^A|\Omega_{T-1})\underset{1}{\Pr}(p_1^A|\Omega_T)]\}
\end{aligned}
$$

Welfare values for the following periods are computed similarly, but taking into account that a trader faces different states of the book depending on the orders submitted in the previous periods. For example, still for the case $b_{T-2} = [22]$, the welfare of trader a, with $a = \{ST, LT\}$, arriving at $T-1$ is computed as follows:

$$
\begin{aligned}
W_{T-1,a} = {} & \tfrac{1}{2}\{\underset{T-2}{\Pr}(\varphi^{*,1}_{LT,[22]})W^{[20,v]}_{T-1,a} + \underset{T-2}{\Pr}(\varphi^{*,6}_{LT,[22]})W^{[02,v]}_{T-1,a} + \underset{T-2}{\Pr}(\varphi^{*,3}_{LT,[22]})W^{[42,v]}_{T-1,a} \\
& + [\underset{T-2}{\Pr}(\varphi^{*,2}_{LT,[22]}) + \underset{T-2}{\Pr}(\varphi^{*,5}_{LT,[22]})]W^{[22,i]}_{T-1,a} + \underset{T-2}{\Pr}(\varphi^{*,4}_{LT,[22]})W^{[24,v]}_{T-1,a}\} + \tfrac{1}{2}\{\underset{T-2}{\Pr}(\varphi^{*,1}_{ST,[22]})W^{[21,v]}_{T-1,a} \\
& + \underset{T-2}{\Pr}(\varphi^{*,4}_{ST,[22]})W^{[12,v]}_{T-1,a} + \underset{T-2}{\Pr}(\varphi^{*,2}_{ST,[22]})W^{[32,v]}_{T-1,a} + \underset{T-2}{\Pr}(\varphi^{*,3}_{ST,[22]})W^{[23,v]}_{T-1,a}\}
\end{aligned}
$$

Proof of Proposition 7

This proof follows the same methodology presented in the proof of Proposition 1. To ease the comparison with the previous framework, we provide again as an example the case $b_T = [20]$,

$inv_{T-2}\ vis_{T-1}$, for both uninformed (u) and informed (i) large traders.[15] Consider first the u-trader, profits from market orders are unchanged and omitted, but profits from $\varphi_u(-2, p_{Mid})$ become:

$$\pi_T^e[\varphi_u(-2, p_{Mid})|\Omega_T^{[20,iv]}] = 2(\tfrac{p_1^A+p_2^B}{2} - \beta_T v)\Pr_{-2}(\tfrac{p_1^A+p_2^B}{2}|\Omega_T)$$
$$= 2(\tfrac{p_1^A+p_2^B}{2} - \beta_T v)[\tfrac{1}{3}\times 1 + \tfrac{1}{3}\sum_{d=i,u} \tfrac{\Pr_{T-2}(\varphi_d^*(-2,p_{Mid}))}{\Pr_{T-2}(\varphi_d^*(+2,p_{Mid}))+\Pr_{T-2}(\varphi_d^*(-2,p_{Mid}))}]$$

where we omit that all probabilities at $T-2$ are conditional to Ω_{T-2}. Notice that the trader rationally anticipates that i-traders observe the state of the DP when selecting their optimal strategies. Consider now i-traders: their profits for $\varphi_i(\pm 2, p_{Mid})$ depend on the actual state of the DP that in this case can be $DP_T = \{\pm 2, \pm 4\}$. We provide as an example profits for $\varphi_i(-2, p_{Mid})$:

$$\pi_T^e[\varphi_i(-2, p_{Mid})|\Omega_T^{[20,iv,+2]}] = \pi_T^e[\varphi_i(-2, p_{Mid})|\Omega_T^{[20,iv,+4]}] = 2(\tfrac{p_1^A+p_2^B}{2} - \beta_T v)$$
$$\pi_T^e[\varphi_i(-2, p_{Mid})|\Omega_T^{[20,iv,-2]}] = \pi_T^e[\varphi_i(-2, p_{Mid})|\Omega_T^{[20,iv,-4]}] = 0$$

So the i-trader faces no uncertainty and perfectly knows before submitting the order whether he will be executed or not. We omit the discussion on periods $T-1$ and $T-2$ as the intuition is similar. Profits at $T-2$ for both the u- and i-trader are presented in Figures A11 and A12-14 respectively.

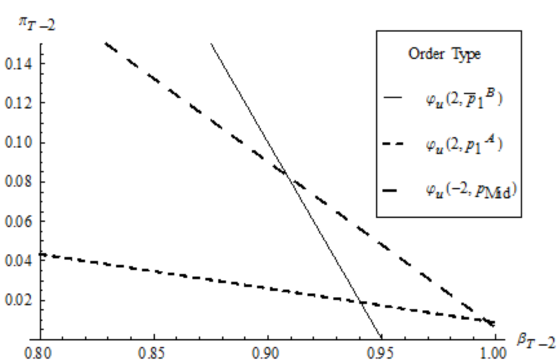

Figure A11. Uninformed LT – b_{T-2}=[22]

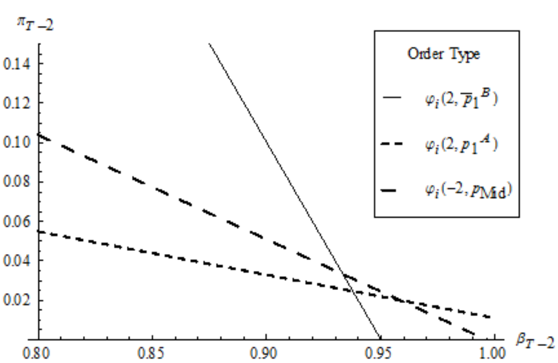

Figure A12. Informed LT – b_{T-2}=[22] - DP_{T-2}=[0]

[15] The case $b_T = [20]$, $vis_{T-2}\ vis_{T-1}$ is not interesting as no one plays on the DP.

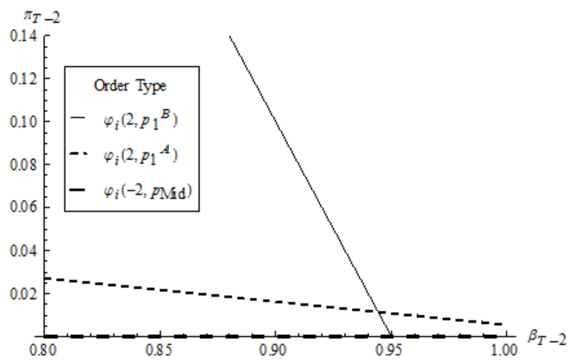

Figure A13. Informed LT – b_{T-2}=[22] - DP_{T-2}=[-6]

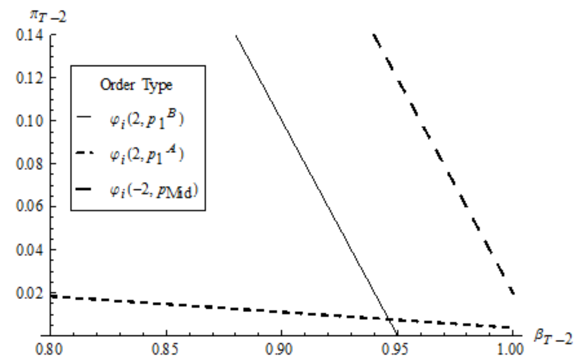

Figure A14. Informed LT – b_{T-2}=[22] - DP_{T-2}=[+6]

Results are derived by comparing numerical values for equilibrium strategies with those obtained in Proposition 1. Results for market quality and welfare are obtained by computing the corresponding numerical values using the formulas presented respectively in the proofs of Proposition 4 and 6, with the addition of i-traders.

References

Admati, A. and P. Pfleiderer, 1991. Sunshine trading and financial market equilibrium, Review of Financial Studies , 4, 443-481.

Barclay M. J., T. Hendershott, and D. T. McCormick, 2003. Competition Among Trading Venues: Information and Trading on Electronic Communication Networks, Journal of Finance, 58, 2637-2665.

Baruch, S., 2005. Who Benefits from and Open Limit-Order Book?, Journal of Business, 78, 4, 1267-1306.

Baruch, S., A. Karolyi, and M. Lemmon, 2007. Multi-Market Trading and Liquidity: Theory and Evidence, Journal of Finance, 62, 5, 2169-2200.

Bennett, P. and L. Wei, 2006. Market structure, fragmentation, and market quality, Journal of Financial Markets, 9, 1, 49-78.

Bessembinder, H. and H. Kaufman, 1997. A cross-exchange comparison of execution costs and information flow for NYSE-listed stocks, Journal of Financial Economics, 46, 3, 293-319.

Bessembinder, H., M. Panayides, and K. Venkataraman. 2009. Hidden Liquidity: An Analysis of Order Submission Strategies in Automated Markets. Journal of Financial Economics, 94, 3, 361-383.

Biais, B., P. Hillion and C. Spatt, 1999. Price Discovery and Learning during the Preopening Period in the Paris Bourse. Journal of Political Economy, 107, 6, 1218-1248.

Bloomfield, R. and M. O'Hara, 1999. Market Transparency: Who Wins and Who Loses?, The Review of Financial Studies, 12, 1, 5-35.

Bloomfield, R. and M. O'Hara, 2000. Can transparent markets survive?, The Journal of Financial Economics, 55, 425-459.

Boehmer, E. and B. Boehmer, 2003. Trading your neighbor's ETFs: Competition or fragmentation? Journal of Banking and Finance, 27, 1667-1703.

Boehmer E., G. Saar and L.Yu, 2005. Lifting the Veil: An Analysis of Pre-trade Transparency at the NYSE. Journal of Finance, 60, 2, 783-815.

Brunnermeier, M. K. and L. H. Pedersen, 2009. Market liquidity and funding liquidity, Review of Financial Studies, 22, 2201–2238.

Buti, S. and B. Rindi, 2008. Hidden Orders and Optimal Submission Strategies in a Dynamic Limit Order Market, Working Paper.

Buti, S., B. Rindi and I. M. Werner, 2010. Diving into Dark Pools. Fisher College of Business, Working Paper Series.

Conrad, J., Johnson, K.and S. Wahal, 2003. Institutional trading and alternative trading systems. Journal of Financial Economics 70, 99-134.

Degryse, H., M.Van Achter and G. Wuyts, 2009. Dynamic order submission strategies with competition between a dealer market and crossing network, Journal of Financial Economics, vol. 91, no. 3, 319-338.

De Winne, R. and C. D'Hondt, 2007. Hide-and-Seek in the Market: Placing and Detecting Hidden Orders, Review of Finance, 11, 663-692.
Donges, J. and F. Heinemann, 2004. Competition for Order Flow as a Coordination Game, working paper.
Dufour, A. and R. F. Engle, 2000. Time and the impact of a trade, Journal of Finance, 55, 2467–2498.
Easley, D., N. M. Kiefer and M. O'Hara, 1996. Cream-skimming or profit-sharing? The curious role of purchased order flow, Journal of Finance, 51, 3, 811-833.
Engle, R. F. and A. Patton, 2004. Impacts of Trades in an Error-Correction Model of Quote Prices, Journal of Financial Markets, 7, 1, 1-25.
Esser, A. and B. Mönch, 2007. The Navigation of an Iceberg: The Optimal Use of Hidden Orders, Finance Research Letters, 4, 68-81.
Fishman, M. and F. Longstaff, 1992. Dual Trading in Futures Markets, Journal of Finance, 47, 643-672.
Fong, K., Madhavan, A., and P. Swan, 2004. Upstairs, Downstairs: Does the Upstairs. Market Hurt the Downstairs? Working Paper, University of New South Wales.
Flood, M. D., R. Huisman, K. C.G. Koedijk and R. Mahieu, 1999. Quote disclosure and price discovery in multiple-dealer markets, Review of Financial Studies, 12, 2, 37-59.
Forster, M. and T. George, 1992. Anonymity in Securities Markets, Journal of Financial Intermediation, 2, 168-206.
Foster, D. P., S. Gervais and K.Ramaswamy, 2007. The Benefits of Volume-Conditional Order-Crossing, Working paper.
Foucault, T., S. Moinas and E. Theissen, 2007. Does Anonymity Matter in Electronic Limit Order Markets? Review of Financial Studies, 20, 5, 1707-1747.
Frey, S. and P. Sandas, 2008. The Impact of Hidden Liquidity in Limit Order Books, working paper, University of Tubingen.
Gabaix, X., P. Gopikrishnan, V. Plerou, and H. Stanley, 2006. Institutional investors and stock market volatility, Quarterly Journal of Economics, 121, 461–504.
Goldstein, M. A., A. V. Shkilko, B. F. Van Ness and R. A. Van Ness, 2008. Competition in the Market for NASDAQ Securities, Journal of Financial Markets, 11, 2, 113-143.
Gresse, C., 2006. The Effect of Crossing-Network Trading on Dealer Market's Bid-Ask Spreads, European Financial Management, 12, 2, 143-160.
Hasbrouck, J., 1991. Measuring the Information Content of Stock Trades, 46, Journal of Finance, 179-207.
Hasbrouck, J. and G. Saar, 2004. Limit Orders and Volatility in a Hybrid Market: The Island ECN, working paper, NYU.
Hendershott, T. and H. Mendelson, 2000. Crossing Networks and Dealer Markets: Competition and Performance, Journal of Finance, 55, 5, 2071-2115.
Hendershott, T. and A. J. Menkveld, 2010. Price Pressures, working paper.

Karolyi, A. G., 2006. The World of Cross-Listings and Cross-Listings of the World: Challenging Conventional Wisdom, Review of Finance, 10, 1, 73-115.
Kratz P., and T. Schoeneborn, 2009. Optimal liquidation in dark pools, working paper, Deutsche Bank Group.
Lee, C. M., 1993. Market Integration and Price Execution for NYSE-Listed Securities, Journal of Finance, 48, 3, 1009-1038.
Madhavan, A., 1995. Consolidation, Fragmentation, and the Disclosure of Trading Information, Review of Financial Studies, 8, 3, 579-603.
Moinas, Sophie, 2006. Hidden Limit Orders and Liquidity in Limit Order Markets, working paper, Toulouse Business School.
Naes, R., and B. A. Odegaard, 2006. Equity Trading by institutional Investors: To Cross or Not to Cross? Journal of Financial Markets, 9, 2, 79-99.
Naes, R., and J. Skjeltorp, 2003. Equity Trading by Institutional Investors: Evidence on Order Submission Strategies, Journal of Banking and Finance, 27, 1779-1817.
Nguyen, V., B. F. Van Ness, and R. A. Van Ness, 2007. Short- and Long-Term Effects of Multimarket Trading, Financial Review, 42, 3, 349-372.
Pagano, M., 1989. Trading volume and asset liquidity, Quarterly Journal of Economics, 104, 2, 255-274.
Pagano, M., and A. Röell, 1996. Transparency and Liquidity: A Comparison of Auction and Dealer Markets with Informed Trading, Journal of Finance, 51, 2, 579-611.
Pardo, A., and R. Pascual, 2006. On the Hidden Side of Liquidity, working paper.
Parlour, C. A., 1998. Price Dynamics in Limit Order Markets, Review of Financial Studies, 11, 789-816.
Parlour, C. A. and D. Seppi, 2008. Limit order markets: A survey, in A.W.A. Boot, and A.V. Thakor, ed.: Handbook of Financial Intermediation & Banking.
Ready, M., 2009. Determinants of Volume in Dark Pools, working paper, University of Wisconsin.
Reiss, P. C., and I. Werner, 2004. Anonymity, Adverse Selection, and the Sorting of Interdealer Trades, Review of Financial Studies, 17, 599-636.
Röell, A., 1991. Dual capacity trading and the quality of the market, Journal of Financial Intermediation, 1, 105-124.
Seppi, D., 1997. Liquidity Provision with Limit Orders and a Strategic Specialist, Review of Financial Studies, 10, 103-150.
Sirri, E. R., 2008. Speech by SEC Staff: Keynote Speech at the SIFMA 2008 Dark Pools Symposium.
Subrahmanyam, A., 1997. Multi-market trading and the informativeness of stock trades: An empirical intraday analysis. Journal of Economics and Business, 49, 6, 515-531.
Theissen, E., 2001. Trader Anonymity, Price Formation and Liquidity, European Finance Review, 7, 1-26.

Tuttle, L., 2006. Hidden orders, trading costs and information, Working Paper, American University of Sharjah.
Ye, M., 2009. A Glimpse into the Dark: Price Formation, Transaction Cost and Market Share of the Crossing Network.Working Paper, Cornell University.
Zhu, H., 2011. Do Dark Pools Harm Price Discovery? Working Paper.

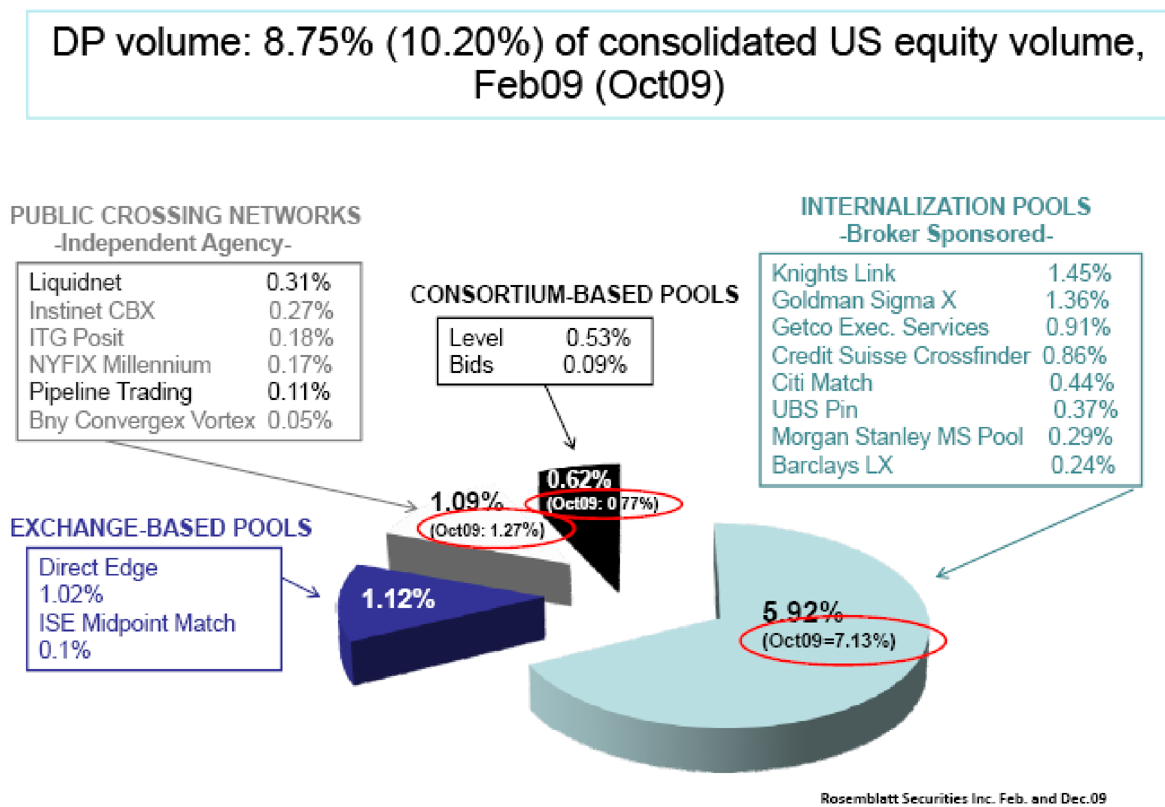

Figure 1 - Consolidated US equity volume, February-October 2009

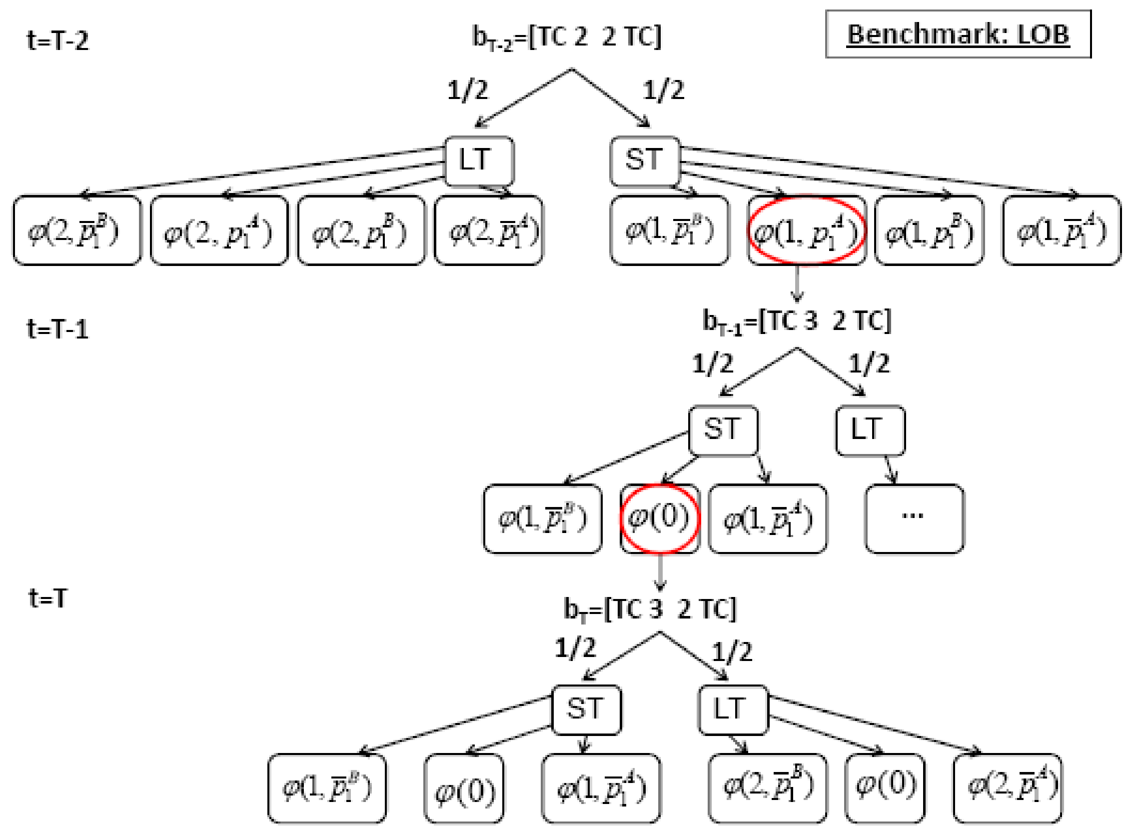

Figure 2 - Benchmark Model of Limit Order Book (LOB): Example for $j = 2$ of the Extensive Form of the Game. The trading crowd is indicated by TC, while small and large traders are named ST and LT respectively. We present only the equilibrium strategies.

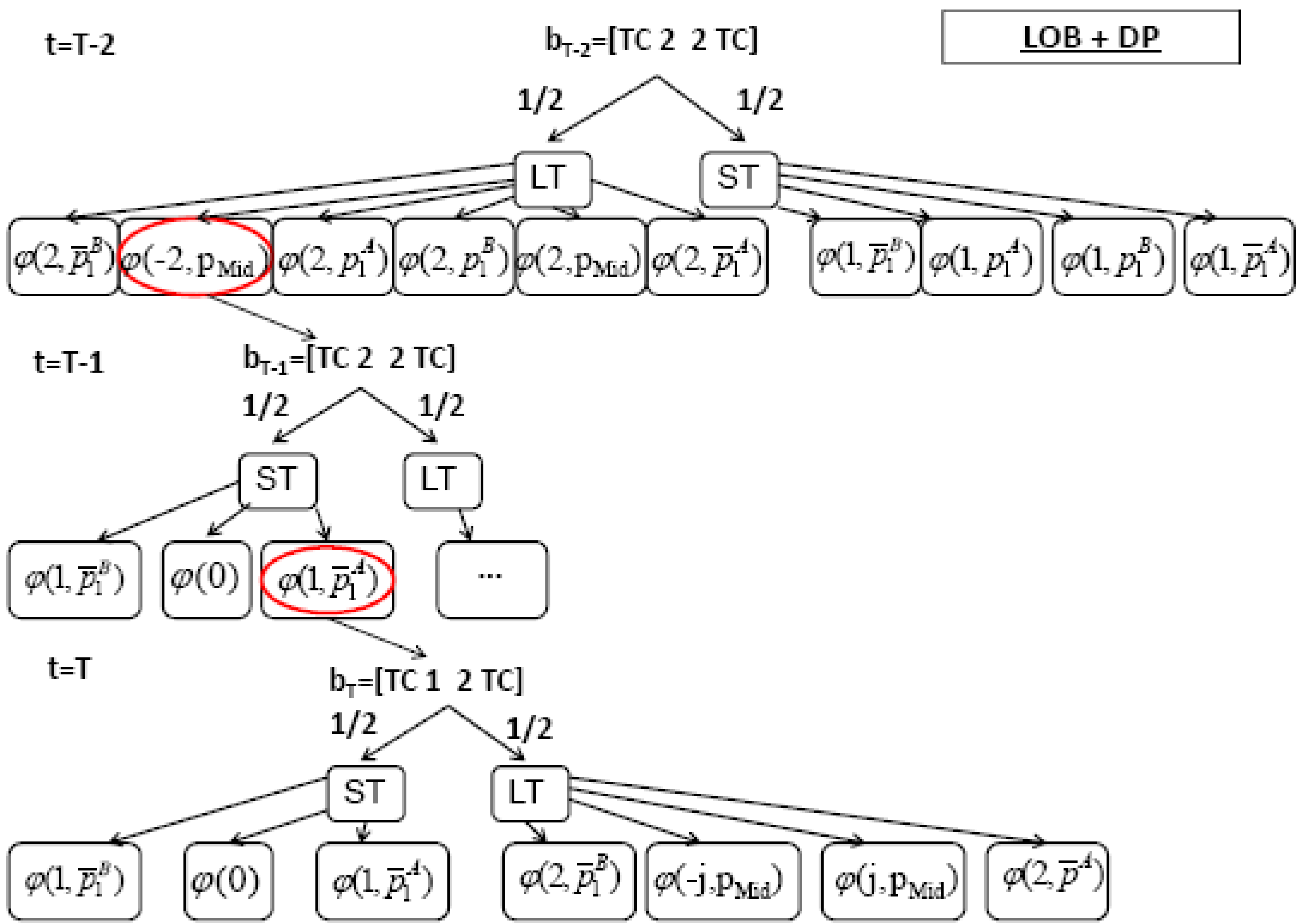

Figure 3 - Limit Order Book and Dark Pool Market (LOB&DP): Example of the Extensive Form of the Game for $j = 2$.

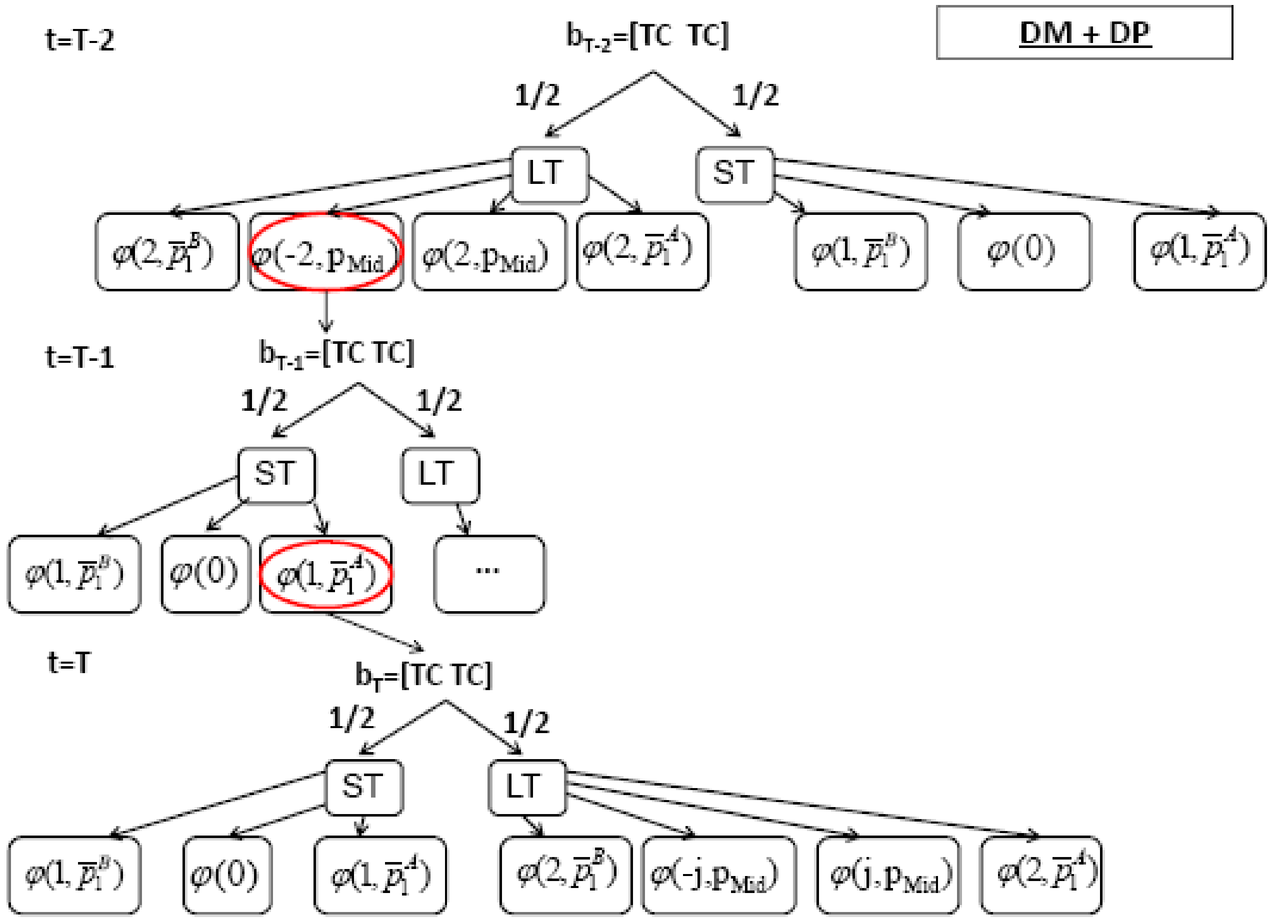

Figure 4 - Dealership Market with a Dark Pool (DM&DP): Example of the Extensive Form of the Game for $j = 2$.

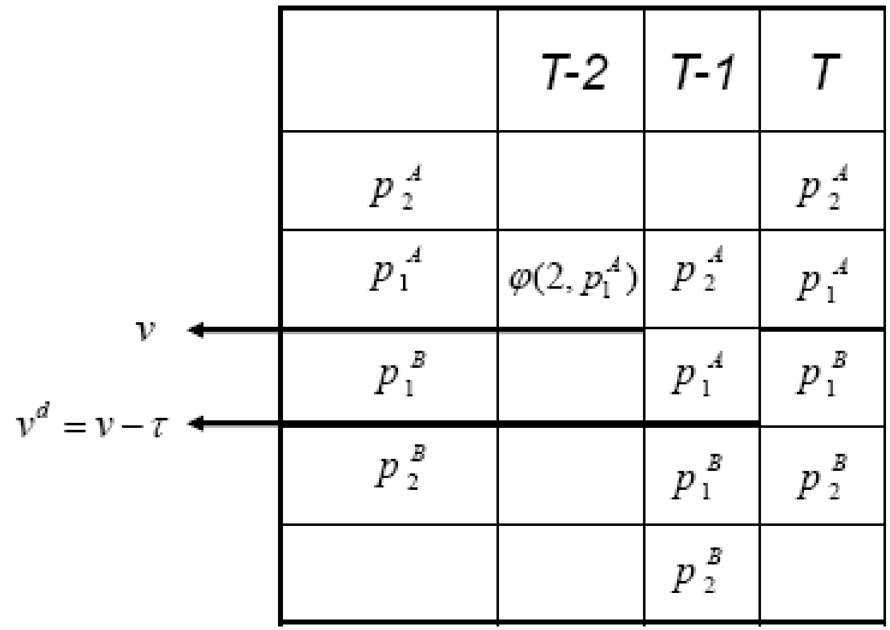

Figure 5 - This Figure reports the mechanics of the LOB when a limit order to sell 2 shares is submitted at $T-2$ that generates a temporary price pressure of 1 tick downward.

Table 1 - Order Submission Probabilities - $b_{T-2,T-1,T} = [22]$

	b_{T-2}= [22]			b_{T-1}= [22] vis_{T-2} (inv_{T-2})			b_T= [22] vis_{T-2} vis_{T-1} (inv_{T-2} vis_{T-1})		
Large Trader	*LOB*	*LOB&DP*	*DM&DP*	*LOB*	*LOB&DP*	*DM&DP*	*LOB*	*LOB&DP*	*DM&DP*
$\varphi(2, \overline{p}_2^B)$									
$\varphi(2, \overline{p}_1^B)$	.4686	.4612	.4617	.4750	.4621 (.4500)	.4621 (.4556)	.4750	.4625 (.4500)	.4625 (.4558)
$\varphi(2, p_1^A)$	.0314	.0109							
$\varphi(-j, p_{Mid})$		.0279	.0383		.0379 (.0500)	.0379 (.0444)		.0375 (.0500)	.0375 (.0442)
$\varphi(0)$				.0500			.0500		
$\varphi(j, p_{Mid})$		.0279	.0383		.0379 (.0500)	.0379 (.0444)		.0375 (.0500)	.0375 (.0442)
$\varphi(2, p_1^B)$	.0314	.0109							
$\varphi(2, \overline{p}_1^A)$	.4686	.4612	.4617	.4750	.4621 (.4500)	.4621 (.4556)	.4750	.4625 (.4500)	.4625 (.4558)
$\varphi(2, \overline{p}_2^A)$									

Table 1: Order Submission Probabilities - $b_{T-2,T-1,T} = [22]$. This Table reports large traders' submission probabilities for the orders listed in column 1 for the benchmark (LOB), for the LOB&DP and for the DM&DP model respectively. Execution probabilities are reported for the state of the book with 2 shares on both sides of the order book and for all periods T, $T-1$ and $T-2$. For example, large agents arriving at the market at $T-1$ after having observed a change in the order book at $T-2$ (vis_{T-2}), submit market orders to sell at B_1 with probability .4750 if they trade in the LOB, with probability .4621 if they trade in the LOB&DP and .4621 in the DM&DP protocol. If instead they do not observe any change in the order book at time $T-2$ (inv_{T-2}), the probabilities are .4500 for LOB&DP and .4556 for DM&DP.

Table 2 - Order Submission Probabilities - $b_{T-2,T-1,T} = [00]$

	$b_{T-2} = [00]$			$b_{T-1} = [00]$ vis_{T-2} (inv_{T-2})			$b_{T} = [00]$ vis_{T-2} vis_{T-1} $(inv_{T-2}$ $vis_{T-1})$		
Large Trader	*LOB*	*LOB&DP*	*DM&DP*	*LOB*	*LOB&DP*	*DM&DP*	*LOB*	*LOB&DP*	*DM&DP*
$\varphi(2, \overline{p}_2^B)$	.2857	.2912		.3697	.3726 (.3500)		.4250	.3875 (.3500)	
$\varphi(2, \overline{p}_1^B)$			.4617			.4621 (.4556)			.4625 (.4558)
$\varphi(2, p_1^A)$	.2143	.2088		.1303	.1274 (.0491)				
$\varphi(-2, p_{Mid})$			.0383		(.1009)	.0379 (.0444)		.1125 (.1500)	.0375 (.0442)
$\varphi(0)$							.1500		
$\varphi(2, p_{Mid})$			.0383		(.1009)	.0379 (.0444)		.1125 (.1500)	.0375 (.0442)
$\varphi(2, p_1^B)$	.2143	.2088		.1303	.1274 (.0491)				
$\varphi(2, \overline{p}_1^A)$			.4617			.4621 (.4556)			.4625 (.4558)
$\varphi(2, \overline{p}_2^A)$	.2857	.2912		.3697	.3726 (.3500)		.4250	.3875 (.3500)	

Table 2: Order Submission Probabilities - $b_{T-2,T-1,T} = [00]$. This Table reports large traders' submission probabilities for the orders listed in column 1 for the benchmark (LOB), for the LOB&DP and for the DM&DP model respectively. Execution probabilities are reported for the state of the book with no shares on both sides of the order book and for all periods T, $T-1$ and $T-2$**.** For example, large agents arriving at the market at T after having observed a change in the order book both at $T-1$ and at $T-2$ (vis_{T-2}, vis_{T-1}), submit orders to sell to the DP with probability .1125 in the LOB&DP market and .0375 in the DM&DP. If instead they observe a change in the order book only at time $T-1$ and no change at $T-2$ (inv_{T-2}, vis_{T-1}), the probabilities are .1500 and .0442 respectively.

Table 3 - Market Depth, Inside Spread: Time T-2

Panel A - Large Trader					
$Market\ [b_{T-2}]\ (tick\ size)$	$LOB\&DP$ [00]	$LOB\&DP$ [22]	$LOB\&DP$ [11] (.1)	$LOB\&DP$ [11] (.15)	$DM\&DP$
$\varphi(2,\overline{p}_2^B)$	.2912				
$\varphi(2,\overline{p}_1^B)$		.4612			.4617
$\varphi(2,\overline{p}^B)$			.4168	.3785	
$\varphi(2,p_1^A)$	.2088	.0109	.0832	.1158	
$\varphi(-2,p_{Mid})$		.0279		.0057	.0383
$\varphi(2,p_{Mid})$		.0279		.0057	.0383
$\varphi(2,p_1^B)$	.2088	.0109	.0832	.1158	
$\varphi(2,\overline{p}^A)$			.4168	.3785	
$\varphi(2,\overline{p}_1^A)$		.4612			.4617
$\varphi(2,\overline{p}_2^A)$	.2912				
Panel B - Small Trader					
$\varphi(1,\overline{p}_2^B)$	.2142				
$\varphi(1,\overline{p}_1^B)$		.4656	.4333	.4054	.4750
$\varphi(1,p_1^A)$	.2858	.0344	.0667	.0946	
$\varphi(0)$					.0500
$\varphi(1,p_1^B)$	.2858	.0344	.0667	.0946	
$\varphi(1,\overline{p}_1^A)$		.4656	.4333	.4054	.4750
$\varphi(1,\overline{p}_2^A)$	.2142				

Table 3: Market Depth and Inside Spread - Time T-2. This Table reports large traders' (Panel A) submission probabilities to the LOB&DP at time $T-2$ for the orders listed in column 1; columns 2 to 5 report results for different states of the book and column 6 gives the equilibrium order submission probabilities for the DM&DP protocol. Panel B reports the order submission probabilities for small traders. The order submission probabilities reported in this Table allow to compare different values of market depth and inside spread.

Table 4 - Large Trader - Time T: $vis_{T-2}\ vis_{T-1}\ (inv_{T-2}\ vis_{T-1})$					
Order Submission Probabilities to compute Market Depth					
$Market\ [b_T]$	$LOB\&DP$ [22]	$LOB\&DP$ [12]	$LOB\&DP$ [21]	$LOB\&DP$ [11]	$DM\&DP$
$\varphi(2, \overline{p}_2^B)$					
$\varphi(2, \overline{p}_1^B)$	.4625 (.4500)	.4625 (.4500)			.4625 (.4558)
$\varphi(2, \overline{p}^B)$			.4250 (.4000)	.4250	
$\varphi(-2, p_{Mid})$	.0375 (.0500)	.0375 (.0500)	.0750 (.1000)	.0750	.0375 (.0442)
$\varphi(2, p_{Mid})$	.0375 (.0500)	.0750 (.1000)	.0375 (.0500)	.0750	.0375 (.0442)
$\varphi(2, \overline{p}^A)$		.4250 (.4000)		.4250	
$\varphi(2, \overline{p}_1^A)$	.4625 (.4500)		.4625 (.4500)		.4625 (.4558)
$\varphi(2, \overline{p}_2^A)$					
Order Submission Probabilities to compute Inside Spread					
$Market\ [b_T]$	$LOB\&DP$ [00]	$LOB\&DP$ [20]	$LOB\&DP$ [02]	$LOB\&DP$ [22]	$DM\&DP$
$\varphi(2, \overline{p}_2^B)$	.3875 (.3500)	.4000 (.3750)			
$\varphi(2, \overline{p}_1^B)$			.4500 (.4250)	.4625 (.4500)	.4625 (.4558)
$\varphi(-2, p_{Mid})$	.1125 (.1500)	.0750 (.1000)	.0750 (.1000)	.0375 (.0490)	.0375 (.0442)
$\varphi(2, p_{Mid})$	.1125 (.1500)	.0750 (.1000)	.0750 (.1000)	.0375 (.0490)	.0375 (.0442)
$\varphi(2, \overline{p}_1^A)$		.4500 (.4250)		.4625 (.4500)	.4625 (.4558)
$\varphi(2, \overline{p}_2^A)$	.3875 (.3500)		.4000 (.3750)		

Table 4: Estimated Depth and Inside Spread -Time T. This Table reports large traders' order submission probabilities to the LOB&DP market at time T for the orders listed in column 1; columns 2 to 5 report results for different states of the book and column 6 gives the equilibrium order submission probabilities for the DM&DP protocol. Values in parenthesis refer to the case where traders only observe a change in the order book at $T-1$, whereas they do not observe any variation at $T-2$ (inv_{T-2}, vis_{T-1}). The Table allows for comparisons among different levels of market depth and different inside spreads.

Table 5 - Price Impact and Price Pressure: $b_{T-2} = 22$				
Large Trader				
$Market \; [b_{T-2}]$	LOB	$LOB\&PP$	$LOB\&DP$	$LOB\&DP\&PP$
$\varphi(2, \overline{p}_1^B)$	.4686	.4704	.4612	.4612
$\varphi(1, p_1^A)$		.0296		
$\varphi(2, p_1^A)$	.0314		.0109	0072
$\varphi(-j, p_{Mid})$			.0279	.0316
$\varphi(j, p_{Mid})$			.0279	.0316
$\varphi(2, p_1^B)$	.0314		.0109	.0072
$\varphi(1, p_1^B)$		.0296		
$\varphi(2, \overline{p}_1^A)$	.4686	.4704	.4612	.4612

Table 5: Price Impact and Price Pressure - $b_{T-2} = [22]$. This Table reports large traders' submission probabilities for the orders listed in column 1 for the benchmark (LOB) and for the LOB&DP model, both with and without price pressure.

Table 6 - Market Quality

Book	**Estimated Spread**			**Estimated Depth**			**Estimated Volume**				
$b_{T-2} = [00]$	LOB	LOB&DP	% Δ	LOB	LOB&DP	% Δ	LOB	LOB&DP	% Δ	DP	LOB&DP+DP
Average	.2651	.2664	.0052	.7483	.7179	$-.0406$	1.0736	1.0665	$-.0066$		
Total				1.4966	1.4358	$-.0406$	3.2208	3.1996	$-.0066$	.0804	3.2800
T	.2469	.2488	.0077	.7650	.7325	$-.0425$	1.3074	1.2386	$-.0526$		
T-1	.2483	.2505	.0089	.7316	.7033	$-.0387$	1.1450	1.1643	.0169		
T-2	.3000	.3000					.7684	.7967	.0368		
$b_{T-2} = [11]$											
Average	.1686	.1689	.0020	1.2282	1.2050	$-.0189$	1.2788	1.2520	$-.0209$		
Total				2.4563	2.4099	$-.0189$	3.8363	3.7561	$-.0209$	.0925	3.8486
T	.2212	.2218	.0027	1.0624	1.0269	$-.0266$	1.3201	1.2525	$-.0512$		
T-1	.1846	.1850	.0022	1.3939	1.3830	$-.0054$	1.2556	1.2367	$-.0151$		
T-2	.1000	.1000					1.2606	1.2669	.0050		
$b_{T-2} = [22]$											
Average	.1449	.1443	$-.0041$	2.2642	2.2349	$-.0130$	1.3658	1.3331	$-.0239$		
Total				4.5284	4.4697	$-.0130$	4.0974	3.9993	$-.0239$	.1043	4.1036
T	.1878	.1868	$-.0053$	1.8326	1.8016	$-.0169$	1.3474	1.2917	$-.0413$		
T-1	.1469	.1461	$-.0054$	2.6958	2.6681	$-.0103$	1.3479	1.3196	$-.0210$		
T-2	.1000	.1000					1.4021	1.3880	$-.0101$		

Table 6: Market Quality. This Table compares estimated inside spread (columns 2 to 4), estimated depth at the BBO (columns 5 to 7) and estimated volumes (columns 8 to 12) for the LOB and the LOB&DP protocol, and reports the percentage variations ($\% \Delta = \frac{LOB\&DP-LOB}{LOB}$) across different books and periods.

Table 7 - Systematic Pattern in Order Flows

	$b_{T-2}=[20]$			$b_{T-2}=[10]$			$\%\ \Delta: \frac{(b_{T-2}=[10])-(b_{T-2}=[20])}{b_{T-2}=[20]}$	
Large Trader	LOB	LOB&DP	$\%\ \Delta: \frac{LOB\&DP-LOB}{LOB}$	LOB	LOB&DP	$\%\ \Delta: \frac{LOB\&DP-LOB}{LOB}$	LOB	LOB&DP
$\varphi(2,\overline{p}_2^B)$	.4132	.3878	$-.0615$	.3805	.3824	.0050	$-.0791$	$-.0139$
$\varphi(2,p_1^A)$	.0692	.0118	$-.8295$	.1116	.1096	$-.0179$	.6127	8.2881
$\varphi(-2,p_{Mid})$		.0827						
$\varphi(2,p_1^B)$	.1174	.1139	$-.0298$	.1679	.1625	$-.0322$	.4302	.4267
$\varphi(2,\overline{p}^A)$				.3400	.3455	.0162		$-.1444$
$\varphi(2,\overline{p}_1^A)$	.4001	.4038	.0092				$-.1502$	
Small Trader								
$\varphi(1,\overline{p}_2^B)$	.4063	.4076	.0032	.3401	.3447	.0135	$-.1629$	$-.1543$
$\varphi(1,p_1^A)$	.0776	.0762	$-.0180$	.1544	.1500	$-.0285$	.9897	.9685
$\varphi(1,p_1^B)$	.1725	.1562	$-.0899$	.1566	.1415	$-.0964$	$-.0922$	$-.0987$
$\varphi(1,\overline{p}_1^A)$	.3436	.3601	.0454	.3488	.3638	.0430	.0151	.0128
Average Probability								
$\frac{1}{2}\sum_{j=1,2}\varphi(j,\overline{p}_2^B)$	.4098	.3977	$-.0294$	.3603	.3636	.0090	$-.1207$	$-.0859$
$\frac{1}{2}\sum_{j=1,2}\varphi(j,p_1^A)$	.0734	.0440	$-.4005$	.1330	.1298	$-.0241$	.8120	1.9500
$\frac{1}{2}\varphi(-2,p_{Mid})$		.0414						
$\frac{1}{2}\sum_{j=1,2}\varphi(j,p_1^B)$	.1450	.1355	$-.0655$	.1623	.152	$-.0632$	.1193	.1222
$\frac{1}{2}\sum_{j=1,2}\varphi(j,\overline{p}_1^A)$	.3719	.3815	.0260	.3444	.3547	.0298	$-.0739$	$-.0702$

Table 7: Systematic Pattern in Order Flows. This Table reports the submission probabilities of the orders listed in column 1 for both large and small traders, as well as the average probability of market orders, limit orders and DP orders. Results are reported for the two states of the book $b_{T-2}=[20]$ and $b_{T-2}=[10]$. For each state values are shown for both the LOB and the LOB&DP, and for the difference of the two ($\%\Delta = \frac{LOB\&DP-LOB}{LOB}$). Columns 8 and 9 show how the submission probabilities of each order change moving from $b_{T-2}=[20]$ to $b_{T-2}=[10]$, both for the LOB and the LOB&DP protocol ($\Delta = \frac{(b_{T-2}=[10])-(b_{T-2}=[20])}{b_{T-2}=[20]}$).

Table 8 - Welfare

	$b_{T-2} = [00]$			$b_{T-2} = [11]$			$b_{T-2} = [22]$		
	LOB	LOB&DP	$\% \Delta : \frac{LOB\&DP - LOB}{LOB * 100}$	LOB	LOB&DP	$\% \Delta : \frac{LOB\&DP - LOB}{LOB * 100}$	LOB	LOB&DP	$\% \Delta : \frac{LOB\&DP - LOB}{LOB * 100}$
Total	1.7443	1.7435	−.0430	1.8224	1.8255	.1729	1.9213	1.9258	.2342
T-2	.5914	.5886	−.4735	.6374	.6371	−.0549	.6778	.6780	.0295
T-1	.5818	.5806	−.2063	.6026	.6030	.0747	.6364	.6378	.2279
T	.5711	.5744	.5691	.5824	.5855	.5237	.6071	.6100	.4694
Institutional	2.3023	2.3071	.2085	2.3755	2.3827	.3031	2.547	2.5555	.3337
T-2	.7768	.7746	−.2832	.8197	.8192	−.0610	.9035	.9040	.0553
T-1	.7685	.7681	−.0520	.7876	.7890	.1778	.8411	.8438	.3210
T	.7570	.7644	.9775	.7682	.7745	.8201	.8024	.8077	.6605
Retail Traders	1.1862	1.1799	−.5311	1.2692	1.2683	−.0709	1.2955	1.296	.0386
T-2	.4060	.4026	−.8374	.4551	.4549	−.0439	.4521	.4520	−.0221
T-1	.3950	.3930	−.5063	.4175	.4170	−.1198	.4316	.4318	.0463
T	.3852	.3843	−.2336	.3966	.3964	−.0504	.4118	.4122	.0971

Table 8: Welfare. This Table reports the results for total welfare as well as institutional and retail traders welfare across different periods and states of the order book. For each period and each state of the book results are reported for both the LOB and the LOB&DP, together with the difference between the two, ($\% \Delta = \frac{LOB\&DP - LOB}{LOB}$).

Table 9 - LOB&DP&IOI - Asymmetric Information on the state of the DP - Large Traders

Panel A.1	b_{T-2}= [22]						
	LOB	*LOB&DP*	*LOB&DP&IOI*				
Large			Uninf	Inf	Inf	Inf	Weighted Average
Trader				(DP=0)	(DP=+6)	(DP=-6)	
$\varphi(2,\overline{p}_1^B)$	.4686	.4612	.4542	.4667		.4721	.3836
$\varphi(2,p_1^A)$	.0314	.0109	.0021	.0210		.0245	.0133
$\varphi(-2,p_{Mid})$		.0279	.0437	.0123	.5034		.1079
$\varphi(0)$							
$\varphi(2,p_{Mid})$		.0279	.0437	.0123		.5034	.1079
$\varphi(2,p_1^B)$	.0314	.0109	.0021	.0210	.0245		.0133
$\varphi(2,\overline{p}_1^A)$	.4686	.4612	.4542	.4667	.4721		.3836
Panel A.2	$b_{\ T-1}$= [22] inv_{T-2}						
	LOB	*LOB&DP*	*LOB&DP&IOI*				
Large			Uninf	Inf (DP=	Inf (DP=	Inf (DP=	Weighted Average
Trader				+2,(-2))	+8 or +4)	-8 or −4)	
$\varphi(2,\overline{p}_1^B)$	.4750	.4500	.4480	(.4750)		.4750	.3824
$\varphi(2,p_1^A)$							
$\varphi(-2,p_{Mid})$		.0500	.0520	.5000	.0503		.1931
$\varphi(0)$	.0500			.0250 (.0250)	.0220	.0220	.0199
$\varphi(2,p_{Mid})$		.0500	.0520	(.5000)		.0503	.1931
$\varphi(2,p_1^B)$							
$\varphi(2,\overline{p}_1^A)$	.4750	.4500	.4480	.4750	.4750		.3824

Table 9: LOB&DP&IOI - Asymmetric Information on the state of the DP. This Table reports in Panel A large traders' submission probabilities for the orders listed in column 1 for three protocols, LOB, LOB&DP and LOB&DP&IOI. For the latter, i.e. the market with IOI messages, the Table also reports results for both uninformed and informed large traders; for the informed traders results are reported for the three states of the DP, namely, empty and full on the sell and on the buy side. Column 8 reports the average submission probability of each order type computed as the weighted average of the order submission probabilities of both uninformed and informed traders. For example, the weighted average of the submission probability of DP orders to sell is equal to: $\frac{1}{2}.0437 + \frac{1}{2}[\frac{1}{3}.0123 + \frac{1}{3}.5034]$. In Panel B small traders' submission probabilities are reported for the LOB&DP and LOB&DP&IOI protocols.

Table 9 - LOB&DP&IOI - Asymmetric Information on the state of the DP - Small Trader

Panel B	$b_{T-2} = [22]$		$b_{\ T-1} = [22] \quad (inv_{T-2})$	
Small Trader	$LOB\&DP$	$LOB\&DP\&IOI$	$LOB\&DP$	$LOB\&DP\&IOI$
$\varphi(1, \overline{p}_1^B)$	.4656	.4675	(.4750)	(.4750)
$\varphi(1, p_1^A)$	.0344	.0325		
$\varphi(0)$			(.0500)	(.0500)
$\varphi(1, p_1^B)$	.0344	.0325		
$\varphi(1, \overline{p}_1^A)$	.4656	.4675	(.4750)	(.4750)

Table 10 - Asymmetric Information and Market Quality					
Book	**Estimated Spread**				
$b_{T-2} = [22]$	LOB	LOB&DP	LOB&DP &IOI	% Δ: LOB vs LOB&DP	% Δ: LOB&DP vs LOB&DP&IOI
Average	.1449	.1443	.1380	−.0041	−.0434
Total					
T	.1878	.1868	.1757	−.0053	−.0594
T-1	.1469	.1461	.1384	−.0054	−.0527
T-2	.1000	.1000	.1000		
	Estimated Depth				
Average	2.2642	2.2349	2.4012	−.0130	.0744
Total	4.5284	4.4697	4.8023	−.0130	.0744
T	1.8326	1.8016	1.9872	−.0169	.1030
T-1	2.6958	2.6681	2.8151	−.0103	.0551
T-2					
	Estimated Volume				
Average	1.3658	1.3331	1.1963	−.0239	−.1026
Total	4.0974	3.9993	3.5890	−.0239	−.1026
T	1.3474	1.2917	1.1695	−.0413	−.0946
T-1	1.3479	1.3196	1.1849	−.0210	−.1021
T-2	1.4021	1.3880	1.2346	−.0101	−.1105
	DP	LOB&DP+DP	DP&IOI	LOB&DP&IOI+DP	
Total	.1043	4.1036	.5812	4.1549	

Table 10: Asymmetric Information and Market Quality. This Table compares estimated inside spread, estimated depth at the BBO and estimated volumes for the three protocols, LOB, LOB&DP and LOB&DP&IOI. Column 5 reports the comparison of the three indicators of market quality between the LOB and the LOB&DP markets ($\% \, \Delta = \frac{LOB\&DP - LOB}{LOB}$), whereas column 6 reports results for the comparison between LOB&DP and LOB&DP&IOI ($\%\Delta = \frac{LOB\&DP\&IOI - LOB\&DP}{LOB\&DP}$). Results are reported for $b_{T-2} = [22]$ and across different periods. The last row reports total volumes for each market protocol.

Table 11 - Asymmetric Information on the state of the DP - Welfare

$b_{T-2} = [22]$

	LOB	LOB&DP	LOB&DP&IOI	% Δ: LOB vs LOB&DP	% Δ: LOB&DP vs LOB&DP&IOI
Total	1.9213	1.9258	1.9686	.2342%	2.2225%
T-2	.6778	.6780	.6878	.4694%	1.4454%
T-1	.6364	.6378	.6488	.2279%	1.7247%
T	.6071	.6100	.6320	.0295%	3.6066%
Retail Traders	1.2955	1.296	1.3039	.0386%	.6096%
T-2	.4521	.4520	.4518	−.0221%	−.0442%
T-1	.4316	.4318	.4350	.0463%	.7411%
T	.4118	.4122	.4171	.0971%	1.1887%
Institutional (Uninformed)	2.547	2.5555	2.5751	.3337%	.7670%
T-2	.9035	.9040	.9055	.0553%	.1659%
T-1	.8411	.8438	.8538	.3210%	1.1851%
T	.8024	.8077	.8158	.6605%	1.0028%
Institutional (Informed)			2.6913		
T-2			.9421		
T-1			.8714		
T			.8778		

Table 11: Asymmetric Information on the state of the DP - Welfare. This Table reports results for total welfare as well as institutional and retail traders welfare across different periods and market protocols. Results are also reported for both uninformed and informed institutional traders. For each period results are given for the LOB, the LOB&DP, and the LOB&DP%IOI together with the difference between the LOB and the LOB&DP (column 5: $\%\Delta = \frac{LOB\&DP - LOB}{LOB}$), and the LOB&DP and the LOB&DP&IOI (column 6: $\%\Delta = \frac{LOB\&DP\&IOI - LOB\&DP}{LOB\&DP}$).