Tick Size Regulation and Sub-Penny Trading*

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Abstract

We show that the effects of a smaller tick size in a public limit order book (PLB) crucially depend on the liquidity of the stock: market quality and welfare fall for illiquid but improve for liquid stocks. When we add competition from an internalization pool (IP) that works like a limit order book characterized by a finer price grid, market quality deteriorates for illiquid, low priced stocks, while it improves for liquid stocks. Because all traders can demand liquidity on the IP, total welfare increases in all cases. The effects are less pronounced when the PLB has itself a smaller tick size.

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1 Introduction

The tick size, the minimum size of an asset price variation, is one of the most relevant factors affecting the level of liquidity of financial securities traded on public limit order books (PLBs). As a consequence, it has been at the top of the regulatory agenda over the past decade (e.g., SEC 2010 and SEC 2012). Decimalization, i.e., the transition to trading and quoting securities in one penny increments, started in 2001 in the US and in 2004 in Europe and had profound consequences both for the level of liquidity, and for the business model of those institutions which support liquidity. The anticipated and unanticipated effects of tick-size variations on the quality of the markets and on the welfare of market participants have been debated extensively.[1] The rationale for tick-size reductions was to encourage trading activity by reducing transaction costs; the unanticipated consequence was that the increased competition for the provision of liquidity reduced the incentive for market participants to supply liquidity.

The comparative assessment of markets with different tick sizes is even more complex today, because the documented outcome of changes in the tick size must be appraised within a global framework in which lit markets compete with dark markets for liquidity. In the latter, more that 17% of consolidated equity volumes (SEC, 2010) are executed through broker-dealers who have access to venues that allow them to undercut the existing liquidity on regulated markets by placing quotes in sub-penny increments, i.e., at fractions of the minimum tick size.

Sub-penny trading is controversial, and the model we present in this paper aims at providing regulators and trading venues with a set of guidelines for selecting the optimal tick size. Our model also informs empirical research on how to evaluate the effects of tick

[1] See Section 2 and 3.

size changes. We do so in an environment where a market that works like a double auction trading platform competes for the provision of liquidity with a dark trading platform where liquidity providers can trade in sub-pennies.

A relevant trade-off to consider when selecting the tick size is between undercutting and liquidity provision. On the one hand, the smaller the tick size, the cheaper is the undercutting for all market participants, thus reducing the profits made by broker-dealers from submitting sub-penny passive limit orders. Hence, a reduction in the tick size is effectively a way to alleviate the effects of sub-penny trading. On the other hand, a smaller tick size discourages liquidity providers from posting limit orders at the top of the book, because of the increased danger of being undercut. Therefore, the critical regulatory issue is to adjust the tick size for each stock to optimally balance the effect it has on sub-penny trading with the effect it has on liquidity provision, which in turn crucially depends on both the initial level of liquidity and the tick-to-price ratio (Goldstein and Kavajecz, 2000).

Starting with a one-market model, we show that the effects of having a different tick size depend both on the liquidity and on the price of the stock. For liquid stocks a smaller tick size increases competition among liquidity suppliers and hence improves both market quality and traders' welfare. When the top of the book is very deep, however, the liquidity pressure at the best bid-offer is so intense that inside depth decreases with smaller tick sizes. For illiquid stocks, instead, a smaller tick size discourages liquidity provision and worsens both market quality and traders' welfare. The effects of introducing a smaller tick size are more significant when the value of the tick size is relatively large compared to the stock price (i.e., for low priced stocks). Based on these findings, we suggest that when setting the minimum price improvement, regulators and market operators should consider both the asset price level and the liquidity of the stock.

We extend the framework to a dual-market model in which a group of broker-dealers can execute customers' orders by choosing between trading on a PLB or on an internalization pool (IP). IPs are dark venues that work like opaque limit order books but have a finer price grid than regulated markets. Regular traders access the liquidity posted on the IP through smart order routers which seek the best execution available on the two markets. This extension allows us to investigate the consequences of sub-penny trading in a setting where the main PLB trades in penny increments. This setup captures the essence of today's stock trading environment where market venues such as NYSE-Euronext and Nasdaq-OMX compete with internalizers and dark pools.

Our results show that competition from the IP reduces the number of both limit and market orders sent to the PLB. The reduction of limit orders causes a reduction in the provision of liquidity on the PLB and hence has a detrimental effect on both market depth and the inside spread. Conversely, the reduction of market orders leads to a decrease in the demand for liquidity. This preserves depth on the PLB, generating a positive effect on liquidity. However, it also reduces the execution probability of limit orders, reducing the incentive traders have to supply liquidity. Our model shows that the effect on the quality of the PLB is dictated by the net result of these two forces, which in turn depends on the initial state of the PLB.

We show that for illiquid stocks the existence of an IP is detrimental for the level of liquidity on the PLB: the provision of liquidity in the PLB decreases to such a degree that both market depth and the inside spread worsen. On the other hand, when competition for the provision of liquidity on the PLB is high, because the book is liquid, the positive effect of the reduced liquidity demand on the PLB dominates. The introduction of the IP induces broker-dealers to trade intensively on the IP: aggressive market orders are intercepted by the

IP away from the PLB, thus preserving liquidity on the PLB. All these effects are stronger for low priced stocks, as they are driven by the tick to price ratio rather than by the absolute value of the tick size.

Further, our results show that even though competition from a dark market with a finer price grid has mixed effects on the quality of the lit market, it makes all traders better off, be they broker-dealers who supply liquidity, or regular traders who can access this extra liquidity via smart order routers. The results also show that gains from trade are higher for low priced stocks, which explains why broker-dealers are more active in these stocks. A word of caution is warranted, though. Regular traders' welfare increases with sub-penny trading provided that smart order routers allow them to benefit from the liquidity posted on the IP. Hence, unsophisticated retail traders, who are likely to be unable to take advantage of this optional liquidity, could be harmed when the quality of the PLB deteriorates, as we observe for illiquid stocks.

Our model contributes to the empirical literature on the impact of tick size changes by providing several new empirical predictions. By selecting a sample of stocks to account for variation in liquidity and price, this new framework could be exploited to shed light on the relationship between tick size changes, market quality, and traders' welfare, when regular exchanges compete with dark markets. In addition, the results from our model show that it is necessary to study the dynamics of order flows to understand how PLBs and IPs interact by affecting both the provision of and the demand for liquidity. Finally, a more in-depth analysis of how order submission strategies change following a tick size reduction in markets characterized by competition from dark pools should help investigating the effect of sub-penny trading on liquidity and traders' welfare.

This paper is related to three strands of the existing theoretical literature, that is to intermarket competition,[2] to the optimal tick size,[3] and to the internalization of order flows by broker-dealers (Battalio and Holden, 2001). To the best of our knowledge, it is the first model that allows researchers to investigate the tick size rule within a framework that takes into account both the asset value and the liquidity of the stock. It also departs from the existing theoretical works as it embeds sub-penny trading through modelling an IP.

The remaining part of this paper is structured as follows: in Section 2, we discuss the regulatory debate on tick size, in Section 3 we overview the related literature. In Section 4 and 5, we focus on the single market model, whereas Section 6 contains the model with an IP. In Section 7, we discuss the effects of tick size changes on traders' welfare. We present the empirical implications in Section 8, and we draw policy conclusions in Section 9. All the proofs appear in the Appendix.

2 Regulatory Debate

As a vast body of empirical literature has shown,[4] when the tick size is reduced spread decreases but depth at the top of the book deteriorates. For this reason, regulators are concerned by those trading strategies that exploit the possibility to submit orders at fractions of the minimum tick size. In 2005 the Securities and Exchange Commission (SEC) introduced the Sub-Penny Rule [adopted Rule 612 under Regulation National Market System (NMS)]. The rule was aimed at protecting displayed limit orders from being undercut by trivial amounts. It prohibits market participants from displaying, ranking, or accepting quotations in NMS stocks that are priced at smaller increments than the allowed minimum price

[2] See, for example, Chowdhry and Nanda (1991), Parlour and Seppi (2003), and Foucault and Menkveld (2008).

[3] See Anshuman and Kalay (1998), Cordella and Foucault (1999), Foucault et al. (2005), Goettler et al. (2005), Kadan (2006), and Seppi (1997).

[4] See Section 3.

variation.

In the years following the introduction of Rule 612, however, the development of dark markets deeply affected intermarket competition, and made the rule ineffective in protecting displayed limit orders. In particular, two features of the rule paved the way for sub-penny trading. First, Rule 612 prohibits market participants from quoting prices in sub-penny, but in the belief that sub-penny trading would not be as detrimental as sub-penny quoting, it expressly allows broker-dealers to provide price improvement to a customer order that resulted in a sub-penny execution, thus allowing sub-penny trading. Second, the Rule 612 prohibition of sub-penny quoting does not apply to dark markets; this means that broker-dealers can exploit IPs to jump the queue by a fraction of a penny and so preempt the National Best Bid Offer (NBBO).

[**Insert Table 1 here**]

Another important factor that facilitates sub-penny trading is the growing importance of fast trading facilities. Using algorithmic programs to generate replications of trading strategies, broker-dealers trading large volumes can make significant profits even though they sacrifice a fraction of a penny in order to step ahead of the PLB.[5] As a result, volumes traded on IPs have steadily increased over time as shown in Table 1. In August 2010 they executed 8.55% of the consolidated US equity volume -the rest being executed in public crossing networks, exchange and consortium-based pools- which is an increase of 25% over the previous year. Furthermore, the proportion of sub-penny trading (queue jumping) has dramatically increased over the past 10 years as shown in Figure 1.

[**Insert Figure 1 here**]

[5] Jarnecic and Snape (2010) suggest that high frequency trading is negatively related to the tick size.

The SEC (2010) has recently proposed a potential solution to the sub-penny issue in the form of the Trade-At Rule that would practically ban sub-penny trading by prohibiting "any trading center from executing a trade at the price of the NBBO unless the trading center was displaying that price at the time it received the incoming contra-side order." By contrast, BATS (2009) proposed to reduce the minimum price increment of publicly displayed market centers to sub-pennies, in order to level the playing field.

More recently, in April 2012, the US Congress passed the Jumpstart Our Business Startup (JOBS) Act which instructed the SEC to study the impact of decimalization on liquidity for small and medium capitalization companies. According to the JOBS Act, if needed, the SEC is allowed to increase the minimum trading increment of emerging growth companies. However, the conclusions of the SEC Report to Congress on Decimalization (2012) discouraged the Commission from proceeding with the rulemaking required to increase the tick size. Our model shows that an increase in the tick size for smaller stocks may be ineffective in today's trading environment. We explain why in Section 9 below.

3 Literature Review

There is extensive research on the relationship between the reduction of the tick size and market quality. Empirical studies from various markets around the world have found that a tick size reduction is associated with a decline in both the spread and depth, and that the spread is not equally affected across stocks.[6] These findings are confirmed by a more recent pilot program implemented by the major European platforms aimed at investigating

[6]See Ahn et al. (1996 and 2007), Bacidore (1997), Bourghelle and Declerck (2004), Cai et al. (2008), Golstein and Kavajecz (2000), Griffiths et al. (1998), Harris (1994), Lau and McInish (1995), Porter and Weaver (1997), and Ronen and Weaver (2001).

the effect of a reduction of the tick size,[7] and are consistent with the early predictions of Angel (1997) and Harris (1994).

Theoretical models have also been developed to study the effect of a tick size variation in different market structures. Seppi (1997) investigates the optimal tick size in a market in which a specialist competes for liquidity provision against a competitive limit order book and finds that large traders may prefer a larger tick size than small traders. Cordella and Foucault (1999) study competition between dealers who arrive at the market sequentially and whose bidding strategy depends of the value of the tick size. They show that a larger tick size can increase the speed at which dealers adjust their quotes towards the competitive price, especially when monitoring costs are high. Hence transaction costs can ultimately decrease following an increase in the tick size. Similarly, competition among dealers for the provision of liquidity to an incoming market order drives the results obtained by Kadan (2006). He shows that when the number of dealers is small, liquidity benefits from a small tick size since this prevents dealers from exploiting their market power. When the number of dealers is instead large, a smaller tick size may still improve liquidity. The reason is that it allows dealers to post quotes as close as possible to their reservation value, thus transferring welfare to liquidity demanders.

Our model departs from all these protocols as we consider a pure order driven market where liquidity provision is endogenously created by market participants who choose limit as opposed to market orders. To evaluate the effects of a tick size variation in pure limit order markets, one has to consider the competitive interaction of both patient traders who supply

[7] In December 2008, BATS Europe, in conjuction with Chi-X, Nasdaq OMX Europe and Turquoise, developed a proposal to standardize the tick size of the pan European trading platforms. Starting June 1, 2009, Chi-X, followed by Turquoise, BATS Europe, and finally the LSE and Nasdaq OMX Europe, reduced the tick size for a number of stocks. This pilot program, aimed at studying the effect of a change in the tick size based on actual market data, showed that following the reduction of the tick size, effective spread, inside spread, inside depth and average trade size decreased (BATS, 2009).

liquidity via limit orders, and impatient traders who demand liquidity via market orders. Our framework shares this feature with the dynamic model of Foucault et al. (2005) who show that a reduction of the tick size may harm market resiliency and have adverse effects on transaction costs. In their model, however, traders cannot refrain from trading and when submitting limit orders they must provide a price improvement. Hence, because patient traders cannot join the queue at the existing best bid or offer, a larger tick size has the effect of making their orders more rather than less aggressive, resulting in an increased resiliency and a narrowed spread. By contrast, in our model traders are free to submit market and limit orders at any level on the price grid, as well as to refrain from trading.

Our protocol is closer to Goettler et al. (2005) who consider an infinite horizon version of Parlour (1998) and model a limit order book as a stochastic sequential game with rational traders arriving and choosing to submit orders at, above or below the existing best quotes. Their model is very rich as it also embeds an asset value shock at each trading period. However, because of its richness, it is analytically intractable and hence solved by numerical simulations. In contrast to Goettler et al. (2005), we focus our analysis on a limited number of periods that allows us to obtain a closed form solution. Within their framework Goettler et al. (2005) show that, by reducing the tick size, regulators achieve an increase of total investors' surplus. We show that the effects of a tick size variation depend on the liquidity of the limit order book and on the price of the security traded. We also study the effects of competition from a dark market with a smaller price grid, and finally we provide intuitive explanations for the interaction between liquidity suppliers and liquidity demanders.

Our paper is also related to the literature on intermarket competition that documents an improvement in market efficiency when competing venues enter a market.[8] Chowdhry and

[8] See, for example, Barclay et al. (2003), Bessembinder and Kaufman (1997), Biais et al. (2010), Foucault and Menkveld (2008), Fink et al. (2006), and Goldstein et al. (2008).

Nanda (1991) extend Kyle (1985) model to accommodate multi-market trading and show that markets with the lowest transaction costs attract liquidity. Closer to our framework, Degryse et al. (2009) analyze the interaction between a dealer market and a crossing network and show that overall welfare is not necessarily enhanced by the introduction of a crossing network. Our setup substantially differs from theirs as we consider a LOB instead of a dealer market and an internalization pool instead of a crossing network. Our model also departs from Buti et al. (2011) who model competition between a PLB and a dark pool by focusing squarely on the tick size. Furthermore, the dark pool that we model has its own discriminatory pricing rule whereas the dark pool in Buti et al. (2011) is based on a derivative pricing rule.

Finally, this paper is related to the literature on broker-dealers' internalization and payment for order flow.[9] Chordia and Subrahmanyam (1995) and Kandel and Marx (1999) show that these practises arise from the existence of the tick size. By contrast, Battalio and Holden (2001) show that when the tick size is set equal to zero, brokers still internalize their clients' orders as they make profits by exploiting their direct relationships with customers. This is consistent with the related empirical works.[10]

4 Single Market Model

In this Section we introduce the single market framework and in the next one we solve the model and compare the results for two different values of the tick size. In Section 6 we add competition from a dark market where broker-dealers can post quotes at sub-penny

[9] Internalization is either the direction of order flows by a broker-dealer to an affiliated specialist, or the execution of order flows by that broker-dealer acting as a market maker.

[10] See Chung et al. (2004a and 2004b), Hansch et al. (1999), He et al. (2006), Hendershott and Jones (2005), and Porter and Weaver (1997).

increments.

4.1 The Market

A market for a security is run over a trading day divided into T periods: $t = 1, ..., T$. At each period t a trader arrives and for simplicity we assume that the size of his order is unitary. Following Parlour (1998), traders have the following linear preferences:

$$U(C_1, C_2; \beta) = C_1 + \beta C_2 \tag{1}$$

where C_1 is the cash inflow from selling or buying the security on day 1, while C_2 is the cash inflow from the asset payment on day 2 and is equal to $+v$ $(-v)$ in case of a buy (sell) order. Traders are risk neutral and have a personal trade-off between consumption in the two days equal to β that is a patience indicator drawn from the uniform distribution $U(\underline{\beta}, \overline{\beta})$, with $0 \leq \underline{\beta} < 1 < \overline{\beta}$. A patient trader has a β close to 1 while an eager one has values of β close either to $\underline{\beta}$ or to $\overline{\beta}$.

Upon arrival at the market in period t, the trader observes the state of the book that is characterized by the number of shares available at each level of the price grid. The latter assembles two prices on the ask $(A_1 < A_2)$ and two on the bid side of the market $(B_1 > B_2)$, symmetrically distributed around the asset value v. The difference between two adjacent prices, which we name τ, is the tick size. It is equal to the minimum price increment and also corresponds to the minimum inside spread. Thus the possible prices are equal to $A_1 = v + \frac{\tau}{2}$, $A_2 = v + \frac{3\tau}{2}$, $B_1 = v - \frac{\tau}{2}$, and $B_2 = v - \frac{3\tau}{2}$. The state of the book that specifies the number of shares Q_t available at each price level is defined as $S_t = [Q_t^{A_2}, Q_t^{A_1}, Q_t^{B_1}, Q_t^{B_2}]$. As in Seppi (1997) and Parlour (1998), we assume that a trading crowd provides liquidity at the highest levels of the limit order book and prevents traders from quoting prices that are too far away from the top of the book. Besides, traders are allowed to submit limit orders queuing in front of the trading crowd. In this parsimonious way, we can extend Parlour (1998) model to include two price levels where traders can submit orders, and, at the same time, keep the strategy space as small as possible. In addition we can investigate the effects of the tick size reduction on depth at different levels of the book.

The market allows two types of orders: limit orders represented by $+1$ and market orders represented by -1. Traders can submit limit orders to buy (sell) one share at different levels of the bid (ask) prices, or market orders which hit the bid (ask) prices and are executed immediately, or they can decide not to trade. Orders cannot be modified or cancelled after submission, and a trader's strategy at time t is defined by H_t. His strategy space is therefore $H = \{\pm 1^i, 0\}$, where $i = A_2, A_1, B_1$, and B_2. The change in the limit order book induced by the trader's strategy H_t is indicated by h_t and defined as:

$$h_t = [h_t^{A_2}, h_t^{A_1}, h_t^{B_1}, h_t^{B_2}] = \begin{cases} [\pm 1, 0, 0, 0] & \text{if } H_t = \pm 1^{A_2} \\ [0, \pm 1, 0, 0] & \text{if } H_t = \pm 1^{A_1} \\ [0, 0, \pm 1, 0] & \text{if } H_t = \pm 1^{B_1} \\ [0, 0, 0, \pm 1] & \text{if } H_t = \pm 1^{B_2} \\ [0, 0, 0, 0] & \text{if } H_t = 0 \end{cases} \tag{2}$$

The state of the book is hence characterized by the following dynamics:

$$S_t = S_{t-1} + h_t \tag{3}$$

and the expected state of the book at time t is given by:

$$E[S_{t|t-1}] = S_{t-1} + E[h_t] \quad (4)$$

where $E[h_t^i] = \int_{\beta \in \{\beta : H_t(\beta) = \pm 1^i\}} H_t(\beta) d\beta$ for $i = A_2, A_1, B_1, B_2$.

4.2 Order Submission Decision

To select his order submission strategy, a trader needs to choose an order type and a price. His goal is to maximize his utility, which in this risk neutral setting is equivalent to maximize his payoff, considering all the available strategies. Market orders guarantee immediate executions but higher price opportunity costs, while limit orders enable traders to get better prices at the cost of uncertain execution. Hence in this market traders face the trade-off between execution costs and price opportunity costs.[11] The payoffs of the different strategies available to traders are listed in Table 2. Equilibrium strategies are derived in the following Section.

[**Insert Table 2 here**]

In Table 2 we denote by A and B with no subscript the best available quotes, so that for example a market buy order executed at the best available price is indicated by -1^A. We indicate by $p_t^*(A_k^{N_{-k},N_k} | S_t)$ (or $p_t^*(B_k^{M_{-k},M_k} | S_t)$) with $k = 1, 2$ the equilibrium execution probability for a limit sell (or buy) order queuing at the N_k (M_k) position at the price level A_k (B_k), with $N_{-k} = \sum_{d<k} N_d$ ($M_{-k} = \sum_{d<k} M_d$) being the number of shares standing at lower (higher) price levels.[12] This execution probability is conditional on the state of the limit

[11] When traders choose a limit rather than a market order, they forgo execution certainty to obtain a better price, and consequently they increase their execution costs. At the same time, however, they reduce their price opportunity cost, which is the cost associated with an execution at a less favourable price.

[12] A star superscript indicates equilibrium values.

order book, and depends on both the price level at which the order is posted and the depth available on the limit order book. An order posted at A_k and queueing at the N_k position, is executed against the $(N_{-k} + N_k)$-th market order only if $(N_{-k} + N_k - 1)$ market orders have already hit both the N_{-k} shares available at lower prices and the $N_k - 1$ shares available at A_k with time priority. If $(N_{-k} + N_k)$ is larger than the number of remaining periods, additional limit orders at that price level will never be executed and $p_t(A_k^{N_{-k},N_k} \,|S_t\,) = 0$. The execution probability also depends on the state of the other side of the limit order book: a deep book on the bid side increases the incentive for a seller to post limit orders as he knows that incoming buyers will be more inclined to post market orders (due to the long queue on the bid side).[13]

4.3 Market Equilibrium

Traders use information from the state of the limit order book to rationally compute different orders' execution probabilities, and then compare the expected payoffs from each order to choose the optimal strategy consistent with their own β.[14] We solve the model by backward induction. At time T, the execution probability for limit orders is zero, and traders submit only market orders or decide not to trade. It can be easily shown that traders' equilibrium strategies are:

[13] To simplify the notation, when the best ask is A_k, we indicate the execution probability of a limit order queuing at A_k by $p_t(A_k^{N_k} \,|S_t\,)$ instead of using $p_t(A_k^{0,N_k} \,|S_t\,)$ and the execution probability of the order standing at the first place of the queue by $p_t(A_k \,|S_t\,)$ instead of using $p_t(A_k^{0,1} \,|S_t\,)$.

[14] Differently from Parlour (1998), we do not assume that traders are ex-ante buyers or sellers but we endogenously derive the trader's decision to buy or to sell the asset.

$$H_T^*(\beta, S_{T-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \frac{B}{v}) \\ 0 & \text{if } \beta \in [\frac{B}{v}, \frac{A}{v}) \\ -1^A & \text{if } \beta \in [\frac{A}{v}, \overline{\beta}] \end{cases} \tag{5}$$

where the best ask and bid prices are equal to $A \in \{A_1, A_2\}$ and $B \in \{B_1, B_2\}$ depending on the state of the book. By using these equilibrium strategies together with the distribution of β, we calculate the equilibrium execution probabilities at the best quotes for limit orders submitted at $T-1$:

$$p_{T-1}^*(A\,|S_{T-1}) = \int\limits_{\beta \in \{\beta : H_T^* | S_{T-1} = -1^A\}} \frac{1}{(\overline{\beta} - \underline{\beta})} d\beta = \frac{\overline{\beta}v - A}{(\overline{\beta} - \underline{\beta})v} \tag{6}$$

$$p_{T-1}^*(B\,|S_{T-1}) = \int\limits_{\beta \in \{\beta : H_T^* | S_{T-1} = -1^B\}} \frac{1}{(\overline{\beta} - \underline{\beta})} d\beta = \frac{B - \underline{\beta}v}{(\overline{\beta} - \underline{\beta})v} \tag{7}$$

These execution probabilities are the dynamic link between period T and $T-1$. A trader arriving at $T-1$ can choose between a market and a limit order, and his choice is driven by his β value. The following Lemma holds:

Lemma 1 *If at time $t \neq T$ at least one limit order strategy has positive execution probability, there will always exist a β value for which a limit order is optimally selected by the incoming trader.*

After substituting the equilibrium execution probabilities at T given by (6) and (7) for the case in which a limit order posted at $T-1$ has a positive execution probability on both sides of the market, i.e., $p_{T-1}^*(A_k^{N_{-k},N_k}\,|S_{T-1}) \neq 0$ and $p_{T-1}^*(B_k^{M_{-k},M_k}\,|S_{T-1}) \neq 0$, we obtain the following optimal strategies for $T-1$:[15]

$$H^*_{T-1}(\beta, S_{T-2}) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}) \\ +1^{A_k} & \text{if } \beta \in [\beta_{1,T-1}, \beta_{3,T-1}) \\ +1^{B_k} & \text{if } \beta \in [\beta_{3,T-1}, \beta_{5,T-1}) \\ -1^{A} & \text{if } \beta \in [\beta_{5,T-1}, \overline{\beta}] \end{cases} \tag{8}$$

where $\beta_{1,T-1} = \frac{B}{v} - \frac{p^*_{T-1}(A_k|S_{T-1})}{1-p^*_{T-1}(A_k|S_{T-1})} \cdot \frac{A_k - B}{v}$, $\beta_{3,T-1} = \frac{p^*_{T-1}(A_k|S_{T-1})A_k + p^*_{T-1}(B_k|S_{T-1})B_k}{p^*_{T-1}(A_k|S_{T-1}) + p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{1}{v}$, and $\beta_{5,T-1} = \frac{A}{v} + \frac{p^*_{T-1}(B_k|S_{T-1})}{1-p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{A-B_k}{v}$. Because only one trader can still arrive at the market at T, a limit order posted at $T-1$ has a positive execution probability only if it undercuts all the orders on the book and gains price priority, i.e., $p^*_{T-1}(A_k^{N_{-k},N_k}|S_{T-1}) \neq 0$ only when $N_{-k} = 0$ and $N_k = 1$. Moreover, the greater the limit order execution probability, $p^*_{T-1}(A_k|S_{T-1})$, the smaller the threshold between market sell orders and limit sell orders, $\beta_{1,T-1}$, and the more likely traders submit limit rather than market orders. More generally, if execution probabilities at time t are high enough that execution costs are lower than price opportunity costs, traders will submit limit orders. If instead execution probabilities are low, they will choose market orders.

The optimal price at which a trader will submit a limit order is the result of a trade-off between price opportunity costs and execution costs: a more competitive price implies a higher execution probability due to both the lower risk of being undercut by incoming traders and the fact that the order becomes more attractive for traders on the opposite side of the market. This is, however, obtained at the cost of a lower revenue once the order is executed. This trade-off crucially depends on the relative tick size, $\frac{\tau}{v}$, as shown in the following Lemma:

[15] The other cases are discussed in the Appendix.

Lemma 2 *At time $t \neq T$ traders' aggressiveness in the provision of liquidity is positively related to the value of $\frac{\tau}{v}$.*

From the equilibrium strategies at $T-1$, we can derive the execution probabilities for limit orders submitted in previous periods, and the corresponding equilibrium strategies. The equilibrium is defined as follows:

Definition 1 *Given an initial book S_0, a dynamic equilibrium is a set of order submission decisions $\{H_t^*\}$ and states of the limit order book $\{S_t\}$ such that at each period the trader maximizes his payoff $U(\cdot)$ (Table 2) according to his Bayesian belief over the execution probabilities $p^*(\cdot)$, i.e.*

$$
\begin{aligned}
&\{H_t^* := \arg\max \ U(\cdot \,|\, S_{t-1}, p_{t-1}^*)\} \\
&\{S_t := S_{t-1} + h_t^*\} \\
&\text{where } h_t^* \text{ is defined by (2)}
\end{aligned}
$$

In order to keep the analysis tractable, from here onwards we focus only on the last three periods of the trading game, starting from $T-2$. For our numerical simulation we also assume that the support of the β distribution is $[0,2]$.

5 Tick Size and Market Quality

We start with the market (LM) characterized by a large tick size equal to τ that we already presented in Section 4, and we compare the resulting equilibrium trading strategies with those obtained when, all else equal, the tick size is set to $\frac{1}{3}\tau$. Both price grids are shown in Table 3: on the LM the price grid is $P^{LM} = \{A_2, A_1, B_1, B_2\}$, while on the small tick market (SM) it has five levels on both the ask and the bid side, a_l and b_l, with $l = 1, .., 5$. For the SM the evolution of the state of the book is still characterized by equations (3) and (4), the main difference being that both S_t^{SM} and h_t^{SM} now consist of ten components instead of four. The trader's strategy space for the SM is much richer thanks to the finer price grid, $H^{SM} = \{\pm 1^j, 0\}$ with $j = \{a_{1:5}, b_{1:5}\}$.

[**Insert Table 3 here**]

To compare the two markets, we build standard indicators of market quality using traders' equilibrium strategies. For each period t, depth is measured by the number of shares available on the book at different price levels. More precisely, for the LM we define the following depth indicators:

$$DP_t^{i,LM} = E[Q_t^i], \text{ with } i = \{A_{1:2}, B_{1:2}\} \tag{9a}$$

$$DPI_t^{LM} = E[Q_t^A + Q_t^B] \tag{9b}$$

$$DPT_t^{LM} = \sum_i E[Q_t^i] \tag{9c}$$

where DP^i is the expected depth at price i, DPI is the average depth at the best quotes, and DPT is total depth measured by the sum of average depth at all price levels. The average spread is computed as the expected difference between the best ask and bid prices:

$$SP_t^{LM} = E[A - B] \tag{10}$$

Finally, volume in period t is measured by the number of orders executed, while liquidity provision is measured by the number of limit orders submitted. Because at each period only one trader arrives, expected volume, VL_t, and liquidity provision, LP_t, are computed as the probability that this trader will submit a market order or a limit order at all price levels:

$$VL_t^{LM} = E[\sum_i \int_{\beta\in\{\beta:H_t=-1^i|S_{t-1}\}} 1 d\beta] \quad (11)$$

$$LP_t^{LM} = E[\sum_i \int_{\beta\in\{\beta:H_t=+1^i|S_{t-1}\}} 1 d\beta] \quad (12)$$

Indicators of market quality for the SM are computed in a similar way, but using $j = \{a_{1:5}, b_{1:5}\}$. To illustrate the effects of different tick sizes on both liquid and illiquid stocks, we use the initial state of the book as a proxy for liquidity and consider three cases: an empty book for illiquid stocks, and a book with either one or two units on the first (second) level of the LM (SM) price grid for liquid stocks.[16] The following Proposition summarizes the effects of a tick size change on both traders' strategies and market quality:

Proposition 1 *When moving from a large to a small tick market, changes in traders' order submission strategies and market quality depend on the initial state of the book.*

- *For liquid stocks*
    - *liquidity provision increases; spread, total depth and inside depth improve;*
    - *the effects are the same when the top of the book is very deep, except for inside depth that worsens.*
- *For illiquid stocks the results are the opposite: liquidity provision decreases and spread, total depth and inside depth deteriorate.*

[16] Notice that to compare markets with different tick size we need to start from the same initial state of the book. This implies that in the SM book the price levels not in common with the LM book must be empty.

- *All the above effects become stronger for low priced stocks.*

[**Insert Table 4**]

Table 4 reports results for market quality under both the large and the small tick size regimes for three different opening states of the limit order book: with 0, 1, or 2 shares on $A_1(a_2)$ and $B_1(b_2)$. By considering books that differ in market depth, we can offer insights on how the effects on market quality of a tick size change can be influenced by the initial level of liquidity.

Consider first a book that opens at $T-2$ with only 1 share on both $A_1(a_2)$ and $B_1(b_2)$. When the tick size is smaller, undercutting is cheaper and competition for the provision of liquidity becomes more intense. This implies that in equilibrium traders switch from market to limit orders that they post at the new best price levels (a_1 and b_1). The enhanced limit order submission increases market depth, both total and at the inside spread, and the increased limit order aggressiveness narrows the spread. These effects become stronger as the stock price decreases. Clearly, when the value of the tick size becomes relatively large compared to the stock price, the benefit of having a finer price grid increases, and the probability that traders switch from market to limit orders posted at a_1 and b_1 becomes larger.

Consider then a book that opens at $T-2$ with 2 shares at $A_1(a_2)$ and $B_1(b_2)$. Compared to the previous case, a smaller tick size produces effects that are more intense and of the same direction except for depth at the inside spread, which decreases rather than increases. In the large tick size regime there is no room for additional limit orders[17] and traders are forced either to use market orders or to refrain from trading. Hence when the tick size becomes

[17] Recall that only one trader arrives at each period and hence between $T-2$ and T at most 2 shares can be executed.

smaller, traders move even more aggressively than before to the top of the book and total depth increases. However, depth at the top of the book decreases as now traders can spread their orders on the additional price levels instead of being clustered at the best prices.

We can therefore conclude that for these two cases a smaller tick size improves liquidity as it narrows the inside spread and increases total depth, but its effect on inside depth depends on the state of the book. If the regime with a deeper book is a good proxy for very liquid stocks, then our results show that for these stocks a smaller tick size can actually decrease depth at the inside spread.

However, for illiquid stocks, that we proxy by the empty book, the effect of a smaller tick size is to worsen both inside spread and depth. Traders do not have enough incentive to undercut aggressively by posting limit orders at the new top of the book: to avoid being undercut by the next trader, they would have to accept a very low execution price (a_1 and b_1). Therefore, they prefer to submit either more market orders, or, despite the lower execution probability, limit orders at higher levels of the book. Also in this case, when the stock price increases, a smaller tick size tends to produce effects that gradually drop off.

6 Dual-Market Model: PLB, IP and Sub-Penny Trading

In this Section we broaden our comparative analysis of markets with different tick sizes to include intermarket competition. We extend the previous framework by introducing competition between a large tick market that works like a PLB and a special type of dark pool, the IP, that differs from the PLB in that it has a smaller tick size and allows only broker-dealers to supply liquidity. This dual-market model is suitable to investigate sub-penny trading. Such a practice is carried out by those broker-dealers who can access IPs to compete on price with the limit orders posted at the top of the PLB.

Regulators are concerned about the ultimate effects of sub-penny trading on market quality. It is not yet clear whether this practice fosters competition for the provision of liquidity and improves market quality, or whether it only allows highly sophisticated dealers to generate considerable returns from using IPs to step in front of the NBBO. Because the better prices available in the IP intercept market orders sent to the PLB, due to the existence of smart order routers, liquidity demand decreases. This generates two opposite effects on the quality of the PLB. On the one hand, when fewer market orders hit the top of the PLB, depth is preserved, keeping the spread tighter. On the other hand, the reduction of market orders decreases the execution probability of limit orders and hence the incentive for traders to post depth at the top of the PLB, resulting in lower market depth and wider spread. The changing pattern of market orders, i.e., migrating from the PLB to the IP, is affected by the state of the PLB as well as by the tick to price ratio of the stock considered. Agents' choice between market and limit orders hinges on the trade-off between price opportunity costs and execution costs, which are both influenced by stock characteristics.

To discuss this setting, we adapt our previous single market model to embed sub-penny trading. We assume that at each trading period one individual out of two groups of traders arrives at the market: with probability α the incoming trader is a broker-dealer and with the complementary probability he is a regular trader. While a regular trader can only observe and provide liquidity to the PLB, a broker-dealer can observe and use both the PLB and the IP. The IP is characterized by a smaller tick size so that the broker-dealer can undercut orders posted by other traders at the top of the PLB. Furthermore, in order to approximate the nature of real IPs, we assume that the IP does not have a trading crowd sitting at a_5

and b_5. Finally, while only broker-dealers can post limit orders at the IP, all traders can take advantage of the liquidity offered by both trading platforms. This assumption is consistent with the existence of a smart order routing technology (SOR)[18] that allows all investors to simultaneously access multiple sources of liquidity (Butler, 2010).

SORs allow traders to search the best quotes on the consolidated limit order book (PLB&IP) but they do not necessarily reveal the state of the IP. In reality, the more sophisticated the SOR technology, the better traders' inference of the state of the IP. Hence, to consider different regimes of IP pre-trade transparency, we will first assume perfect inference on the state of the IP, and then extend the model to include partial inference and Bayesian learning.

To sum up, we consider two protocols: the benchmark (PLB), where only one trading platform is available to all traders, and the PLB&IP framework, where an IP competes with the PLB. The latter case further differentiates into a transparent and an opaque setting, where the lack of transparency refers to the IP market. To introduce a certain degree of uncertainty on the state of the IP, we assume that at $T-2$ the IP opens either empty or with one unit on the first level of the book with equal probability. The following Proposition presents the results.

Proposition 2 *When an IP is added to a PLB, trade migrates to the IP, and traders' order submission strategies and market quality change as follows:*

- *for illiquid stocks, PLB market quality, measured by depth and inside spread, deteriorates because liquidity provision decreases. The effects are stronger both for low priced stocks and when the IP market is opaque;*

[18] Examples are ITG Dark Aggregator and Smartrade Liquidity Aggregator.

- *for liquid stocks, the effect of sub-penny trading is to foster price competition so that in the PLB spread and depth improve, yet liquidity provision worsens. The effects are weaker both for low priced stocks and when the IP market is opaque.*

- *The IP is used more intensively by broker-dealers when the stock is both liquid and low priced and when their proportion (α) increases.*

- *When considering markets with a smaller tick size, all the effects diminish.*

Table 5 focuses on the case in which the PLB opens empty at $T-2$, that we use to proxy illiquid stocks, while Table 6 shows results for a book that opens with one share on A_1 and B_1 which should offer intuitions for more liquid stocks. Table 5 shows that on the PLB both depth and limit orders decrease when the IP is added. When traders perceive the potential competition from broker-dealers, they react by supplying less liquidity to the PLB. Furthermore, this effect generally outweighs the reduction in market orders resulting from their interception by the IP, so that overall the inside spread worsens. These effects become weaker as the stock price increases: when the tick to price ratio becomes very small, the profitability of liquidity provision, and hence of undercutting, declines so that the IP competition becomes less relevant. When instead the IP is opaque, these effects become stronger: the uncertainty on the IP depth and on the actual level of competition makes traders even more reluctant to post limit orders at the PLB.

[**Insert Tables 5 and 6 here**]

For more liquid stocks (Table 6) the main effect of sub-penny trading is to foster price competition. When the IP platform is introduced, broker-dealers submit limit orders to the IP at a_1 to undercut the existing depth at A_1, thus intercepting incoming market orders away from the PLB. The resulting reduction of liquidity demand improves spread and depth on the PLB. However, the switch of market orders from the PLB to the IP as well as of broker-dealers' limit orders implies that volume and liquidity provision worsen on the public venue.

In this case, when the stock price increases, the effects of the IP competition become stronger due to the reaction of both market orders and limit orders. When the tick to price ratio decreases, market orders become central to the choice of traders' optimal submission strategies and therefore their increased reduction has a greater effect on the PLB quality. This positive effect is reinforced by the fact that the change in limit orders becomes almost irrelevant, and hence, compared to the $v = 1$ case, their reduction almost disappears. When instead the IP market becomes opaque, uncertainty increases for regular traders and therefore the effects on both limit and market orders lessen. As a result, the IP positive effect on PLB market quality decreases.

By comparing traders' equilibrium strategies for illiquid and liquid stocks, we observe that broker-dealers post orders at the IP more intensively for liquid stocks in which competition for liquidity provision in the PLB is higher. This effect is stronger for low priced stocks in which providing liquidity is more convenient than taking liquidity, due to the larger tick to price ratio. Finally, when the proportion α of broker-dealers is increased from 10% to 20% (Table 7), the sub-penny activity builds up in the IP and hence the effects on the PLB spread and depth are magnified.

[**Insert Table 7 here**]

Finally, our model allows us to evaluate how different tick size values affect competition for the provision of liquidity between regular and dark markets. Table 8 shows that when the tick size is $\tau/3$ rather than τ, all the effects of sub-penny trading decrease. A smaller tick size, in fact, reduces the profits for broker-dealers from posting orders in the IP and consequently their activity on this trading platform.

[**Insert Table 8 here**]

7 Welfare

We now turn to investigate how different tick size values can affect the welfare of market participants. Following Degryse et al. (2009) and Goettler et al. (2005), we measure total welfare (W) as the sum of the expected welfare in each period $E(W_t)$, i.e., $W = \sum_t E(W_t)$, where $E(W_t)$ is the sum of agents' expected gains from both market (first term) and limit orders (second term):

$$E(W_t) = \sum_i \int_{\beta \in \{\beta : H_t = -1^i | S_{t-1}\}} |i - \beta v| \cdot f(\beta) d\beta + \sum_i \int_{\beta \in \{\beta : H_t = +1^i | S_{t-1}\}} |i - \beta v| \cdot p_t^*(i\,|S_t) \cdot f(\beta) d\beta \tag{13}$$

As this measure only allows us to calculate the absolute change in welfare, to obtain a relative indicator, we compute the expected gains from trade accruing to market participants in a market without frictions, $\overline{W}$. In a frictionless market a first-best allocation of resources is achieved and all orders are executed at the fundamental value of the asset:

$$\overline{W} = \sum_t E(\overline{W}_t) = \int_{\underline{\beta}}^{\overline{\beta}} |v - \beta v| \; f(\beta) \, d\beta \tag{14}$$

The results obtained are summarized in the following Proposition.

Proposition 3 .

- *In the single market model a smaller tick size makes traders better off when the stock is liquid and worse off when it is illiquid. Changes in traders' welfare increase with the tick to price ratio.*
- *Competition from an IP makes all traders better off, and the improvement decreases with the tick to price ratio.*

The effects on welfare of a change in the tick size depend on the state of the book (Table 9). As we have previously shown, when the book opens shallow, a smaller tick size worsens liquidity with the result that traders' welfare decreases. When instead the market opens deep, a smaller tick size improves the spread and makes the market deeper, thus increasing traders' gains from trade. Finally, consistently with the results obtained on market quality, the effects of a tick size reduction are stronger for low priced stocks, whereas they tend to vanish for high priced stocks.

[**Insert Table 9 here**]

To evaluate the effects on welfare of the opportunity to trade in an IP, we consider broker-dealers and regular traders separately. As reported in Table 10, competition from the IP increases total welfare for both liquid and illiquid stocks, and for both broker-dealers and regular traders. In fact, even though only broker-dealers can post limit orders to the IP, due to the existence of smart order routers, all traders can take advantage of the liquidity offered. This effect decreases with the tick size because broker-dealers have a lower incentive to provide liquidity on the IP.

[**Insert Table 10 here**]

8 Empirical implications

The equilibrium strategies derived from our model generate several empirically testable predictions. We find that the behaviour of traders crucially depends on the liquidity of the limit order book and on the price of the stock considered. Starting with the single market model, we show that a tick size reduction can improve or worsen the indicators of market quality, depending on the initial state of the book:

> **Prediction 1** When the tick size is smaller, depth worsens if the book is shallow and improves if it is deep. For the inside spread the effect is not monotonic with respect to book liquidity: this market indicator worsens for shallow books, improves for deep ones but worsens again when the book is extremely deep.

Most of the existing empirical evidence does not distinguish between liquid and illiquid stocks, and shows that when the tick size is reduced, the inside spread decreases but depth does not necessarily improve.[19] On the other hand, when stocks are classified according to their degree of liquidity, as in Bourghelle and Declerck (2004), results show that as liquidity decreases, the percentage spread increases and the quoted depth decreases. Similarly, our prediction is in line with the empirical findings of Golstein and Kavajecz (2000), who sort NYSE stocks according to their levels of liquidity and price. They show that as the tick size is reduced, there is an overall improvement of the quoted spread and depth for liquid stocks, whereas market quality deteriorates for illiquid ones. Golstein and Kavajecz also show that both effects become stronger as the stock price decreases, and that for low priced illiquid

[19] See Ahn et al. (1996), Bacidore (1997), Bessembinder (2003), Harris (1994), and Porter and Weaver (1997).

stocks the negative effect on the quoted spread and depth is the greatest. The latter results are consistent with our second prediction:

Prediction 2 Both in deep and in shallow markets, the effect of a smaller tick size gradually increases with the tick to price ratio.

Both predictions can be directly tested by sorting stocks by their price and the level of liquidity and applying an event study of tick size change. The key is to select a sample of stocks with enough independent variation in liquidity and in price.

When we extend the analysis to a framework which includes inter-market competition between a primary market and an IP, we obtain additional testable implications. The endogenous strategic interaction between regular traders and more sophisticated broker-dealers having access to dark platforms generates distinctive empirical patterns for liquidity supply and liquidity demand. More precisely, competition from an IP affects traders' desire to supply liquidity which generates empirical predictions about the dynamics of order flows. These again depend on the stocks' characteristics.

Prediction 3 When the primary market is deep, the IP attracts market orders, thus preserving liquidity and ultimately enhancing depth and spread on the PLB. When the primary market is shallow, competition from the IP deters liquidity provision on the PLB where depth and spread worsen. All these effects decrease as the tick size gets smaller.

Prediction 4 For illiquid stocks, a high tick to price ratio amplifies the effects on market quality of competition from IPs, while the opposite holds for liquid stocks.

These theoretical findings are directly testable either cross-sectionally, by selecting stocks with different degrees of liquidity and price levels; or by using time series of order data. A final empirical implication of our model is that:

Prediction 5 Broker-dealers' activity on IPs is more intensive on liquid and low priced stocks.

This is an interesting topic for future research, when data on dark markets becomes available.

9 Policy Discussion and Conclusions

Today's financial markets are characterized by extensive intermarket competition, both between fully transparent venues and between transparent and dark pools of liquidity. In such a world, the regulation of the minimum price increment is extremely important, as it defines how markets compete for liquidity. In this paper, we analyze the impact of the tick size for market quality and traders' welfare in a public limit order market. We also analyze how the entry of a dark venue with a finer price grid (trading in sub-pennies) affects traders' strategies, and therefore results in changes in market quality and traders' welfare.

First of all, we discuss the effects of a smaller tick size within the context of a limit order book and show that these effects depend on the liquidity of the stock and on the underlying value of the asset. In this respect, the model's results show that for illiquid stocks market quality worsens with a smaller tick size, whereas for liquid stocks it improves. Such effects are relevant for low priced stocks, whereas they vanish as the price of the security increases. These results suggest that the objective of tick size regulation should be the definition of a minimum price change that is consistent with the stock's main attributes and should be related to its liquidity and price.

The model is then extended to include competition from an IP. This dual-market model accounts for the changing nature of liquidity provision around the world, which is now dominated by competition among regular and dark markets, and in which a growing number of dark pools allows broker-dealers to post limit orders at fractions of the tick size. This extension is well suited to investigate the issue of sub-penny trading, which is one of the main concerns addressed by the SEC in the April 2010 concept release on Equity Market Structure. Our model suggests that sub-penny trading undertaken by broker-dealers on IPs can have negative effects on the market quality for illiquid stocks, while it is beneficial to the quality of the market for liquid stocks.

By drawing on our model we can also comment the conclusions of the SEC Report on Decimalization (2012), which was required by the JOBS Act (2012) and which states that "the Commission should not proceed with the specific rulemaking to increase the tick size." The JOBS Act advised the SEC to consider the possibility of increasing the tick size for low priced stocks and of eventually undertaking a pilot program, under which some small and mid-cap stocks would trade at increments greater than a penny. The objective of this proposal was to restore the economic incentive to support liquidity for these stocks, that over time had experienced a reduction of the inside spread. Our model reveals that a greater tick size for illiquid low-price stocks could on the one hand stimulate competition for the provision of liquidity and result in smaller rather than wider spreads, while on the other it could lead to an increase in broker-dealers profits from sub-penny trading. According to our results, it would therefore appear that an increase of the tick size would not necessarily restore the interest of liquidity providers for these stocks.

Appendix

A Proof of Lemma 1

Proof. At any period $t \neq T$, a trader selects his optimal strategy H_t^* by comparing the payoffs of all the strategies described in Table 2, $H_t = \{-1^B, +1^{A_k}, +1^{B_k}, -1^A, 0\}$. Assume that $p_t^*(A_k^{N_{-k},N_k}) > 0$.[20] We compute the threshold $\beta_{-1^B,0}$ between $H_t = -1^B$ and $H_t = 0$ by equating the profits from the two strategies $(B - \beta v = 0)$ and we obtain $\beta_{-1^B,0} = \frac{B}{v}$. Similarly, we compute the threshold between $H_t = -1^B$ and $H_t = +1^{A_k}$ and obtain $\beta_{-1^B,1^{A_k}} = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$. Under the assumption that $p_t^*(A_k^{N_{-k},N_k}) > 0$, then $\beta_{-1^B,1^{A_k}} < \beta_{-1^B,0}$ and hence there always exists a value for $\beta \in (\beta_{-1^B,1^{A_k}}, \beta_{-1^B,0})$ such that $H_t^*(\beta, S_{t-1}) = +1^{A_k}$. A similar result holds for the bid side. Clearly, traders' equilibrium β-thresholds, and hence strategies, crucially depend on the state of the book which affects the execution probability of limit orders. There are four possible scenarios. When there is room for limit orders on both sides of the market, equilibrium traders' strategies for $t \neq T$ are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_3) \\ +1^{B_k} & \text{if } \beta \in [\beta_3, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) \neq 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) \neq 0 \tag{15a}$$

where $\beta_1 = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$, $\beta_3 = \frac{p_t^*(A_k^{N_{-k},N_k})A_k + p_t(B_k^{M_{-k},M_k})B_k}{p_t^*(A_k^{N_{-k},N_k}) + p_t(B_k^{M_{-k},M_k})} \cdot \frac{1}{v}$ and $\beta_5 = \frac{A}{v} + \frac{p_t^*(B_k^{M_{-k},M_k})}{1-p_t^*(B_k^{M_{-k},M_k})} \cdot \frac{A - B_k}{v}$. When instead the book opens full either on the ask or on the bid side, the equilibrium strategies are respectively:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\beta_1, \beta_4) \\ +1^{B_k} & \text{if } \beta \in [\beta_4, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) = 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) \neq 0 \tag{15b}$$

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_2) \\ 0 & \text{if } \beta \in [\beta_2, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) \neq 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) = 0 \tag{15c}$$

[20] To simplify the notation, we omit to write that all the execution probabilities are conditional on S_t.

where $\beta_2 = \frac{A_k}{v}$ and $\beta_4 = \frac{B_k}{v}$. Finally, when the book is full on both sides, equilibrium strategies are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\beta_1, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if } p_t^*(A_k^{N_{-k},N_k}) = 0 \ \& \ p_t^*(B_k^{M_{-k},M_k}) = 0 \qquad \text{(15d)}$$

If $p_t^*(A_k^{N_{-k},N_k}) = p_t^*(B_k^{M_{-k},M_k}) = 0$, then $H_t^*(\beta, S_{t-1}) = H_T^*(\beta, S_{t-1})$. ■

B Proof of Lemma 2

Proof. As an example, we consider the ask side in period $T-1$; the other cases can be derived in a similar way. At $T-1$ limit orders have a positive execution probability on both A_1 and A_2 only when the book opens empty on both sides, $S_{T-1} = [0,0,0,0]$.[21] In this case traders can optimally select their level of price aggressiveness. Profits from the two available limit order strategies are:

$$\begin{array}{lcl} H_{T-1} = +1^{A_2} & : & (A_2 - \beta v) \cdot p_{T-1}^*(A_2 \,|[1000]\,) = (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta}) v} \\ H_{T-1} = +1^{A_1} & : & (A_1 - \beta v) \cdot p_{T-1}^*(A_1 \,|[0100]\,) = (A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta}) v} \end{array}$$

A limit order at A_1 is optimal if $\exists \beta$ such that $(A_1 - \beta v) \cdot p_{T-1}^*(A_1 \,|[0100]\,) > \max\{B_2 - \beta v, (A_2 - \beta v) \cdot p_{T-1}^*(A_2 \,|[1000]\,)\}$; in this case the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_1}$ is smaller than the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_2}$. Specifically, as $\beta_{-1^B,1^{A_k}} = \beta_1 \,|_{B=B_2, A_k} = \frac{B_2}{v} - \frac{p_{T-1}^*(A_k|S_t)}{1 - p_{T-1}^*(A_k|S_t)} \cdot \frac{A_k - B_2}{v}$, in order for $\beta_{-1^B,1^{A_1}} < \beta_{-1^B,1^{A_2}}$ the lower selling price (by one tick, τ) must be compensated by a higher execution probability. As $p_{T-1}^*(A_1 \,|[0100]\,) - p_{T-1}^*(A_2 \,|[1000]\,) = \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta}) v} - \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta}) v} = \frac{\tau}{(\overline{\beta} - \underline{\beta}) v}$ is an increasing function of the relative tick size, for $H_{T-1} = +1^{A_1}$ to be an optimal strategy, $\frac{\tau}{v}$ must be larger than $\widehat{\frac{\tau}{v}}$, where $\widehat{\frac{\tau}{v}}$ solves $(A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta}) v} - (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta}) v} = 0$. ■

C Proof of Proposition 1

Proof. We consider the illiquid stocks for which the book opens at $T-2$ as [0000]. We analyze first a LM where at $T-2$ traders' strategy space is $\{-1^B, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$; each strategy corresponds to an opening book at $T-1$ equal to [0000], [1000], [0100], [0010], [0001], and [0000] respectively. Consider the book that opens at $T-1$ with one share on A_2,

[21] From here onwards, to simplify the notation, we omit commas, i.e., [0000] represents an empty limit order book.

i.e., [1000]. Traders' strategy space is $\{-1^B, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$ and the corresponding payoffs are:

$H_{T-1} = -1^B$	:	$B_2 - \beta v$
$H_{T-1} = +1^{A_1}$	:	$(A_1 - \beta v) \cdot p^*_{T-1}(A_1 \,\vert [1100])$
$H_{T-1} = +1^{B_1}$	:	$(\beta v - B_1) \cdot p^*_{T-1}(B_1 \,\vert [1010])$
$H_{T-1} = +1^{B_2}$	:	$(\beta v - B_2) \cdot p^*_{T-1}(B_2 \,\vert [1001])$
$H_{T-1} = -1^A$	:	$\beta v - A_2$
$H_{T-1} = 0$	:	0

where the execution probabilities are given by (6) and (7). After comparing these payoffs, we obtain the equilibrium strategies at $T-1$ that depend on the relative tick size $\frac{\tau}{v}$, as shown in Lemma 2. If the relative tick size is small such that traders are not aggressive and post limit orders at higher levels of the book, the equilibrium strategies at $T-1$ are given by Eq. (8):

$$H^{*LM}_{T-1}(\beta, [1000]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}\,|_{A_1,B=B_2}) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-1}\,|_{A_1,B=B_2}, \beta_{3,T-1}\,|_{A_1,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-1}\,|_{A_1,B_2}, \beta_{5,T-1}\,|_{A=A_2,B_2}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}\,|_{A=A_2,B_2}, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-1}\,|_{A_1,B=B_2} = \frac{B_2}{v} - \frac{p^*_{T-1}(A_1|[1100])}{1-p^*_{T-1}(A_1|[1100])} \cdot \frac{A_1-B_2}{v}$, $\beta_{3,T-1}\,|_{A_1,B_2} = \frac{p^*_{T-1}(A_1|[1100])A_1 + p^*_{T-1}(B_2|[1001])B_2}{p^*_{T-1}(A_1|[1100]) + p^*_{T-1}(B_2|[1001])} \cdot \frac{1}{v}$ and $\beta_{5,T-1}\,|_{A=A_2,B_2} = \frac{A_2}{v} + \frac{p^*_{T-1}(B_2|[1001])}{1-p^*_{T-1}(B_2|[1001])} \cdot \frac{A_2-B_2}{v}$. This allows us to compute the execution probability of the strategy $H_{T-2} = +1^{A_2}$ that turns the book into [1000] at $T-1$:

$$\begin{aligned} p^*_{T-2}(A_2 \,|[1000]) \;=\; & \frac{\overline{\beta} - \beta_{5,T-1}\big|_{A=A_2,B_2}}{\overline{\beta} - \underline{\beta}} + \frac{\beta_{5,T-1}\big|_{A=A_2,B_2} - \beta_{3,T-1}\big|_{A_1,B_2}}{\overline{\beta} - \underline{\beta}} \cdot p^*_{T-1}(A_2 \,|[1001]) \\ & + \frac{\beta_{1,T-1}\big|_{A_1,B=B_2} - \underline{\beta}}{\overline{\beta} - \underline{\beta}} \cdot p^*_{T-1}(A_2 \,|[1000]) \end{aligned}$$

The first term is the execution probability at $T-1$, whereas the other two represent the execution probability at T. Similarly, we compute the execution probabilities for the other order types to compare all possible trader's payoffs at $T-2$:

$H_{T-2} = -1^B$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_2}$	:	$(A_2 - \beta v) \cdot p^*_{T-2}(A_2 \,\vert [1000]\,)$
$H_{T-2} = +1^{A_1}$	:	$(A_1 - \beta v) \cdot p^*_{T-2}(A_1 \,\vert [0100]\,)$
$H_{T-2} = +1^{B_1}$	:	$(\beta v - B_1) \cdot p^*_{T-2}(B_1 \,\vert [0010]\,)$
$H_{T-2} = +1^{B_2}$	:	$(\beta v - B_2) \cdot p^*_{T-2}(B_2 \,\vert [0001]\,)$
$H_{T-2} = -1^A$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

Hence equilibrium strategies at $T-2$ are:

$$H^{*LM}_{T-2}(\beta, [0000]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_2,B=B_2}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_2,B=B_2}\,, \beta_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}\,, \beta_{5,T-2}\,|_{A=A_2,B_2}\,) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_2,B_2}\,, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-2}\,|_{A_2,B=B_2} = \frac{B}{v} - \frac{p^*_{T-2}(A_2|[1000]\,)}{1-p^*_{T-2}(A_2|[1000]\,)} \cdot \frac{A_2-B}{v}$, $\beta_{3,T-2}\,|_{A_2,B_2} = \frac{p^*_{T-2}(A_2|[1000]\,)A_2+p^*_{T-2}(B_2|[0001]\,)B_2}{p^*_{T-2}(A_2|[1000]\,)+p^*_{T-2}(B_2|[0001]\,)} \cdot \frac{1}{v}$, $\beta_{5,T-2}\,|_{A=A_2,B_2} = \frac{A}{v} + \frac{p^*_{T-2}(B_2|[0001]\,)}{1-p^*_{T-2}(B_2|[0001]\,)} \cdot \frac{A-B_2}{v}$.

When instead $\frac{\tau}{v}$ is large, submitting limit orders on the first level of the book becomes an optimal strategy, and the equilibrium strategies at $T-2$ are derived from a modified version of Eq. (15a):

$$H^{*LM}_{T-2}(\beta, [0000]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_1,B=B_2}\,) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_1,B=B_2}\,, \beta_{6,T-2}\,|_{A_1,A_2}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta_{6,T-2}\,|_{A_1,A_2}\,, \beta_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}\,, \beta_{7,T-2}\,|_{B_1,B_2}\,) \\ +1^{B_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{B_1,B_2}\,, \beta_{5,T-2}\,|_{A=A_2,B_1}\,) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_2,B_1}\,, \overline{\beta}] \end{cases}$$

where $\beta_{6,T-2}\,|_{A_1,A_2} = \frac{p^*_{T-2}(A_1|[0100]\,)A_1-p^*_{T-2}(A_2|[1000]\,)A_2}{p^*_{T-2}(A_1|[0100]\,)-p^*_{T-2}(A_2|[1000]\,)} \cdot \frac{1}{v}$, $\beta_{7,T-2}\,|_{B_1,B_2} = \frac{p^*_{T-2}(B_1|[0010]\,)B_1-p^*_{T-2}(B_2|[0001]\,)B_2}{p^*_{T-2}(B_1|[0010]\,)-p^*_{T-2}(B_2|[0001]\,)} \cdot \frac{1}{v}$.

We solve the same problem for the SM and directly present the equilibrium strategies at $T-2$ for the case with an empty book, that we indicate with $S_t = [0]$, and a small relative tick size value, $\frac{\tau}{v}$:

$$H^{*SM}_{T-2}(\beta, [0]) = \begin{cases} -1^b & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_5,b=b_5}\,) \\ +1^{a_5} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_5,b=b_5}\,, \beta_{3,T-2}\,|_{a_5,b_5}\,) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}\,, \beta_{5,T-2}\,|_{a=a_5,b_5}\,) \\ -1^a & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_5,b_5}\,, \overline{\beta}] \end{cases}$$

When instead $\frac{\tau}{v}$ is large, the equilibrium strategies at $T-2$ are modified as follows:

$$H^{*SM}_{T-2}(\beta,[0]) = \begin{cases} -1^b & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_1,b=b_5}) \\ +1^{a_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_1,b=b_5}, \beta_{6,T-2}\,|_{a_1,a_5}) \\ +1^{a_5} & \text{if } \beta \in [\beta_{6,T-2}\,|_{a_1,a_5}, \beta_{3,T-2}\,|_{a_5,b_5}) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}, \beta_{7,T-2}\,|_{b_1,b_5}) \\ +1^{b_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{b_1,b_5}, \beta_{5,T-2}\,|_{a=a_5,b_1}) \\ -1^a & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_5,b_1}, \overline{\beta}] \end{cases}$$

To compare the LM with the SM in terms of liquidity provision and executed volume, it is sufficient to compare the thresholds that make the trader indifferent between submitting a market order and the most attractive among the available limit order strategies:

$$\widehat{\beta}^{LM}_{5,T-2}\,|_{A=A_2,B_k} = \max_{B_k} \beta_{5,T-2}\,|_{A=A_2,B_k} = \max_{B_k}\{\tfrac{A_2}{v} + \tfrac{p^*_{T-1}(B_k|S_{T-1})}{1-p^*_{T-1}(B_k|S_{T-1})} \cdot \tfrac{A_2-B_k}{v}\}$$

$$\widehat{\beta}^{SM}_{5,T-2}\,|_{a=a_5,b_l} = \max_{b_l} \beta_{5,T-2}\,|_{a=a_5,b_l} = \max_{b_l}\{\tfrac{a_5}{v} + \tfrac{p^*_{T-1}(b_l|S_{T-1})}{1-p^*_{T-1}(b_l|S_{T-1})} \cdot \tfrac{a_5-b_l}{v}\}$$

After substituting the equilibrium execution probabilities, we find that for a market buy order $\widehat{\beta}^{SM}_{5,T-2}\,|_{a=a_5,b_l} < \widehat{\beta}^{LM}_{5,T-2}\,|_{A=A_2,B_k}$, and for a market sell order $\widehat{\beta}^{SM}_{1,T-2}\,|_{a_l,b=b_5} > \widehat{\beta}^{LM}_{1,T-2}\,|_{A_k,B=B_2}$. Consequently, volume is higher in the SM:

$$VL^{SM}_{T-2} = \tfrac{\overline{\beta}-\widehat{\beta}^{SM}_{5,T-2}|_{a=a_5,b_l}}{\overline{\beta}-\underline{\beta}} + \tfrac{\widehat{\beta}^{SM}_{1,T-2}|_{a_l,b=b_5} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} > \tfrac{\overline{\beta}-\widehat{\beta}^{LM}_{5,T-2}|_{A=A_2,B_k}}{\overline{\beta}-\underline{\beta}} + \tfrac{\widehat{\beta}^{LM}_{1,T-2}|_{A_k,B=B_2} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} = VL^{LM}_{T-2}$$

When the book opens empty at $T-2$, no trading ($H_{T-2}=0$) is never optimal and hence in this single market model the submission probabilities of market and limit orders are complements. Thus, as $VL^{LM}_{T-2} < VL^{SM}_{T-2}$, we obtain that $LP^{LM}_{T-2} > LP^{SM}_{T-2}$. Furthermore, inside depth is equal to both total depth and liquidity provision, $DPI_{T-2} = DPT_{T-2} = LP_{T-2}$, where:

$$LP_{T-2} = \tfrac{\widehat{\beta}_{5,T-2}-\widehat{\beta}_{1,T-2}}{\overline{\beta}-\underline{\beta}}$$

Consequently, also total and inside depth are lower in the SM: $DPI^{LM}_{T-2} = DPT^{LM}_{T-2} > DPI^{SM}_{T-2} = DPT^{SM}_{T-2}$. Finally, in order to compute the spread, we consider two cases. When $\frac{\tau}{v}$ is large, we obtain:

$$\begin{aligned} SP^{LM}_{T-2} &= E[A-B] = 3\tau - (\tfrac{\beta_{6,T-2}|_{A_1,A_2} - \beta_{1,T-2}|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}} + \tfrac{\beta_{5,T-2}|_{A=A_2,B_2} - \beta_{7,T-2}|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}) \cdot \tau \\ &< 3\tau - (\tfrac{\beta_{6,T-2}|_{a_1,a_5} - \beta_{1.T-2}|_{a_1,b=b_5}}{\overline{\beta}-\underline{\beta}} + \tfrac{\beta_{5,T-2}|_{a=a_5,b_1} - \beta_{7,T-2}|_{b_1,b_5}}{\overline{\beta}-\underline{\beta}}) \cdot \frac{4\tau}{3} = SP^{SM}_{T-2} \end{aligned}$$

where, for example, $\frac{\beta_{6,T-2}|_{A_1,A_2} - \beta_{1,T-2}|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit sell order posted at A_1 and $\frac{\beta_{5,T-2}|_{A=A_2,B_2} - \beta_{7,T-2}|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit buy order posted at B_1.When instead $\frac{\tau}{v}$ is small, so that traders in equilibrium post limit orders only at $A_2 = a_5$ and $B_2 = b_5$, we obtain that $SP^{LM}_{T-2} = SP^{SM}_{T-2} = 3\tau$.

The proofs for liquid, [0110], and very liquid stocks, [0220], are obtained following the same procedure and are available from the authors upon request. ■

D Proof of Proposition 2

Proof. We provide a proof for liquid stocks, i.e., a PLB that opens at $T-2$ as [0110]. The proof for illiquid stocks follows a similar procedure and is available from the authors upon request.

1) Liquid stocks, transparent IP

When the IP is transparent, regular traders' (RT) equilibrium strategies depend on the initial state of the IP. We consider two cases: an empty IP, [0110]&[0], and an IP with one share on the first level of the book, [0110]&[1]. To obtain market quality indicators that are comparable with the opaque IP case, we take the average of the values obtained for the two cases.

(1.1) When the opening book is [0110]&[0]**,** at $T-2$ the overall trader's strategy space, considering both broker-dealers (BD) and RT, is $\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. Therefore, at the beginning of $T-1$ there are 17 possible states of the books: one share added to the i-th level of the PLB and no shares added to the IP (4 cases), one share added to the j-th level of the IP and no shares added to the PLB (10 cases), one share taken from the PLB (2 cases) or no trading (1 case). We compute the optimal equilibrium strategy for different types of traders conditional on each case. Because at T all traders can observe the best available price, the equilibrium strategies of RT and BD are the same (market orders only). As a result the orders' execution probabilities at $T-1$ do not depend on the type of trader arriving at T, for example: $p^{*RT}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*BD}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1})$. Suppose $H^*_{T-2} = +1^{A_2}$ so that at $T-1$ the book opens [1110]&[0]. If a RT arrives, his payoffs are:

$H_{T-1} = -1^B$	:	$B_1 - \beta v$
$H_{T-1} = -1^A$	:	$\beta v - A_1$
$H_{T-1} = 0$	:	0

and his equilibrium strategies are:

$$H_{T-1}^{*RT}(\beta,[1110]\&[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}^{RT}\,|_{B=B_1}) \\ 0 & \text{if } \beta \in [\beta_{1,T-1}^{RT}\,|_{B=B_1}, \beta_{5,T-1}^{RT}\,|_{A=A_1}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}^{RT}\,|_{A=A_1}, \overline{\beta}] \end{cases}$$

By using the optimal β thresholds associated with these strategies, we compute the execution probability $H_{T-2} = +1^{A_2}$ conditional on a RT arriving at $T-1$:

$$p_{T-2}^{*RT}(A_2\,|[1110]\&[0]\,) = \frac{\overline{\beta}-\beta_{5,T-1}^{RT}\big|_{A=A_1}}{\overline{\beta}-\underline{\beta}} \cdot p_{T-1}^{*}(A)\,|_{A=A_2}$$

If instead a BD arrives at $T-1$, his payoffs are:

$H_{T-1} = -1^B$	:	$B_1 - \beta v$
$H_{T-1} = +1^{a_l}$	:	$(a_l - \beta v) \cdot p_{T-1}^{*}(a_l \| S_{T-1}^{PLB}, S_{T-1}^{IP})$
$H_{T-1} = +1^{b_l}$	:	$(\beta v - b_l) \cdot p_{T-1}^{*}(b_l \| S_{T-1}^{PLB}, S_{T-1}^{IP})$
$H_{T-1} = -1^A$	:	$\beta v - A_1$
$H_{T-1} = 0$	:	0

and his equilibrium strategies are:

$$H_{T-1}^{*BD}(\beta,[1110]\&[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}^{BD}\,|_{a_1,B=B_1}) \\ +1^{a_1} & \text{if } \beta \in [\beta_{1,T-1}^{BD}\,|_{a_1,B=B_1}, \beta_{2,T-1}^{BD}\,|_{a_1,a_2}) \\ +1^{a_2} & \text{if } \beta \in [\beta_{2,T-1}^{BD}\,|_{a_1,a_2}, \beta_{3,T-1}^{BD}\,|_{a_2,b_2}) \\ +1^{b_2} & \text{if } \beta \in [\beta_{3,T-1}^{BD}\,|_{a_2,b_2}, \beta_{4,T-1}^{BD}\,|_{b_1,b_2}) \\ +1^{b_1} & \text{if } \beta \in [\beta_{4,T-1}^{BD}\,|_{b_1,b_2}, \beta_{5,T-1}^{BD}\,|_{A=A_1,b_1}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}^{BD}\,|_{A=A_1,b_1}, \overline{\beta}] \end{cases}$$

It follows that, conditional on a BD arriving at $T-1$, the execution probability of the limit order posted at A_2 is:

$$p_{T-2}^{*BD}(A_2\,|[1110]\&[0]\,) = \frac{\overline{\beta}-\beta_{5,T-1}^{BD}\big|_{A=A_1,b_1}}{\overline{\beta}-\underline{\beta}} \cdot p_{T-1}^{*}(A)\,|_{A=A_2}$$

We compute the total execution probability of the limit order posted at A_2 at $T-2$ as the weighted average of the two conditional probabilities:

$$p_{T-2}^{*}(A_2\,|[0110]\&[0]\,) = \alpha p_{T-2}^{*BD}(A_2\,|[1110]\&[0]\,) + (1-\alpha) p_{T-2}^{*RT}(A_2\,|[1110]\&[0]\,)$$

Similarly, we compute the equilibrium strategies for all the other possible states of the book at $T-1$ and obtain the execution probabilities of the different order types available at $T-2$.

If a RT arrives at $T-2$, his strategy space is $\{-1^B, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$. His payoffs are:

$H_{T-2} = -1^B$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_k}$	:	$(A_k - \beta v) \cdot p^*_{T-2}(A_k \mid [0110]\&[0])$
$H_{T-2} = +1^{B_k}$	:	$(\beta v - B_k) \cdot p^*_{T-2}(B_k \mid [0110]\&[0])$
$H_{T-2} = -1^A$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

where, for example, $p^*_{T-2}(A_1|[0110]\&[0]) = \alpha p^{*RT}_{T-2}(A_1\,|[0210]\&[0]\,)+(1-\alpha)p^{*BD}_{T-2}(A_1\,|[0210]\&[0]\,)$. By comparing these payoffs, we obtain:

$$H^{*RT}_{T-2}(\beta,[0110]\&[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}\,, \beta^{RT}_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta^{RT}_{3,T-2}\,|_{A_2,B_2}\,, \beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}\,) \\ -1^A & \text{if } \beta \in [\beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}\,, \overline{\beta}] \end{cases}$$

If instead a BD arrives at $T-2$, he also selects a trading venue. Thus his strategy space is $\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. His payoffs are:

$H_{T-2} = -1^B$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_k}$	:	$(A_k - \beta v) \cdot p^*_{T-2}(A_k \mid [0110]\&[0])$
$H_{T-2} = +1^{a_l}$	:	$(a_l - \beta v) \cdot p^*_{T-2}(a_l \mid [0110]\&[0])$
$H_{T-2} = +1^{b_l}$	:	$(\beta v - b_l) \cdot p^*_{T-2}(b_l \mid [0110]\&[0])$
$H_{T-2} = +1^{B_k}$	:	$(\beta v - B_k) \cdot p^*_{T-2}(B_i \mid [0110]\&[0])$
$H_{T-2} = -1^A$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

where for example $p^*_{T-2}(a_l|[0110]\&[0]) = \alpha p^{*BD}_{T-2}(a_l\,|[0110]\&Q^{a_l}=1)+(1-\alpha)p^{*RT}_{T-2}(a_l\,|[0110]\&Q^{a_l}=1)$. If he submits a limit order to the PLB, his order's execution probability is the same as the one of RT. His equilibrium strategies are:

$$H^{*BD}_{T-2}(\beta,[0110]\&[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}\,|_{a_1,B=B_1}\,) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}\,|_{a_1,B=B_1}\,, \beta^{BD}_{2,T-2}\,|_{a_1,a_2}\,) \\ +1^{a_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}\,|_{a_1,a_2}\,, \beta^{BD}_{2,T-2}\,|_{a_2,A_2}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}\,|_{a_2,A_2}\,, \beta^{BD}_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta^{BD}_{3,T-2}\,|_{A_2,B_2}\,, \beta^{BD}_{4,T-2}\,|_{B_2,b_2}\,) \\ +1^{b_2} & \text{if } \beta \in [\beta^{BD}_{4,T-2}\,|_{B_2,b_2}\,, \beta^{BD}_{4,T-2}\,|_{b_2,b_1}\,) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{4,T-2}\,|_{b_2,b_1}\,, \beta^{BD}_{5,T-2}\,|_{A=A_1,b_1}\,) \\ -1^A & \text{if } \beta \in [\beta^{BD}_{5,T-2}\,|_{A=A_1,b_1}\,, \overline{\beta}] \end{cases}$$

At $T-2$ expected volume on the PLB is:

$$VL^{PLB}_{T-2}([0110]\&[0]) = \alpha[\Pr(H^{*BD}_{T-2} = -1^{A_1}) + \Pr(H^{*BD}_{T-2} = -1^{B_1})] + (1-\alpha)[\Pr(H^{*RT}_{T-2} = -1^{A_1}) + \Pr(H^{*RT}_{T-2} = -1^{B_1})]$$
$$= \alpha\left(\frac{\beta^{BD}_{1,T-2}\big|_{a_1,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\beta^{BD}_{5,T-2}\big|_{A=A_1,b_1}}{\overline{\beta}-\underline{\beta}}\right) + (1-\alpha)\left(\frac{\beta^{RT}_{1,T-2}\big|_{A_2,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\beta^{RT}_{5,T-2}\big|_{A=A_1,B_2}}{\overline{\beta}-\underline{\beta}}\right)$$

Because there is no volume executed in the IP, we obtain that:

$$LP^{PLB}_{T-2}([0110]\&[0]) = 1 - VL^{PLB}_{T-2}([0110]\&[0]) - LP^{IP}_{T-2}([0110]\&[0])$$

We also compute the other market indicators for the PLB:

$$\begin{aligned}
DPI^{PLB}_{T-2}([0110]\&[0]) &= 2 - VL^{PLB}_{T-2}([0110]\&[0]) \\
DPT^{PLB}_{T-2}([0110]\&[0]) &= 2 + LP^{PLB}_{T-2}([0110]\&[0]) - VL^{PLB}_{T-2}([0110]\&[0]) \\
SP^{PLB}_{T-2}([0110]\&[0]) &= \tau\{LP^{PLB}_{T-2}([0110]\&[0]) + LP^{IP}_{T-2}([0110]\&[0])\} + 2\tau VL^{PLB}_{T-2}([0110]\&[0])
\end{aligned}$$

(1.2) When the opening book at $T-2$ is $[0110]\&[1]$, we derive the equilibrium strategies for both a RT and a BD in a similar way:

$$H^{*RT}_{T-2}(\beta,[0110]\&[1]) = \begin{cases} -1^{b} & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-2}\,|_{B=b_1}) \\ 0 & \text{if } \beta \in [\beta^{RT}_{1,T-2}\,|_{B=b_1}, \beta^{RT}_{5,T-2}\,|_{A=a_1}) \\ -1^{a} & \text{if } \beta \in [\beta^{RT}_{5,T-2}\,|_{A=a_1}, \overline{\beta}] \end{cases}$$

$$H^{*BD}_{T-2}(\beta,[0110]\&[1]) = \begin{cases} -1^{b} & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}\,|_{a_1,B=b_1}) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}\,|_{a_1,B=b_1}, \beta^{BD}_{3,T-2}\,|_{a_1,b_1}) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{3,T-2}\,|_{a_1,b_1}, \beta^{BD}_{5,T-2}\,|_{A=a_1,b_1}) \\ -1^{a} & \text{if } \beta \in [\beta^{BD}_{5,T-2}\,|_{A=a_1,b_1}, \overline{\beta}] \end{cases}$$

We also compute the corresponding market quality indicators: $VL^{PLB}_{T-2}([0110]\&[1])$, $DPI^{PLB}_{T-2}([0110]\&[1])$, $DPT^{PLB}_{T-2}([0110]\&[1])$, and $SP^{PLB}_{T-2}([0110]\&[1])$.

(1.3) The market quality indicators for the transparent IP case (T) are equal to the average of those obtained in (1.1) and (1.2). For example:

$$VL^{PLB,T}_{T-2} = \frac{VL^{PLB}_{T-2}([0110]\&[0])}{2} + \frac{VL^{PLB}_{T-2}([0110]\&[1])}{2}$$

We compare these market indicators with those obtained for the single market model. We first analyze separately the two cases, and start with $[0110]\&[0]$. Without broker-dealers ($\alpha = 0$), the PLB&IP model converges to the single market model. When α is positive, instead, limit orders submitted at $T-2$ have a lower execution probability in the PLB&IP

model, as they are more frequently undercut by BD. Therefore both BD and RT submit market orders with a higher probability. From Proposition 1 we know that:

$$VL_{T-2}^{LM} = \frac{\widehat{\beta}_{1,T-2}^{LM}\big|_{A_k,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{LM}\big|_{A=A_1,B_k}}{\overline{\beta}-\underline{\beta}}$$

So we obtain that:

$$\begin{aligned} \beta_{1,T-2}^{RT}\,|_{A_2,B=B_1} &> \beta_{1,T-2}^{BD}\,|_{a_1,B=B_1} > \widehat{\beta}_{1,T-2}^{LM}\,|_{A_k,B=B_1} \\ \beta_{5,T-2}^{RT}\,|_{A=A_1,B_2} &< \beta_{5,T-2}^{BD}\,|_{A=A_1,b_1} < \widehat{\beta}_{5,T-2}^{LM}\,|_{A=A_1,B_k} \end{aligned}$$

This implies that $VL_{T-2}^{PLB}([0110]\&[0]) > VL_{T-2}^{LM}$. It is straightforward to show that:

$$\begin{aligned} LP_{T-2}^{PLB}([0110]\&[0]) &\leq 1 - VL_{T-2}^{PLB}([0110]\&[0]) < 1 - VL_{T-2}^{LM} = LP_{T-2}^{LM} \\ DPI_{T-2}^{PLB}([0110]\&[0]) &> DPI_{T-2}^{LM} \\ DPT_{T-2}^{PLB}([0110]\&[0]) &> DPT_{T-2}^{LM} \\ SP_{T-2}^{PLB}([0110]\&[0]) &< SP_{T-2}^{LM} \end{aligned}$$

In the case with [0110]&[1] all incoming market orders at $T-2$ are executed in the IP, so that $VL_{T-2}^{PLB}([0110]\&[1]) = 0$. Furthermore, limit orders are only posted to the IP as on the PLB they have a zero execution probability. It follows that $LP_{T-2}^{PLB}([0110]\&[1]) = 0$ and $SP_{T-2}^{PLB}([0110]\&[1]) = \tau$.

Averaging over the two cases, [0110]&[0] and [0110]&[1], we obtain $VL_{T-2}^{PLB,T} < VL_{T-2}^{LM}$, $LP_{T-2}^{PLB,T} < LP_{T-2}^{LM}$, $DPI_{T-2}^{PLB,T} > DPT_{T-2}^{LM}$, $SP_{T-2}^{PLB,T} < SP_{T-2}^{LM}$.

2) Liquid stocks, opaque IP

In this proof we only highlight the differences with the transparent IP framework, therefore we focus only on RT. Consider again the PLB that opens [1110] at $T-1$. If a RT arrives, he will infer the state of the IP from the observed PLB. The RT knows that $H_{T-2} = +1^{A_2}$ is never an equilibrium strategy for a BD if the state of the IP is [1]. So he will update the probabilities associated with $IP = [0]$ and $IP = [1]$ from 1/2 to:

$$\Pr\{S_{T-2}^{IP} = [0]\,\big|S_{T-2}^{PLB} = [1110]\} = \frac{\frac{1}{2}[\alpha \Pr(H_{T-2}^{*RT} = +1^{A_2}) + (1-\alpha)\Pr(H_{T-2}^{*BD} = +1^{A_2})]}{\frac{1}{2}\alpha \Pr(H_{T-2}^{*RT} = +1^{A_2}) + (1-\alpha)\Pr(H_{T-2}^{*BD} = +1^{A_2})} > \frac{1}{2}$$

$$\Pr\{S_{T-2}^{IP} = [1]\,\big|S_{T-2}^{PLB} = [1110]\} = \frac{\frac{1}{2}(1-\alpha)\Pr(H_{T-2}^{*BD} = +1^{A_2})}{\frac{1}{2}\alpha \Pr(H_{T-2}^{*RT} = +1^{A_2}) + (1-\alpha)\Pr(H_{T-2}^{*BD} = +1^{A_2})} < \frac{1}{2}$$

To select his optimal trading strategy, he will then compute his expected payoffs using the Bayesian updated probabilities. For example:

$$H_{T-1} = -1^B : B_1 \Pr\{S_{T-2}^{IP} = [0] \,|[1110]\} + b_1 \Pr\{S_{T-2}^{IP} = [1] \,|[1110]\} - \beta v$$

A similar reasoning applies to a RT arriving at T, his payoffs are:

$$\boxed{\begin{array}{lcl} H_T = -1^B & : & E(B\,|S_{T-1}, h_{T-1}, h_{T-2}) - \beta v \\ H_T = -1^A & : & \beta v - E(A\,|S_{T-1}, h_{T-1}, h_{T-2}) \\ H_T = 0 & : & 0 \end{array}}$$

where $E(B\,|S_{T-1}, h_{T-1}, h_{T-2})$ is for example the expected execution price of a market sell order given the actual state of the book and the order submissions observed in the previous periods. Differently from the transparent IP framework, here RT and BD strategies are not the same at T. Finally, to analyze the changes in the PLB after the introduction of an IP, we compare the market indicators with those computed both in the proof of Proposition 1 and the case with a transparent IP:

$$\begin{array}{rcl} DPI_{T-2}^{PLB,T} & > & DPI_{T-2}^{PLB,O} > DPI_{T-2}^{LM} \\ DPT_{T-2}^{PLB,T} & > & DPT_{T-2}^{PLB,O} > DPT_{T-2}^{LM} \\ SP_{T-2}^{PLB,T} & < & SP_{T-2}^{PLB,O} < SP_{T-2}^{LM} \end{array}$$

■

E Proof of Proposition 3

Proof. Following Eq. (13) and (14), we define the change in welfare as:

$$\begin{aligned} \sum_t E(W_t - \overline{W}_t) = & \sum_t (-1) \cdot \left[\sum_i |i - v| \cdot \int_{\beta \in \{\beta : H_t = -1^i | S_{t-1}\}} f(\beta) d\beta \right. \\ & \left. + \sum_i |i - v| \cdot \int_{\beta \in \{\beta : H_t = +1^i | S_{t-1}\}} p_t^*(i\,|S_t) \cdot f(\beta) d\beta \right] \end{aligned}$$

where, for example, in the single market model $|i - v|$ takes values $\{\frac{3\tau}{6}, \frac{9\tau}{6}\}$ for the LM and $\{\frac{\tau}{6}, \frac{3\tau}{6}, \frac{5\tau}{6}, \frac{7\tau}{6}, \frac{9\tau}{6}\}$ for the SM. The results in Tables 9 and 10 come from a comparison of both the equilibrium welfare and the change in welfare with respect to the FB for different values of τ and v. ■

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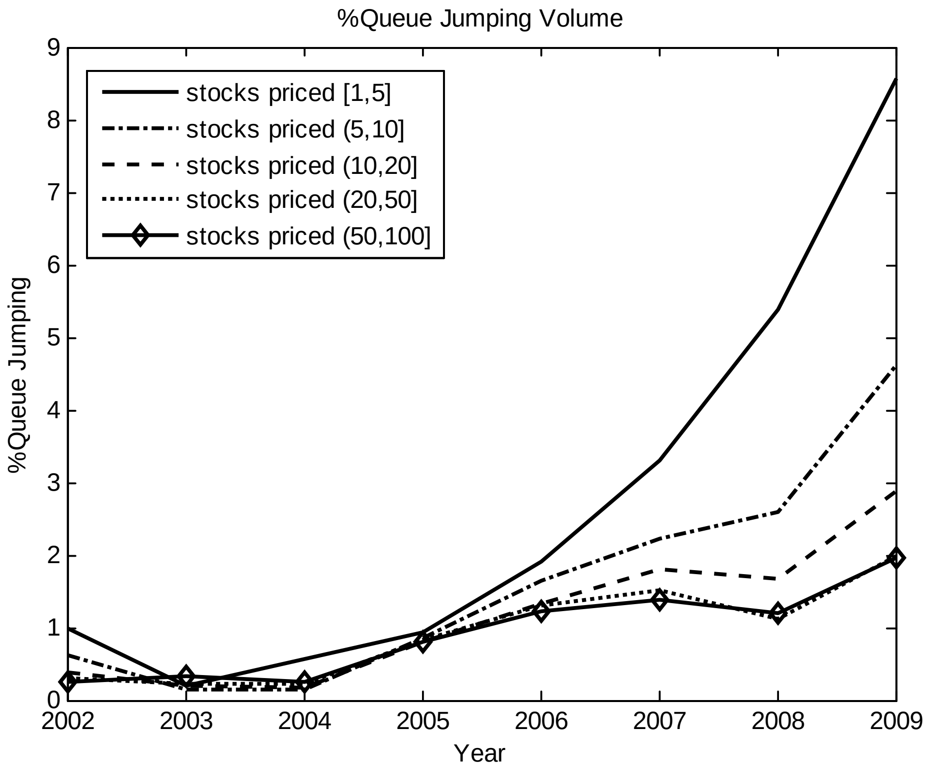

Figure 1: **NASDAQ Stocks - Queue Jumping.** This Figure shows the evolution of sub-penny trading over the past 10 years for different priced NASDAQ stocks. %Queue Jumping indicates the percentage of volume executed by queue jumping over total volume. We use weekly statistics from Delassus and Tyc (2010), computed from Thomson Reuters tick-by-tick historical.

Table 1: US Dark Pool Volume (% Consolidated US Equity Volume). This Table shows the recent evolution of U.S. dark pool volume. Dark pools are classified as Internalization Pools, Public Crossing Networks and Consortium-based Pools. Source: Rosenblatt Securities Inc.

	Time	Jun.08	Aug. 08	May 09	Jun.09	Jul.09	Aug. 09	May 10	Jun.10	Jul. 10	Aug. 10
Internalized Pool (Broker-sponsored) (IP)	Credit Suisse Crossfinder	.81	.88	1.31	1.42	1.55	1.55	2.10	2.08	2.11	2.10
	Goldman Sachs Sigma X	1.37	3.00	1.09	1.19	1.29	1.44	1.57	1.47	1.56	1.55
	Knight Link	.98	1.17	1.23	1.05	1.08	1.23	1.62	1.71	1.44	1.38
	Getco Execution Service	.41	.64	1.17	1.06	1.08	1.11	1.02	1.03	1.16	1.28
	Morgan Stanley MS Pool	.26	.31	.44	.53	.50	.48	.52	.58	.61	.78
	Barclays LX	.47	-	.18	.20	.34	.25	.54	.57	.64	.67
	UBS Pin	.29	.58	.39	.42	.45	.43	.37	.37	.38	.41
	Citi Match	.43	.50	.33	.35	.38	.37	.35	.31	.38	.38
	%IP	5.02	7.08	6.14	6.22	6.67	6.86	8.09	8.12	8.28	8.55
Public Crossing Networks (PC)	ITG Posit	.32	.29	.20	.21	.24	.24	.25	.34	.39	.44
	Instinet CBX	.20	.27	.20	.19	.23	.24	.43	.41	.42	.41
	Liquidnet	.39	.41	.27	.26	.33	.28	.22	.21	.24	.25
	Convergex Millennium	.31	.30	.15	.16	.17	.16	.16	.21	.22	.18
	Convergex Vortex	.05	.06	.06	.08	.06	.05	.08	.07	.09	.10
	Pipeline Trading	.20	.15	.11	.09	.09	.12	.09	.08	.09	.09
	% PC	1.47	1.48	.99	.99	1.12	1.09	1.23	1.32	1.45	1.47
Consortium-based Pools (CBP)	Level	.40	.56	.48	.50	.58	.10	.73	.79	.86	.93
	Bid Trading	.13	.10	.10	.10	.14	.53	.21	.22	.27	.31
	%CBP	.53	.66	.58	.60	.72	.63	.94	1.01	1.13	1.24
	ALL	7.02	9.22	7.71	7.81	8.51	8.58	10.26	10.45	10.86	11.26

Table 2: Order Submission Strategy Space. This Table reports in column 3 the payoffs, $U(\cdot)$, of the order strategies H_t listed in column 1.

Strategy	H_t	$U(\cdot)$
Market Sell Order	-1^B	$B - \beta v$
Limit Sell Order	1^{A_k}	$p_t^*(A_k^{N_{-k},N_k}\|S_t) \cdot (A_k - \beta v)$
No Trade	0	0
Limit Buy Order	1^{B_k}	$p_t^*(B_k^{M_{-k},M_k}\|S_t) \cdot (\beta v - B_k)$
Market Buy Order	-1^A	$\beta v - A$

Table 3: **Price Grid.** This Table shows the price grid for both the large tick market (LM) and the small tick market (SM), where v indicates the asset value and τ the tick size.

LM	**Price**	**SM**
A_2	$v + \frac{9}{6}\tau$	a_5
	$v + \frac{7}{6}\tau$	a_4
	$v + \frac{5}{6}\tau$	a_3
A_1	$v + \frac{3}{6}\tau$	a_2
	$v + \frac{1}{6}\tau$	a_1
	$v - \frac{1}{6}\tau$	b_1
B_1	$v - \frac{3}{6}\tau$	b_2
	$v - \frac{5}{6}\tau$	b_3
	$v - \frac{7}{6}\tau$	b_4
B_2	$v - \frac{9}{6}\tau$	b_5

Table 4: Tick Size Change ($\tau = 0.1$). This Table compares the large tick market (LM) with the small tick market (SM), and focuses on three cases that represent an illliquid, liquid, and very liquid state of the book respectively: (1) both books open empty at $T-2$, $S_{T-2} = [0000] = [00000\,00000]$ (2) LM opens with one share on the first level of the book (A_1, B_1), and SM opens with one share on the second level (a_2, b_2), $S_{T-2} = [0110] = [00010\,01000]$ (3) LM opens with two shares on the first level (A_1, B_1) and SM with two shares on the second level (a_2, b_2), $S_{T-2} = [0220] = [00020\,02000]$. The Table reports the following statistics for different asset values, $v = \{1, 5, 10, 50\}$: liquidity provision, i.e., probability of limit order submission (LP); trading volume, i.e., probability of market order submission (VL); spread, depth at the BBO and total depth. Columns 11-14 report the difference between the two markets (SM-LM).

		LM [$Tick\ Size = \tau$]				SM [$Tick\ Size = \frac{\tau}{3}$]				($\Delta \times 100$)			
S_{T-2}	Asset Price = v	1	5	10	50	1	5	10	50	1	5	10	50
Illiquid	Limit Order = LP	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
	Market Order=VL	.4140	.8260	.9048	.9794	.4414	.8290	.9058	.9795	2.74	.30	.10	.01
	Spread	.2751	.3000	.3000	.3000	.2763	.3000	.3000	.3000	.12	0	0	0
	Depth at BBO	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
	Total Depth	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
Liquid	Limit Order = LP	.1006	.0226	.0114	.0024	.2334	.0492	.0248	.0050	13.28	2.66	1.34	.26
	Market Order=VL	.8994	.9774	.9886	.9976	.7666	.9508	.9752	.9950	-13.28	-2.66	-1.34	-.26
	Spread	.1899	.1977	.1989	.1998	.1699	.1937	.1968	.1994	-2.00	-.40	-.21	-.04
	Depth at BBO	1.1006	1.0226	1.0115	1.0023	1.2334	1.0493	1.0248	1.0050	13.28	2.67	1.33	.27
	Total Depth	1.2012	1.0453	1.0230	1.0047	1.4671	1.0986	1.0496	1.0100	26.59	5.33	2.66	.53
Very Liquid	Limit Order = LP	0	0	0	0	.2334	.0492	.0248	.0050	23.34	4.92	2.48	.50
	Market Order=VL	.9500	.9900	.9950	.9990	.7666	.9508	.9752	.9950	-18.34	-3.92	-1.98	-.40
	Spread	.1000	.1000	.1000	.1000	.0922	.0984	.0992	.0998	-.78	-.16	-.08	-.02
	Depth at BBO	3.0500	3.0100	3.005	3.0010	3.0000	3.0000	3.0000	3.0000	-5.00	-1.00	-.50	-.10
	Total Depth	3.0500	3.0100	3.005	3.0010	3.4668	3.0986	3.0496	3.0100	41.68	8.86	4.46	.90

Table 5: Sub-penny Trading - Illiquid stock ($\tau = 0.1$, $\alpha = 10\%$). This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens empty, $S_{T-2} = [0000]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

		S_{T-2} = [0000]&[00000 00000] or [0000]&[00001 10000] with equal probabilities									
		$v=1$					$v=10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
LP (IP)			0	(0)	0	(0)		0	(0)	0	(0)
VL (PLB)		.4140	.2128	(-20.12)	.3544	(-5.96)	.9048	.4524	(-45.24)	.4753	(-42.95)
VL (IP)			.4696	(46.96)	.3852	(38.52)		.4964	(49.64)	.4799	(47.99)
No Trading		0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread (PLB)		.2751	.2887	(1.36)	.3000	(2.49)	.3000	.3000	(0)	.3000	(0)
Depth BBO (PLB)		.5860	.3176	(-26.84)	.2605	(-32.55)	.0952	.0512	(-4.40)	.0448	(-5.04)
Total Depth (PLB)		.5860	.3176	(-26.84)	.2605	(-32.55)	.0952	.0512	(-4.40)	.0448	(-5.04)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.1684	.1025	0	.1302	0	.0476	.0256	0	.0224	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	.1246	.0563	0	0	0	0	0	0	0	0
Order	a_1	-	-	0	-	0	-	-	0	-	0
Sub	b_1	-	-	0	-	0	-	-	0	-	0
Prob.	$B_1 = b_2$	.1246	.0563	0	0	0	0	0	0	0	0
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.1684	.1025	0	.1302	0	.0476	.0256	0	.0224	0

Table 6: Sub-penny Trading - Liquid stock ($\tau = 0.1$, $\alpha = 10\%$). This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens with one share on the first level of the book, $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

$S_{T-2} = [0110]\&[00000\,00000]$ or $[0110]\&[00001\,10000]$ with equal probabilities											
		$v=1$					$v=10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.1006	.0462	(-5.44)	.0499	(-5.07)	.0114	.0054	(-.60)	.0056	(-.58)
LP (IP)			.0122	(1.22)	.0120	(1.20)		.0011	(0.11)	.0011	(0.11)
VL (PLB)		.8994	.4429	(-45.65)	.4638	(-43.56)	.9886	.4936	(-49.50)	.4961	(-49.25)
VL (IP)			.4911	(49.11)	.4743	(47.43)		.4991	(49.91)	.4972	(49.72)
No Trading		0	.0076	(.76)	0	(0)	0	.0008	(.08)	0	(0)
Spread (PLB)		.1899	.1443	(-4.56)	.1464	(-4.35)	.1989	.1494	(-4.95)	.1496	(-4.93)
Depth BBO (PLB)		1.1006	1.5571	(45.65)	1.5362	(43.56)	1.0115	1.5064	(49.49)	1.5039	(49.24)
Total Depth (PLB)		1.2012	1.6034	(40.22)	1.5617	(36.05)	1.0230	1.5118	(48.88)	1.5040	(48.10)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.0503	.0231	0	.0249	0	.0057	.0027	0	.0028	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	0	0	.0005	0	.0006	0	0	.0000	0	.0000
Order	a_1	-	-	.0056	-	.0054	-	-	.0006	-	.0006
Sub	b_1	-	-	.0056	-	.0054	-	-	.0006	-	.0006
Prob.	$B_1 = b_2$	0	0	.0005	0	.0006	0	0	.0000	0	.0000
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.0503	.0231	0	.0249	0	.0057	.0027	0	.0028	0

Table 7: Sub-penny Trading - Liquid stock ($\tau = 0.1$, $\alpha = 20\%$). This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 20\%$. At $T-2$ the PLB opens with one share on the first level of the book, $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

$S_{T-2} = [0110]\&[00000\,00000]$ or $[0110]\&[00001\,10000]$ with equal probabilities											
		$v=1$					$v=10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.1006	.0422	(-5.84)	.0454	(-5.52)	.0114	.0050	(-.64)	.0052	(-.62)
LP (IP)			.0244	(2.44)	.0240	.0240		.0022	(0.22)	.0022	(0.22)
VL (PLB)		.8994	.4362	(-46.32)	.4548	(-44.46)	.9886	.4932	(-49.54)	.4952	(-49.34)
VL (IP)			.4904	(49.04)	.4758	.4758		.4990	(49.90)	.4974	(49.74)
No Trading		0	.0068	(.68)	0	(0)	0	.0006	(0.06)	.0	(0)
Spread (PLB)		.1899	.1436	(-4.63)	.1455	(-4.44)	.1989	.1493	(-4.96)	.1495	(-4.94)
Depth BBO (PLB)		1.1006	1.5639	(46.33)	1.5452	(44.46)	1.0115	1.5071	(49.56)	1.5048	(49.33)
Total Depth (PLB)		1.2012	1.6061	(40.49)	1.5690	(36.78)	1.0230	1.5121	(48.91)	1.5077	(48.47)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.0503	.0211	0	.0227	0	.0057	.0025	0	.0026	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	0	0	.0009	0	.0011	0	0	.0000	0	.0000
Order	a_1	-	-	.0113	-	.0109	-	-	.0011	-	.0011
Sub	b_1	-	-	.0113	-	.0109	-	-	.0011	-	.0011
Prob.	$B_1 = b_2$	0	0	.0009	0	.0011	0	0	.0000	0	.0000
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.0503	.0211	0	.0227	0	.0057	.0025	0	.0026	0

Table 8: Sub-penny Trading - Illiquid and Liquid stocks - Small Tick Size ($\tau = \frac{0.1}{3}$, $\alpha = 10\%$). This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens either empty (Panel A), $S_{T-2} = [0000]$, or with one share on the first level of the book (Panel B), $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [0000][00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. For both the PLB and the IP liquidity provision (LP), i.e., probability of limit order submission, and trading volume (VL), i.e., probability of market order submission, are reported. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

Panel A: $S_{T-2} = [0000]\&[00000\,00000]$ or $[0000]\&[00001\,10000]$ with equal probabilities										
	$v = 1$					$v = 10$				
	PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
LP (IP)		0	(0)	0	(0)		0	(0)	0	(0)
VL (PLB)	.4140	.2128	(-20.12)	.3544	(-5.96)	.9048	.4524	(-45.24)	.4753	(-42.95)
VL (IP)		.4696	(46.96)	.3852	(38.52)		.4964	(49.64)	.4799	(47.99)
No Trading	0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread (PLB)	.2751	.2887	(1.36)	.3000	(2.49)	.3000	.3000	(0)	.3000	(0)
Depth BBO (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
Total Depth (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
Panel B: $S_{T-2} = [0110]\&[00000\,00000]$ or $[0110]\&[00001\,10000]$ with equal probabilities										
	$v = 1$					$v = 10$				
	PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)	.1006	.0462	(-5.44)	.0499	(-5.07)	.0114	.0054	(-.60)	.0056	(-.58)
LP (IP)		.0122	(1.22)	.0120	(1.20)		.0012	(.12)	.0012	(.12)
VL (PLB)	.8994	.4429	(-45.65)	.4638	(-43.56)	.9886	.4936	(-49.50)	.4961	(-49.25)
VL (IP)		.4911	(49.11)	.4743	(47.43)		.4990	(49.90)	.4972	(49.72)
No Trading	0	.0076	(.76)	0	(0)	0	.0008	(.08)	0	(0)
Spread (PLB)	.1899	.1443	(-4.56)	.1464	(-4.35)	.1989	.1494	(-4.95)	.1496	(-4.93)
Depth BBO (PLB)	1.1006	1.5571	(45.65)	1.5362	(43.56)	1.0114	1.5064	(49.50)	1.5039	(49.25)
Total Depth (PLB)	1.2012	1.6033	(40.21)	1.5861	(38.49)	1.0228	1.5118	(48.90)	1.5040	(48.67)

Table 9: Welfare - Single Market Model ($\tau = 0.1$, $\alpha = 20\%$)**.** This Table reports the expected gains from trade over the three trading periods for both the single market model (PLB) and the market without frictions, i.e. the first best (FB). Results are reported for both the large tick market (LM), and the small tick market (SM); columns 10 to 13 report the difference in welfare between the two trading protocols (SM-LM). In addition, results are reported for four different values of the asset (v), and for three different initial states of the book, respectively with 0, 1, and 2 shares on the first level of LM (A_1, B_1) and the second level of SM (a_2, b_2).

	LM [$Tick\ Size = \tau$]				SM [$Tick\ Size = \frac{\tau}{3}$]				($\Delta \times 100$)			
v	1	5	10	50	1	5	10	50	1	5	10	50
FB	1.5	7.5	15	75	1.5	7.5	15	75				
$S_{T-2} = [0000] = [00000\,00000]$												
PLB	1.1761	7.0841	14.5684	74.5539	1.1723	7.0837	14.5683	74.5539	-.38	-.04	-.01	-.00
$\frac{PLB-FB}{FB}$	-.2159	-.0555	-.0288	-.0059	-.2185	-.0555	-.0288	-.0059	-.38	-.04	-.01	-.00
$S_{T-2} = [0110] = [00010\,01000]$												
PLB	1.2616	7.2337	14.7295	74.7259	1.2881	7.2395	14.7324	74.7265	2.65	.58	.29	.06
$\frac{PLB-FB}{FB}$	-.1589	-.0355	-.0180	-.0037	-.1413	-.0347	-.0178	-.0036	2.65	.58	.29	.06
$S_{T-2} = [0220] = [00020\,02000]$												
PLB	1.3335	7.3268	14.8258	74.8251	1.3535	7.3313	14.8283	74.8257	2.00	.45	.25	.06
$\frac{PLB-FB}{FB}$	-.1110	-.0231	-.0116	-.0023	-.0977	-.0225	-.0114	-.0023	2.00	.45	.25	.06

Table 10: Welfare - PLB&IP Market ($\tau = \{0.1, \frac{0.1}{3}\}$, $\alpha = 10\%$)**.** This Table reports the expected gains from trade over the three trading periods for both the single market model (PLB) and the model in which the PLB competes with the internalization pool (PLB&IP). At $T-2$ the PLB opens either empty (Panel A), $S_{T-2} = [0000]$, or with one share on the first level of the book (Panel B), $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [0000][00000 00000], or with one unit on the first level of the book, [00001 10000]. Welfare in the PLB&IP framework is computed for the two categories of agents - broker-dealers (BD) and regular traders (RT) -, and for the two protocols in which the IP is either transparent (PLB&IP_tra) or opaque (PLB&IP_op) to regular traders. The Table also reports in square brackets the percentage difference with the welfare computed for the PLB in the single market model. In addition, results are reported for two different values of the asset, $v = \{1, 10\}$.

Panel A: S_{T-2} = [0000]&[00000 00000] or [0000]&[00001 10000] with equal probabilities $\left[\frac{\Delta}{PLB}\times 100\right]$								
v	τ	PLB	PLB&IP_tra	BD	RT	PLB&IP_op	BD	RT
1	0.1	1.1761	1.3555 [15.2538]	1.3566 [15.3473]	1.3554 [15.2453]	1.2222 [3.9197]	1.2570 [6.8787]	1.2183 [3.5881]
10	0.1	14.5684	15.9143 [9.2385]	15.9143 [9.2385]	15.9143 [9.2385]	14.6165 [.3302]	14.6797 [.7640]	14.6095 [.2821]
1	$\frac{0.1}{3}$	1.3673	1.5199 [11.1607]	1.5200 [11.1680]	1.5199 [11.1607]	1.3791 [.8630]	1.4021 [2.5452]	1.3766 [.6802]
10	$\frac{0.1}{3}$	14.8522	16.1358 [8.6425]	16.1358 [8.6425]	16.1358 [8.6425]	14.8700 [.1198]	14.8904 [.2572]	14.8677 [.1044]
Panel B: S_{T-2} = [0110]&[00000 00000] or [0110]&[00001 10000] with equal probabilities $\left[\frac{\Delta}{PLB}\times 100\right]$								
v	τ	PLB	PLB&IP_tra	BD	RT	PLB&IP_op	BD	RT
1	0.1	1.2616	1.3276 [5.2315]	1.3315 [5.5406]	1.3272 [5.1997]	1.3365 [5.9369]	1.3305 [5.4613]	1.3372 [5.9924]
10	0.1	14.7295	14.8071 [.5268]	14.8075 [.5295]	14.8070 [.5262]	14.8195 [.6110]	14.8072 [.5275]	14.8209 [.6205]
1	$\frac{0.1}{3}$	1.4130	1.4376 [1.7410]	1.4381 [1.7764]	1.4376 [1.7410]	1.4415 [2.0170]	1.4379 [1.7622]	1.4419 [2.0453]
10	$\frac{0.1}{3}$	14.9088	14.9351 [.1764]	14.9351 [.1764]	14.9351 [.1764]	14.9393 [.2046]	14.9351 [.1764]	14.9398 [.2079]