Notes, Comments, and Letters to the Editor

Mutual absolute continuity of multiple priors

Larry G. Epstein[a], Massimo Marinacci[b],*

[a]*Department of Economics, University of Rochester, USA*
[b]*Dipartimento di Statistica e Matematica Applicata and Collegio Carlo Alberto, Università di Torino, Italy*

Abstract

This note provides a behavioral characterization of mutually absolutely continuous multiple priors.
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1. Introduction

Beginning with Epstein and Schneider [2], in recent years there has been a growing interest in dynamic versions of the multiple-priors (MP) model of Gilboa and Schmeidler [4], motivated by the importance of dynamic considerations in many economic applications. In these dynamic versions of the MP model a very convenient property that is often assumed is mutual absolute continuity of the priors, that is, their agreement on which events are null. This agreement makes possible dynamically consistent Bayesian updating of the priors, without having to deal with updating on events that only some priors regard as null, something that greatly simplifies the analysis of these dynamic MP models.[1]

Here we provide a behavioral characterization of this agreement property of the priors. Interestingly, the relevant behavioral condition turns out to be the translation into the Savage framework of a condition introduced by Kreps [5] in his seminal paper on menu choices.

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* Corresponding author.

E-mail addresses: lepn@troi.cc.rochester.edu (L.G. Epstein), massimo.marinacci@unito.it (M. Marinacci).

[1] Similar considerations apply to the closely related topic of dynamic coherent risk measures (see [1]).

2. Setting

We use a standard Savage-style setting. Throughout, Σ is an event algebra in a state space S. Subsets of S are understood to be in Σ even where not stated explicitly.

We denote by Δ the set of all priors, that is, the set of all finitely additive probability measures P on Σ. Two such measures P_1 and P_2 are *mutually absolutely continuous* when, for all $E \in \Sigma$, $P_1(E) = 0$ if and only if $P_2(E) = 0$.

We denote by X the space of *consequences*. An *act* is a map $f : S \to X$ and it is *simple* when it is finite valued; L_0 denotes the set of all simple Σ-measurable acts.

The decision maker has a preference relation $\succsim$ on L_0, which in turn induces a preference over X, obtained in the standard way by identifying consequences with constant acts.

Say that an event E is null (resp. universal) if $xEy \sim y$ (resp. $xEy \sim x$) for all $x, y \in X$ such that $x \succ y$. Here xEy is the act that pays x if E obtains, and y otherwise.

A binary relation $\succsim$ on L_0 is a MP preference relation if there exists a utility index $u : X \to \mathbb{R}$, and a non-empty weak*-compact set $C \subseteq \Delta$ such that $\succsim$ is represented by the preference functional $V : L_0 \to \mathbb{R}$ defined by

$$V(f) = \min_{P \in C} \int u(f(s)) \, dP(s) \quad \forall f \in L_0. \tag{1}$$

Subjective expected utility (SEU) preferences are the special case of singleton sets of priors C.

For MP preferences, an event E is null iff $P(E) = 0$ for *some* $P \in C$, while E is universal iff $P(E) = 1$ for *all* $P \in C$. Hence, the complement of a universal event is null, but the converse is in general false.

3. The axiom and results

Given any two consequences x and y, let

$$x \vee y = \begin{cases} x & \text{if } x \succsim y, \\ y & \text{otherwise,} \end{cases}$$

and given any two acts f and g, define the act $f \vee g$ by

$$(f \vee g)(s) = f(s) \vee g(s) \quad \forall s \in S.$$

The next axiom translates the Kreps axiom (see [5, Eq. (1.5)]) into our setting.

Axiom 1 (*Generalized Kreps (GK)*). For all $f, f', g \in L_0$,

$$f \sim f \vee f' \implies f \vee g \sim (f \vee g) \vee f'.$$

In every state, the act $f \vee f'$ gives the better of the two outcomes associated with f and f'. Thus say that $f \vee f'$ weakly improves f in 'direction' f'. GK requires that if an improvement of f in direction f' has no value, then the same must be true for an improvement in direction f' of any act (here $f \vee g$) that improves f. The scope of this seemingly innocuous axiom will be fully understood in light of the results to follow.

First, observe that SEU preferences always satisfy the GK axiom: $f \sim f \vee f' \Longrightarrow$

$$\int u(f)\, dP = \int u(f \vee f')\, dP = \int (u(f) \vee u(f'))\, dP \Longrightarrow$$
$$u(f'(s)) \leqslant u(f(s)) \quad P\text{-a.e.} \Longrightarrow$$
$$u(f'(s)) \vee u(g(s)) \leqslant u(f(s)) \vee u(g(s)) \quad P\text{-a.e.} \Longrightarrow$$
$$f \vee g \sim (f \vee g) \vee f'.$$

In general, MP preferences do not satisfy GK, as the next example shows.

Example 2. Let $S = \{1, 2\}$ and $X \subseteq \mathbb{R}$. Consider acts $f = (x_1, x_2)$, $f' = (x_1', x_2')$ and $g = (y_1, y_2)$, where

$$x_1' < y_2 < x_1 < x_2 < x_2' < y_1.$$

We have $f \vee f' = (x_1, x_2')$, $f \vee g = (y_1, x_2)$, and $(f \vee g) \vee f' = (y_1, x_2')$.

Consider the case of complete ignorance and risk neutrality: $u(x) = x$ and $C = \{(p, 1 - p) : p \in [0, 1]\}$. Then, contrary to GK,

$$V(f) = V(f \vee f') = x_1 \quad \text{and} \quad V(f \vee g) = x_2 < x_2' = V(f \vee g \vee f').$$

The following result characterizes MP preferences that satisfy GK, and it shows that the latter is the sought-after behavioral characterization of the mutual absolute continuity of priors.

Theorem 3. *For any MP preference $\succsim$, the following properties are equivalent*:
(i) *$\succsim$ satisfies GK*;
(ii) *all measures in C are mutually absolutely continuous*;
(iii) *for every event E, E is null iff its complement E^c is universal*.

Proof. The equivalence of (ii) and (iii) is easily established. We prove that (i) and (ii) are equivalent.

(ii) implies (i): Assume that all measures in C are mutually absolutely continuous. Suppose that $f \sim f \vee f'$. Then there exists $P^* \in C$ such that

$$\min_{P \in C} \int u(f \vee f')\, dP = \min_{P \in C} \int (u(f) \vee u(f'))\, dP = \int (u(f) \vee u(f'))\, dP^*$$

$$\geqslant \int u(f)\, dP^* \geqslant \min_{P \in C} \int u(f)\, dP = \min_{P \in C} \int u(f \vee f')\, dP.$$

Hence,

$$\int u(f)\, dP^* = \int (u(f) \vee u(f'))\, dP^* \Longrightarrow$$
$$u(f'(s)) \leqslant u(f(s)) \quad P^*\text{-a.e.} \Longrightarrow$$
$$u(f'(s)) \vee u(g(s)) \leqslant u(f(s)) \vee u(g(s)) \quad P^*\text{-a.e.} \Longrightarrow$$
$$u(f'(s)) \vee u(g(s)) \leqslant u(f(s)) \vee u(g(s)) \quad P^*\text{-a.e.} \ \forall P \in C \Longrightarrow$$
$$u(f'(s)) \vee u(g(s)) \vee u(f(s)) = u(f(s)) \vee u(g(s)) \quad P^*\text{-a.e.} \ \forall P \in C \Longrightarrow$$
$$f \vee g \sim (f \vee g) \vee f'.$$

(i) implies (ii): Assume that $\succsim$ satisfies GK but that not all measures in C are mutually absolutely continuous. Then there exist an event E and $P_1, P_2 \in C$ such that $P_1(E) = 0 < P_2(E)$. Let $\overline{P} \in C$ be such that $\overline{P}(E^c) \leqslant P(E^c)$ for all $P \in C$. Then $\overline{P}(E) \geqslant P_2(E) > 0$, and so $\overline{P}(E^c) < 1$.

Wlog let x and y be outcomes such that $u(x) = 1$ and $u(y) = 0$. Since $(xEy) \vee y = xEy$, we have $V(xEy \vee y) = V(xEy) = P_1(E) = 0$. Hence, $xEy \vee y \sim y$. On the other hand, $xEy \vee y \vee xE^c y = x$ and $y \vee xE^c y = xE^c y$, so that

$$V(xEy \vee y \vee xE^c y) = 1 > \overline{P}(E^c) = V(y \vee xE^c y),$$

and $xEy \vee y \vee xE^c y \succ y \vee xE^c y$, contradicting GK. $\square$

Remark 4. Partial versions of Theorem 3 holds for more general classes of preferences. For example, (i) implies (iii) for the biseparable preferences of [3], while (ii) implies (i) for the variational preferences of [6] (in this case condition (ii) becomes the mutual absolute continuity of the probabilities belonging to the effective domain of the function c used in their representation).

It is natural to understand GK as follows: suppose that $f \sim f \vee f'$. Then it must be that the set of states where $f'(s)$ is strictly preferred to $f(s)$ is "impossible," and therefore that the event $\{s : f(s) \succsim f'(s)\}$ is "certain." Then the larger event $\{s : f(s) \vee g(s) \succsim f'(s) \vee g(s)\}$ must also be "certain." Conclude that $f \vee g \sim f \vee g \vee f'$, as required by GK. The next result makes precise this intuitive connection between GK, "impossibility," and "certainty," at least for MP preferences.

Theorem 5. *Let $\succsim$ be a MP preference. Given any f and f' in L_0, consider the following properties:*
  (i) $f \sim f \vee f'$;
 (ii) *the event $\{s : f'(s) \succ f(s)\}$ is null;*
(iii) *the event $\{s : f(s) \succsim f'(s)\}$ is universal.*
  Then (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii)*; and the three properties are equivalent for every f and f' in L_0 iff $\succsim$ satisfies GK.*

Proof. (i) implies (ii): If $f \sim f \vee g$, then there is $P^* \in C$ such that

$$\min_{P \in C} \int u(f)\, dP = \min_{P \in C} \int u(f \vee f')\, dP = \int u(f \vee f')\, dP^* = \int \left(u(f) \vee u(f')\right) dP^*$$
$$\geqslant \int u(f)\, dP^* \geqslant \min_{P \in C} \int u(f)\, dP,$$

so that $\int u(f)\, dP^* = \int \left(u(f) \vee u(f')\right) dP^*$. Therefore, $P^* \left(\{s : f'(s) \succ f(s)\}\right) = 0$, which implies that $\{s : f'(s) \succ f(s)\}$ is null.
  (iii) implies (i): If $\{s : f(s) \succsim f'(s)\}$ is universal, then $P\left(\{s : f(s) \succsim g(s)\}\right) = 1\ \forall P \in C$, and $u(f'(s)) \leqslant u(f(s))$ P-a.e. $\forall P \in C$. It follows that $f \sim f \vee g$.
  Let $\succsim$ satisfy GK. By Theorem 3, $\{s : f'(s) \succ f(s)\}$ is null iff $\{s : f(s) \succsim f'(s)\}$ is universal. Thus, given GK, (ii) implies (iii) and (i)–(iii) are equivalent. Conversely, assume (i)–(iii) are equivalent for any f and f' in L_0. Then $f \sim f \vee f' \Longrightarrow P\left(\{s : f(s) \succsim f'(s)\}\right) = 1\ \forall P \in C = \Longrightarrow P\left(\{s : (f \vee g)(s) \succsim (f' \vee g)(s)\}\right) = 1\ \forall P \in C \Longrightarrow f \vee g \sim (f \vee g) \vee f'$. $\square$

We conclude with a characterization of GK for all monotone (not necessarily MP) preferences. Say that the preference $\succsim$ on L_0 is *monotone* if $f \succsim g$ whenever $f(s) \succsim g(s)$ for each $s \in S$.
  Inspired by [5, p. 568], define a relation $\geqslant^*$ on L_0 by

$$f \geqslant^* f' \quad \text{if } f \sim f \vee f'.$$

For monotone preferences, GK is equivalent to transitivity of $\geqslant^*$. In particular, under GK the symmetric part of $\geqslant^*$ partitions L_0 into equivalence classes of acts that are indistinguishable according to $\succsim$.

Theorem 6. *Let $\succsim$ be monotone. Then $\succsim$ satisfies GK iff $\geqslant^*$ is transitive.*

Proof. Suppose $\geqslant^*$ is transitive. By monotonicity, $f \vee g \geqslant^* f$. Therefore, since $\geqslant^*$ is transitive, $f \geqslant^* f' \Longrightarrow f \vee g \geqslant^* f'$, which proves GK.

Conversely, assume GK.[2] Suppose $f \geqslant^* g$ and $g \geqslant^* h$, that is, $f \sim f \vee g$ and $g \sim g \vee h$. We have $f \sim f \vee g \succsim g \sim g \vee h$ by monotonicity. By GK, $g \sim g \vee h$ implies $g \vee f \sim g \vee h \vee f$, and, by monotonicity, $g \vee h \vee f \succsim h \vee f$. Therefore,

$$f \sim f \vee g \sim g \vee h \vee f \succsim h \vee f \succsim f,$$

which implies $f \sim h \vee f$, that is, $f \geqslant^* h$. $\quad\square$

Acknowledgments

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[2] We use an argument of Kreps [5, p. 569].