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PROBABILISTIC SOPHISTICATION AND MULTIPLE PRIORS

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1. INTRODUCTION

IN THIS PAPER we consider two well-known choice models, the maximin expected utility (MEU) model of Gilboa and Schmeidler (1989) and the probabilistically sophisticated model of Machina and Schmeidler (1992).

Machina and Schmeidler (1992) model choices based on beliefs that satisfy standard de Finetti and Savage qualitative probability axioms and that can therefore be represented by convex-ranged additive numerical probabilities.[2] Their purpose is to provide a subjective Savage-like foundation of the non-expected utility models under risk dealing with Allais-type paradoxes.

On the other hand, there are important cases, exemplified by the classic Ellsberg Paradox, in which choice behavior cannot be represented through "additive" beliefs because of the vagueness that decision makers perceive in their beliefs. In these cases, decision makers face risk, as well as ambiguity. Gilboa and Schmeidler (1989) model this Ellsberg-type behavior by assuming that decisions makers base their decisions on sets of probabilities, sets that are in general nonsingleton because of vagueness.

These two choice models can overlap. For instance, in choice situations featuring ambiguity there may well exist subcollections of "unambiguous" events, involving only risk and no ambiguity. This is the case, for example, in the Ellsberg paradox itself. Over these collections of unambiguous events, a MEU decision maker may well exhibit a probabilistically sophisticated behavior. This is assumed, for example, by some recent models studied in Chen and Epstein (1998), Epstein (1999), Epstein and Zhang (2001), where probabilistic sophistication was assumed in order to deal with possible Allais-type phenomena arising in the unambiguous events collection.

In view of these recent papers, it is especially interesting to understand to what extent these two important models can coexist. Our main result shows that under fairly mild conditions, such a coexistence is possible only inside the standard subjective expected utility (SEU) model of Savage (1954). That is, a decision maker who is both probabilistically sophisticated and maximin expected utility, has to be a subjective expected utility decision maker.

Hence, once we model Ellsberg-type phenomena with maximin expected utility, it is no longer possible to deal with Allais-type phenomena via probabilistic sophistication, even "locally" on the collections of unambiguous events. This is a significant feature of the MEU model as one might be interested in studying both kinds of phenomena, without

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[2] A probability $P : \Sigma \to [0, 1]$ is convex-ranged if, for all $P(A) > 0$ and all $\alpha \in (0, 1)$, there exists $\Sigma \ni B \subseteq A$ such that $P(B) = \alpha P(A)$.

being forced to choose a priori one of the two. It is therefore important to know that the MEU model is not a suitable choice model if one is interested in a "joint" analysis of these two classic problems. In contrast, Section 4 shows that our result only partly extends to the Choquet Expected Utility model of Schmeidler (1989), leaving in that model some room for the analysis of Allais-type phenomena.

Our result rests on several factors. There is a basic tension between probabilistic sophistication, in which beliefs are probabilistic (i.e., represented by a single probability), and the MEU model, in which beliefs are represented by sets of probabilities. However suggestive, this basic tension is only a part of the story, and, indeed, Proposition 2 provides examples of MEU preferences that are probabilistically sophisticated without being SEU. To hold, our result also needs the existence of at least an unambiguous event, an assumption that cannot be dropped (see Proposition 2). Another factor that plays an important role in our result is the range convexity of the probability that underlies probabilistic sophistication. Range convexity is a widely assumed property, common to all Savage-type axiomatizations, like Machina and Schmeidler (1992). Without this property, our result in general fails. This is the case, for example, in finite state spaces. It is easy to see that under very mild conditions, in two-state spaces all MEU preferences have probabilistic beliefs, and so our result completely fails in that setting.

We close by observing that, besides its conceptual interest, our result is useful in studying ambiguity attitudes in the MEU model since it shows that it is without loss of generality to assume SEU as the benchmark model for the absence of ambiguity, and there is no need to consider the more general probabilistically sophisticated preferences. This somewhat simplifies the analysis of these attitudes, an issue recently studied by Epstein (1999), Epstein and Zhang (2001), and Ghirardato and Marinacci (1997).

The paper is organized as follows. Section 2 contains some preliminaries, and Section 3 states and discusses the main result. Section 4 discusses the extension of our results to the Choquet expected utility model. All proofs are in the Appendix.

2. PRELIMINARIES

2.1. *Mathematics*

In this paper we focus on λ-systems, the appropriate structure for modeling collections of unambiguous events (see Zhang (1996), Epstein (1999), and Ghirardato and Marinacci (1997)).

DEFINITION 1: A class Σ of subsets of a set S is a λ-*system* if:
1. $S \in \Sigma$,
2. $E^c \in \Sigma$ when $E \in \Sigma$,
3. $\bigcup_{i=1}^{\infty} E_n \in \Sigma$ for any sequence $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets of Σ.

A σ-algebra is a λ-system closed under intersections (i.e., $E \cap E' \in \Sigma$ when $E, E' \in \Sigma$). In particular, Σ is a σ-algebra if and only if it is both a λ-system and an algebra.

Let Σ be a λ-system. A nonnegative set-function $P : \Sigma \to [0,1]$ is a finitely additive probability if $P(S) = 1$ and $P(E \cup E') = P(E) + P(E')$ for all pairwise disjoint sets $E, E' \in \Sigma$, while it is countably additive if $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ for all sequences $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets of Σ.

A finitely additive probability $P : \Sigma \to [0,1]$ is convex-ranged if, for all $P(E) > 0$ and all $\alpha \in (0,1)$, there exists $\Sigma \ni E' \subseteq E$ such that $P(E') = \alpha P(E)$. As is well known, if Σ is a σ-algebra and P is countably additive, then P is convex-ranged if and only if it is nonatomic, i.e., for all $E \in \Sigma$ such that $P(E) > 0$ there exists $\Sigma \ni E' \subseteq E$ such that $0 < P(E') < P(E)$.

2.2. *Setting*

The set-up consists of a set S of states of the world, a collection Σ of subsets of S, and a set X of consequences. An act $f : S \to X$ is a finite-valued and Σ-measurable function.

We assume that a preference relation $\succsim$ represents the decision maker's preferences on the set of all acts, which we denote by $\mathcal{F}$.

We will need the following Monotone Continuity Axiom, due to Arrow (1970). The condition will be discussed in the next section.

AXIOM (Monotone Continuity): *Given any acts $f \succ g$, consequence $x \in X$, and sequence of events $\{E_n\}_{n>1} \subseteq \Sigma$ with $E_n \supseteq E_{n+1}$ for all $n \geq 1$ and $\bigcap_{n=1}^{\infty} E_n = \varnothing$, for all n sufficiently large we have*:

$$\begin{bmatrix} x & \text{if } s \in E_n \\ f(s) & \text{if } s \notin E_n \end{bmatrix} \succ g \quad \text{and} \quad f \succ \begin{bmatrix} x & \text{if } s \in E_n \\ g(s) & \text{if } s \notin E_n \end{bmatrix},$$

whenever these modified acts are Σ-measurable.

2.3. *Models*

A binary relation $\succeq$ is an *α-maximin expected utility* (α-MEU) preference relation if there exist a utility index $u : X \to \mathbb{R}$, a nonempty, weak*-compact and convex set C of finitely additive probabilities $P : \Sigma \to [0, 1]$, and a constant $\alpha \in [0, 1]$ such that, for all $f, g \in \mathcal{F}$, $f \succsim g$ if and only if

$$\begin{aligned} &\alpha \min_{P \in C} \int_S u(f(s))\, dP + (1-\alpha) \max_{p \in C} \int_S u(f(s))\, dP \\ &\quad \geq \alpha \min_{P \in C} \int_S u(g(s))\, dP + (1-\alpha) \max_{p \in C} \int_S u(g(s))\, dP. \end{aligned}$$

Moreover, we assume that the range $u(X)$ of u is not a nowhere dense subset of $\mathbb{R}$, that is, we assume that the interior of the closure $\overline{u(X)}$ of $u(X)$ is nonempty. For example, this is the case if $u(X)$ is dense in an interval of $\mathbb{R}$. This assumption implies that $u(X)$ is at least countably infinite, and so X as well has to be at least countably infinite.

In general, α-MEU preferences behave quite differently from SEU preferences. For instance, if $X \subseteq \mathbb{R}$ and C is the set of all finitely additive probabilities on $\Sigma = 2^S$, then, for all $f \in \mathcal{F}$,

$$\begin{aligned} &\alpha \min_{P \in C} \int_S u(f(s))\, dP + (1-\alpha) \max_{p \in C} \int_S u(f(s))\, dP \\ &\quad = \alpha \inf_{s \in S} u(f(s)) + (1-\alpha) \sup_{s \in S} u(f(s)). \end{aligned}$$

For $\alpha = 1$, α-MEU preferences are the standard maximin expected utility preferences axiomatized by Gilboa and Schmeidler (1989) in an Anscombe-Aumann setting.[3] The set

[3] Recently, Casadesus-Masanell, Klibanoff, and Ozdenoren (2000) have provided an axiomatization of this model in a Savage setting.

C represents the decision makers' subjective priors, and it is not a singleton because of the vagueness of decision makers' beliefs. When C is a singleton, $\succsim$ is the classic *subjective expected utility* (SEU) relation of Savage (1954).

Some properties of α-MEU preferences have been studied by Ghirardato, Klibanoff, and Marinacci (1988). An important issue is whether C is unique for given $\alpha \in [0,1]$ and u. By the representation result of Gilboa and Schmeidler (1989), C is unique in the important case $\alpha \in \{0,1\}$, while later it will be shown that for $\alpha = 1/2$ such a uniqueness does not hold. It is not known, however, if C is unique when $\alpha \in (0,1)$ and $\alpha \neq 1/2$, though recently Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001) have shown that this is the case for symmetric sets of priors.

We now introduce probabilistic beliefs. Each preference relation $\succsim$ introduces a likelihood relation $\succsim_l$ on Σ, where $E \succsim_l E'$ if there exist consequences $x^* \succ x_*$ such that

$$\begin{bmatrix} x^* & \text{if } s \in E \\ x_* & \text{if } s \notin E \end{bmatrix} \succsim \begin{bmatrix} x^* & \text{if } s \in E' \\ x_* & \text{if } s \notin E' \end{bmatrix}.$$

Throughout the paper we assume that the preference relations $\succsim$ satisfy the standard axiom P4 of Savage (1954), so that the induced likelihood relations do not depend on x^* and x_*. In particular, $\succsim_l$ is represented in the α-MEU model by the set function $\rho : \Sigma \to [0,1]$ defined, for all $E \in \Sigma$, by

$$(1) \qquad \rho(E) \equiv \alpha \min_{P \in C} P(E) + (1-\alpha) \max_{P \in C} P(E).$$

A preference relation $\succsim$ has *weak probabilistic beliefs* if there exists a convex-ranged finitely additive probability $P : \Sigma \to [0,1]$ such that, for all $E, E' \in \Sigma$,

$$(2) \qquad P(E) = P(E') \Longrightarrow E \sim_l E',$$

while $\succsim$ has *probabilistic beliefs* if

$$(3) \qquad P(E) \geq P(E') \Longleftrightarrow E \succsim_l E'.$$

The most important class of preferences exhibiting probabilistic beliefs are the probabilistically sophisticated preferences of Machina and Schmeidler (1992). Besides probabilistic beliefs, they also require some additional conditions that are superfluous for our purposes, and that therefore we do not consider.[4]

3. MAIN RESULT

DEFINITION 2: Let $\succsim$ be an α-MEU preference relation with multiple priors set C. A set $A \in \Sigma$ is *nontrivial and unambiguous* if $0 < \min_{P \in C} P(A) = \max_{P \in C} P(A) < 1$.

We denote by Λ the set of all events that are nontrivial and unambiguous for $\succsim$. Since all priors agree on Λ, there is no vagueness in the beliefs over events belonging to Λ. This is why the events in Λ are called unambiguous.

Epstein and Zhang (2001) and Ghirardato and Marinacci (1997) discuss at length alternative notions of unambiguous events and their behavioral foundations. In any case,

[4] See Machina and Schmeilder (1992, p. 755), as well as Epstein and Le Breton (1993, p. 8) and Grant (1995, p. 163). All these papers provide axiomatizations for probabilistic sophistication.

according to all of them, the events in Λ are classified as unambiguous and, loosely speaking, behaviorally A is unambiguous when $\succsim$, restricted to bets defined on the partition $\{A, A^c\}$, is SEU.[5]

We are now ready to state our main result.

THEOREM 1: *Let $\succsim$ be a monotone continuous α-MEU preference relation defined on the set $\mathcal{F}$ of acts measurable with respect to a λ-system Σ. If $\Lambda \neq \varnothing$ and $\alpha \neq 1/2$, the following two statements are equivalent*:

(i) *$\succsim$ has weak probabilistic beliefs.*

(ii) *$\succsim$ is a subjective expected utility preference relation.*

Theorem 1 shows that, under mild behavioral assumptions, probabilistic beliefs and multiple priors can coexist only in Savage's subjective expected utility model. In the introduction we mentioned that an important case when MEU preferences have probabilistic beliefs is over the collection of unambiguous events. This case is covered by $\Lambda = \Sigma$ in the approach of Ghirardato and Marinacci (1997), while, in contrast, in Epstein and Zhang's (2001) approach the collection of all nontrivial unambiguous events is a λ-system that might well be larger than Λ. It is therefore important that our result only requires that $\varnothing \neq \Lambda \subseteq \Sigma$ and that Σ be a λ-system and not necessarily a σ-algebra.

In Theorem 1 we use three conditions: $\Lambda \neq \varnothing$, $\alpha \neq 1/2$, and the monotone continuity of $\succsim$. We now discuss them one at a time.

The condition $\Lambda \neq \varnothing$ is conceptually the more significant among the conditions we assume, and Proposition 2 below shows that it is not possible to omit it in Theorem 1. It is a fairly mild behavioral assumption as it only requires the existence of a single nontrivial unambiguous event. With conventional additive probabilities, *all* events are unambiguous, and excluding the existence of even a single nontrivial unambiguous event seems in general a very demanding behavioral assumption. Moreover, in decision theory several axiomatizations enlarge the state space by assuming the existence of an external random device with given probabilities (say, a coin flip or a roulette wheel). Let $m : \mathcal{R} \to [0, 1]$ be a probability measure representing the random device, defined on a suitable σ-algebra $\mathcal{R}$. The set of product measures $\{P \otimes m : \Sigma \otimes \mathcal{R} \to [0, 1]$, with $P \in C\}$ has always nontrivial unambiguous events (for example, all events of the form $\Omega \times A$, where $A \in \mathcal{R}$).

Theorem 1 requires $\alpha \neq 1/2$ and we do not know whether it holds for the special case $\alpha = 1/2$. On the other hand, consider the following example. Let P and Q be any two probabilities, and let $C = \{\alpha P + (1-\alpha)Q : \alpha \in [0, 1]\}$. We have $\rho = (P+Q)/2$ for the 1/2-MEU preference $\succsim$ whose set of priors is C, and so $\succsim$ is SEU even though C is not a singleton. This shows that the case $\alpha = 1/2$ has some problematic features. In fact, the nonsingleton nature of the set of priors C associated with an α-MEU preference relation is usually interpreted as a result of the vagueness that decision makers perceive in their beliefs, and, to be consistent, this standard interpretation requires that an α-MEU preference be SEU if and only if its set of priors C is a singleton. This is indeed trivially true in the standard MEU model axiomatized by Gilboa and Schmeidler (1989), which corresponds to $\alpha = 1$, and it can be proved that this is also true for all $\alpha \neq 1/2$ under standard topological assumptions.[6] But, the example shows that this is not the case for $\alpha = 1/2$.

[5] See Ghirardato and Marinacci (1997) for details.

[6] The result is available from the author upon request.

The last assumption we make in Theorem 1 is monotone continuity, a technical condition that implies that all priors in C are countably additive, a property we need to prove Theorem 1.[7] Monotone continuity was introduced by Arrow (1970), who showed that it is the behavioral condition underlying the use of countably additive probabilities in SEU theory. In his opinion, "the assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem" (Arrow (1970), first lecture). In economic applications Monotone Continuity is, implicitly or explicitly, widely assumed since countable additivity is a most useful technical property on which rest many of the fundamental results of probability theory (cf. Machina and Schmeidler (1992, p. 770)).

In any event, the next result shows that it is possible to dispense with this technical assumption in the important case of $\alpha \in \{0, 1\}$ and probabilistic beliefs.

PROPOSITION 1: *Let $\succsim$ be an α-MEU preference relation defined on the set $\mathcal{F}$ of acts measurable with respect to a λ-system Σ. Suppose that $\Lambda \notin \varnothing$ and $\alpha \in \{0, 1\}$. Then the following two statements are equivalent*:

(i) *$\succsim$ has probabilistic beliefs.*
(ii) *$\succsim$ is a subjective expected utility preference relation.*

We close this section by showing that, as announced, it is not possible to omit the condition $\Lambda \neq \varnothing$ in Theorem 1. In fact, consider a nonatomic countably additive probability P^* defined on a σ-algebra Σ, and assume that $X = [0, 1]$. Assume also that $u(x) = x$ for all $x \in X$, so that $\mathcal{F}$ consists of all finite-valued functions $f : S \to [0, 1]$. Let C be the set of all finitely additive probabilities P such that $P(E) \geq (P^*(E))^3$ for all $E \in \Sigma$, and let $\succsim^\alpha$ be the α-MEU preference relation represented by

$$V_\alpha(f) \equiv \alpha \min_{P\in C} \int f\, dP + (1-\alpha) \max_{P\in C} \in f\, dP.$$

PROPOSITION 2: *For each $\alpha \in [0, 1]$, the preference $\succsim^\alpha$ is a monotone continuous α-MEU preference relation with $\Lambda = \varnothing$. It has probabilistic beliefs, but it does not admit a SEU representation.*

REMARKS: (i) The preferences $\succsim^\alpha$ are actually probabilistically sophisticated in the sense of Machina and Schmeidler (1992). (ii) The result also shows that we need condition $\Lambda \neq \varnothing$ even if in Theorem 1 we strengthened point (i) by requiring probabilistic beliefs rather than just weak probabilistic beliefs. Consequently, in Proposition 1 as well, where for $\alpha \in \{0, 1\}$ we have such a strengthening, it is not possible to omit the condition $\Lambda \neq \varnothing$.

4. CONCLUDING REMARKS

We have shown how under mild behavioral assumptions α-MEU preferences have (weak) probabilistic beliefs if and only if they are actually conventional subjective expected utility preferences. In part, a similar result holds for the closely related Choquet expected utility (CEU) model of Schmeidler (1989), where the likelihood relation $\succsim_l$ is represented by a normalized and monotone set function $\rho : \Sigma \to [0, 1]$, called capacity. In this model,

[7] For $\alpha = 1$, Chateauneuf, Marinacci, and Tallon (2000) study in detail the implications of monotone continuity for MEU preferences.

the vagueness of decision makers' beliefs is modeled through the nonadditivity of the representing capacity ρ.

To see to what extent Theorem 1 (and similarly Proposition 1) holds for CEU preferences, notice that now the appropriate set of unambiguous events Λ is $\{A \in \Sigma : \rho(A) + \rho(A^c) = 1\}$. It is easy to see that Theorem 1 holds for monotone continuous CEU preferences whose associated capacities ρ have the form $\rho = \alpha\nu + (1-\alpha)\bar{\nu}$ for some $\alpha \in [0, 1]$, provided that $\Lambda \neq \varnothing$, $\alpha \neq 1/2$, and that ν is an exact capacity.[8]

The capacity ν has to be exact, a popular property that is however not satisfied by several behaviorally interesting capacities. Without exactness, the CEU counterpart of Theorem 1 fails in general, and, therefore, there are CEU preferences for which our results do not hold and that have probabilistic beliefs without being SEU.[9]

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APPENDIX: Proofs and Related Analysis

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A.1. *Theorem 1*

The proof of Theorem 1 is based on a uniqueness theorem for convex-ranged measures proved in Marinacci (2000), and on two lemmas.

A.1.1. A.1.1. *A Uniqueness Theorem*

THEOREM 2: *Let P and Q be two finitely additive probabilities on a λ-system Σ. Suppose that P is convex-ranged. If there exists a set $A \in \Sigma$ with $0 < P(A) < 1$ and such that*

$$P(A) = P(E) \Longleftrightarrow Q(A) = Q(E) \tag{4}$$

whenever $E \in \Sigma$, then $P = Q$. Moreover, if Q is countably additive, we can replace (4) *with the weaker condition*

$$P(A) = P(E) \Longrightarrow Q(A) = Q(E). \tag{5}$$

For a proof, we refer the interested reader to Marinacci (2000). Note that we only require the existence of a single set $A \in \Sigma$ for which conditions (4) and (5) have to be satisfied. No requirement on Q is made if condition (4) holds, while only countable additivity is required when the weaker condition (5) holds. Therefore, the result shows a remarkably strong property: a minimal agreement between two probabilities, at least one of them being convex-ranged, forces them to be equal.

A couple of remarks are in order: (i) some applications of Theorem 2 in decision theory are contained in Epstein and Zhang (2001), Marinacci (1999), and Ghirardato and Marinacci (2000); (ii) Mongin (1996) contains a special case of the first part of Theorem 2 in which both P and Q are nonatomic countably additive probabilities defined on a σ-algebra, and $P(A) = Q(A) = 1/2$.

[8] The dual capacity $\bar{\nu}$ of ν is defined by $\bar{\nu}(E) = 1 - \nu(E^c)$ for all $E \in \Sigma$. The capacity ν is exact if $\nu(E) = \min_{P \in \text{core}(\nu)} P(E)$ for all $E \in \Sigma$, where $\text{core}(\nu)$ is the set of all finitely additive probabilities that setwise dominate ν, and it is assumed to be nonempty. Convex capacities are an important example of exact capacities. Since $\alpha\nu + (1-\alpha)\bar{\nu}$ is a capacity, it can be integrated with respect to a standard Choquet integral, and so there is no need of talking of α-CEU preferences.

[9] A simple example is available from the author upon request.

A.1.2. *Two Lemmas*

LEMMA 1: *Let $\succsim$ be a monotone continuous α-MEU preference relation defined on the set $\mathcal{F}$ of acts measurable with respect to a λ-system Σ. If $\succsim$ satisfies monotone continuity, then all $P \in C$ are countably additive.*

PROOF: Since $u(X)$ is not a nowhere dense subset of $\mathbb{R}$, there is an element $u(\bar{x})$ of $u(X)$ such that, by taking appropriate subsequences if needed, both the two following conditions hold:

(i) there exists a sequence $\{x_k\}_{k\geq 1} \subseteq X$ such that $x_k \succ \bar{x}$ for all $k \geq 1$, and $\lim_{k\to\infty} u(x_k) = u(\bar{x})$;
(ii) there exists a sequence $\{x_k\}_{k\geq 1} \subseteq X$ such that $x_k \prec \bar{x}$ for all $k \geq 1$, and $\lim_{k\to\infty} u(x_k) = u(\bar{x})$.

In fact, let $x \in \text{int}(\overline{u(X)})$. There is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \text{int}(\overline{u(X)})$ and, since $x \in \overline{u(X)}$, there is a sequence in $u(X)$ converging to x. Eventually, the sequence will belong to $(x-\varepsilon, x+\varepsilon)$, and so there is $\bar{x} \in u(X)$ such that $\bar{x} \in (x-\varepsilon, x+\varepsilon)$. Then, there is $\bar{\varepsilon} > 0$ such that $(\bar{x}-\bar{\varepsilon}, \bar{x}+\bar{\varepsilon}) \subseteq \text{int}(\overline{u(X)})$. Now, suppose (ii) does not hold for $\bar{x}$. Then, there is $a \in (\bar{x}-\bar{\varepsilon}, \bar{x})$ such that $(a, \bar{x}) \subseteq u(X)^c$. Since $(a, \bar{x})$ is open, we then have $(a, \bar{x}) \subseteq \text{int}((u(X))^c) \subseteq (\overline{u(X)})^c$. But, $(a, \bar{x}) \subseteq (\bar{x}-\bar{\varepsilon}, \bar{x}+\bar{\varepsilon}) \subseteq \text{int}(\overline{u(X)}) \subseteq \overline{u(X)}$, a contradiction, and we conclude that (ii) holds. A similar argument holds for (i).

Suppose $\alpha \neq 1$ and consider (i). There exists some $y \in X$ such that $y \succ \bar{x}$. W.l.o.g., set $u(\bar{x}) = 0$ and $u(y) = 1$. Let $\{E_n\}_{n>1} \subseteq \Sigma$ be a monotone decreasing sequence with $\bigcap_{n=1}^{\infty} E_n = \varnothing$. Let g be the constant act such that $g(s) = \bar{x}$ for all $s \in S$, and let $\{g_n\}_{n\geq 1}$ be defined as follows: $g_n(s) = \bar{x}$ for all $s \in E_n^c$ and $g_n(s) = y$ for all $s \in E_n$. For each $k \geq 1$, $x_k \succ g$ and so, by Monotone Continuity, for all n sufficiently large we have $x_k \succ g_n$. Hence, $\alpha \min_{P\in C} P(E_n) + (1-\alpha)\max_{P\in C} P(E_n) < u(x_k)$ for all n sufficiently large. Since $\lim_{k\to\infty} u(x_k) = u(\bar{x}) = 0$, we conclude that

$$\lim_{n\to\infty}\Big(\alpha \min_{P\in C} P(E_n) + (1-\alpha)\max_{P\in C} P(E_n)\Big) = 0. \tag{6}$$

Equation (6) in turn implies that $\lim_{n\to\infty} \min_{P\in C} P(E_n) = 0$ because

$$0 \leq \min_{P\in C} P(E_n) \leq \alpha \min_{P\in C} P(E_n) + (1-\alpha)\max_{P\in C} P(E_n).$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty}\max_{P\in C} P(E_n) &= \lim_{n\to\infty} \frac{\alpha \min_{P\in C} P(E_n) + (1-\alpha)\max_{P\in C} P(E_n)}{1-\alpha} \\
&\quad - \frac{\alpha}{1-\alpha}\lim_{n\to\infty}\min_{P\in C} P(E_n) \\
&= 0.
\end{aligned}$$

This proves that $\lim_{n\to\infty} P(E_n) = 0$ for all $P \in C$. Let $\{E_n\}_{n\geq 1} \subseteq \Sigma$ be a monotone decreasing (increasing, resp.) sequence with $\bigcap_{n=1}^{\infty} E_n = E \in \Sigma$ ($\bigcup_{n=1}^{\infty} E_n = E$, resp.). Then $E_n - E \in \Sigma$ and $\bigcap_{n=1}^{\infty}(E_n - E) = \varnothing$ ($\bigcap_{n=1}^{\infty}(E - E_n) = \varnothing$, resp.), so that $\lim_n P(E_n) = P(E)$ for all $P \in C$. Finally, a similar argument, based on property (ii), proves the result for $\alpha = 1$. *Q.E.D.*

LEMMA 2: *Let $\succsim$ be an α-MEU preference relation. If $\alpha \neq 1/2$, then $A \in \Lambda$ if and only if $\rho(A) + \rho(A^c) = 1$ and $\rho(A) \in (0,1)$.*

REMARK: This result does not hold when $\alpha = 1/2$. In fact, when $\alpha = 1/2$, it holds that $\rho(E) + \rho(E^c) = 1$ for all events $E \in \Sigma$. This failure is the reason why we need the condition $\alpha \neq 1/2$ in Theorem 1.

PROOF: We only prove the "if" part, the other being trivial. Suppose that $\rho(A) + \rho(A^c) = 1$ and that $\rho(A) \in (0,1)$. Since $\min_{P\in C} P(E) = 1 - \max_{P\in C} P(E^c)$ for all $E \in \Sigma$, we can write:

$$\begin{aligned}
1 &= \alpha\Big[\min_{P\in C} P(A) + \min_{P\in C} P(A^c)\Big] + (1-\alpha)\Big[\max_{P\in C} P(A) + \max_{P\in C} P(A^c)\Big] \\
&= (2\alpha - 1)\Big[\min_{P\in C} P(A) + \min_{P\in C} P(A^c)\Big] + 2(1-\alpha),
\end{aligned}$$

and so $(2\alpha-1)\left[\min_{P\in C} P(A)+\min_{P\in C} P(A^c)\right]=2\alpha-1$. Conclude that, if $\alpha\neq 1/2$, then $\min_{P\in C} P(A)+\min_{P\in C} P(A^c)=1$, and so $A\in\Lambda$. *Q.E.D.*

A.1.3. *Proof of Theorem 1*

(ii) trivially implies (i).[10] As to the converse, since $\succsim$ is weakly probabilistically sophisticated, there exists a convex-ranged finitely additive probability P^* on Σ such that, for all $E, E'\in\Sigma$,

$$P^*(E)=P^*(E')\Longrightarrow \rho(E)=\rho(E'). \tag{7}$$

Hence, there exists a function $\phi:[0,1]\to[0,1]$, with $\phi(0)=0$ and $\phi(1)=1$, such that $\rho(E)=\phi(P^*(E))$ for all $E\in\Sigma$. Let $A\in\Lambda$. It is easy to check that $\rho(A)\in(0,1)$, and so $P^*(A)\in(0,1)$ as otherwise either $\rho(A)=\phi(P^*(A))=\phi(0)=0$ or $\rho(A)=\phi(P^*(A))=\phi(1)=1$. Now, let $E\in\Sigma$ be such that $P^*(E)=P^*(A)$. Suppose that $\alpha\neq 1/2$. By (7), $\rho(A)=\rho(E)$ and, since $P^*(E^c)=P^*(A^c)$, $\rho(E^c)=\rho^*(A^c)$. Hence, $\rho(E)+\rho(E^c)=\rho(A)+\rho(A^c)=1$, and so, by Lemma 2, $E\in\Lambda$. We then have

$$\min_{P\in C} P(E)=\min_{P\in C} P(A)=P(A)=\rho(A)=\max_{P\in C} P(A)=\max_{P\in C} P(E).$$

Moreover, $A\in\Lambda$ implies $P(A)\in(0,1)$. We conclude that, for all $P\in C$ and all $E\in\Sigma$, $P^*(E)=P^*(A)\Rightarrow P(E)=P(A)$, with $0<P^*(A)$, $P(A)<1$. On the other hand, by Lemma 1, all $P\in C$ are countably additive. Hence, Theorem 2 implies that $P^*=P$ for all $P\in C$, and so $C=\{P^*\}$. This shows that $\succsim$ is SEU. *Q.E.D.*

A.2. *Proposition* 1

We only prove that (i) implies (ii), the converse being trivial. Consider $\alpha=1$ (the case $\alpha=0$ is similar), and let A and ϕ be as in the previous proof. Because $\succsim$ has now probabilistic beliefs, ϕ is strictly increasing. Let $P\in C$ and suppose that $P(A)=P(E)\in(0,1)$. Then

$$\phi(P^*(A))=\min_{P\in C} P(A)=P(A)=P(E)\geq \min_{P\in C} P(E)=\phi(P^*(E)),$$

so that $P^*(A)\geq P^*(E)$ and $P^*(A)\in(0,1)$ because ϕ is strictly increasing. A similar argument shows that $P(A^c)=P(E^c)$ implies $P^*(A^c)\geq P^*(E^c)$. We conclude that $P^*(A)=P^*(E)$. On the other hand, the implication $P^*(E)=P^*(A)\Rightarrow P(E)=P(A)$ of the previous proof still holds, and so, for all $E\in\Sigma$ and all $P\in C$, we have

$$P^*(A)=P^*(E)\Longleftrightarrow P(A)=P(E).$$

Hence, condition (4) of Theorem 2 holds, and so $P^*=P$, which in turn implies $C=\{P^*\}$. *Q.E.D*

A.3. *Proposition* 2

Since $P^*\in C$, $C\neq\varnothing$; it is also easy to check that C is convex and weak*-compact. Because $(P^*)^3$ is continuous at S, all $P\in C$ are countably additive. Hence, each $\succsim^\alpha$ is monotone continuous by a result of Chateauneuf, Marinacci, and Tallon (2000). Since $(P^*)^3$ is a convex capacity, $\min_{P\in C} P(E)=(P^*(E))^3$ for all $E\in\Sigma$. Some simple algebra then shows that $\Lambda=\varnothing$ for all $\alpha\in[0,1]$, and that for all $E\in\Sigma$, $\alpha\min_{P\in C} P(E)+(1-\alpha)\max_{P\in C} P(E)=\phi_\alpha(P^*(E))$, where $\phi_\alpha:[0,1]\to[0,1]$ is defined by $\phi_\alpha(x)=x^3+3(1-\alpha)(x-x^2)$ for each $x\in[0,1]$. Hence, for all $E, E'\in\Sigma$, $E\succsim_l^\alpha E'\Leftrightarrow \phi_\alpha(P^*(E))\geq \phi_\alpha(P^*(E'))$. Since ϕ_α is strictly increasing for each $\alpha\in[0,1]$, we have $E\succsim_l^\alpha E'\Leftrightarrow P^*(E)\geq P^*(E')$, and so $\succsim^\alpha$ has probabilistic beliefs. By a simple result of Villegas (1964), P^* is the only additive representation of each $\succsim_l^\alpha$, and so the only SEU preference functional that can possibly represent each $\succsim^\alpha$ is $U(f)\equiv\int f\,dP^*$. It remains to show that U does not represent any $\succsim^\alpha$. Consider the

[10] It is easy to prove that under standard topological assumptions, (ii) implies $\Lambda\neq\varnothing$.

partition $\{A_1, A_2, A_3\}$ such that $P^*(A_i) = 1/3$ for $i = 1, 2, 3$, and the act f defined as follows: $f(s) = 1$ if $s \in A_1$, $f(s) = \beta \in (0, 1)$ if $s \in A_2$, and $f(s) = 0$ if $s \in A_3$. Simple algebra shows that $U(f) = V_\alpha(f)$ iff $\alpha = (-1/9)\beta + (5/9)$. Hence, if we take $\beta = 1/2$, $U(f) \neq V_\alpha(f)$ for all $\alpha \neq 1/2$, while for $\beta = 1/4$, $U(f) \neq V_\alpha(f)$ for all $\alpha \neq 19/36$. Then, for each α there is $f \in \mathcal{F}$ with $U(f) \neq V_\alpha(f)$. Suppose that $U(f) < V_\alpha(f)$ (a similar argument holds for $U(f) > V_\alpha(f)$). Let g and g' be the constant acts such that $g(s) = U(f)$ and $g'(s) = V_\alpha(f)$ for all $s \in S$. We have $g \prec^\alpha f \sim^\alpha g'$, while $U(g) = U(f) < U(g')$, and so U does not represent $\succsim^\alpha$. *Q.E.D.*

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