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Commutativity, comonotonicity, and Choquet integration of self-adjoint operators

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In this work, we propose a definition of comonotonicity for elements of $B(H)_{sa}$, i.e. bounded self-adjoint operators defined over a complex Hilbert space H. We show that this notion of comonotonicity coincides with a form of commutativity. Intuitively, comonotonicity is to commutativity as monotonicity is to bounded variation. We also define a notion of Choquet expectation for elements of $B(H)_{sa}$ that generalizes quantum expectations. We characterize Choquet expectations as the real-valued functionals over $B(H)_{sa}$ which are comonotonic additive, c-monotone, and normalized.

Keywords: Comonotonicity; commutativity; nonlinear Gleason theorem; Gleason theorem; Choquet integral.

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1. Introduction

In this work, we bridge the ideas of Choquet integration and quantum expectation. In particular, we show how the notion of Choquet expectation can be naturally defined also in the space $B(H)_{sa}$. This notion naturally generalizes the one of quantum expectation/mixed state.

Since readers may know only one of these two concepts, we start the Introduction by briefly describing both. We then proceed to highlight our main contributions

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and offer a physical point of view for some of them. We conclude the Introduction by discussing the related literature and the organization of the paper.

Choquet Integral and Mathematical Economics. Comonotonicity and comonotonic additivity are at the base of the theory of Choquet integration and they both had a huge impact in Mathematical Economics and Decision Theory.[a] In a decision theoretic setting, the primitives are a measurable space $(\Omega, \mathcal{F})$ and a functional $V : B(\mathcal{F}) \to \mathbb{R}$, where $B(\mathcal{F})$ is the space of real-valued, bounded, and $\mathcal{F}$-measurable functions. The functional V is supposed to represent the preferences of an agent over uncertain prospects, modeled as random variables.

Two functions $f, g \in B(\mathcal{F})$ are said to be *comonotonic* if and only if

$$[f(\omega) - f(\omega')][g(\omega) - g(\omega')] \geq 0 \quad \forall\, \omega,\ \omega' \in \Omega. \tag{1}$$

In turn, the functional V is said to be *comonotonic additive* if and only if

$$f \text{ and } g \text{ are comonotonic} \Rightarrow V(f+g) = V(f) + V(g).$$

The celebrated theorem of Schmeidler [23] shows that normalized,[b] monotone, and comonotonic additive functionals are tightly connected to normalized capacities. A set function $\nu : \mathcal{F} \to [0,1]$ is a (normalized) capacity if and only if

(1) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;[c]
(2) If $E, F \in \mathcal{F}$ and $F \supseteq E$, then $\nu(F) \geq \nu(E)$.

Clearly, a finitely additive probability is a capacity, while the converse typically does not hold.

Theorem 1 (Schmeidler). *Let V be a functional from $B(\mathcal{F})$ to $\mathbb{R}$. The following statements are equivalent*:

(i) *V is normalized, monotone, and comonotonic additive*;
(ii) *There exists a capacity $\nu : \mathcal{F} \to [0,1]$ such that*

$$V(f) = \int_0^\infty \nu(f \geq t)dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(\Omega)]dt \quad \forall\, f \in B(\mathcal{F}). \tag{2}$$

Moreover, ν is unique.

The right-hand side of (2) is also known as the Choquet integral of f with respect to ν and we will denote it by $\int f d\nu$.[d]

[a] For applications in Economics, see [20], for applications in Mathematical Finance, see [12], for applications in Statistics see [21] as well as [18].
[b] That is, $V(1_\Omega) = 1$.
[c] In general, a capacity need not be normalized, that is, $\nu(\Omega) \neq 1$. As one might suspect, the normalization property is rather innocuous in developing the theory of Choquet integration. For an introduction to the subject, we refer readers to [20].
[d] As it is customary, we set for each $f \in B(\mathcal{F})$ and for each $t \in \mathbb{R}$, $(f \geq t) = \{\omega \in \Omega : f(\omega) \geq t\}$. The two integrals, $\int_0^\infty$ and $\int_{-\infty}^0$, are Riemann integrals.

The Choquet integral is a generalization of the usual notion of integral. For example, if ν is a countably additive probability measure, then it is a standard measure theory result — see, e.g., [6, p. 275 and p. 280] — to show that the right-hand side of (2) becomes the usual Lebesgue's integral. One can also show that for each $f \in B(\mathcal{F})$ there exists a (possibly finitely additive) probability P_f such that

$$\int f d\nu = \int f dP_f.$$

In other words, Choquet expectation can be viewed as a standard expectation where the probability used depends on the integrand. We highlight three features of comonotonicity and Choquet expectations:

(1) If f and g are comonotonic, then their Choquet expectations coincide with a standard expectation computed using a common probability. More formally (see [20, Proof of Theorem 4.3]), we have that

$$f \text{ and } g \text{ are comonotonic} \Rightarrow P_f \text{ and } P_g \text{ can be chosen to be the same.} \tag{3}$$

(2) The notion of comonotonicity has a useful characterization.[e] Two functions f and g in $B(\mathcal{F})$ are comonotonic if and only if their covariance is positive for each (finitely additive) probability, that is,

$$Cov_P(f, g) \geq 0 \quad \text{for all } P \in \Delta. \tag{4}$$

(3) If $f = \sum_{j=1}^{p_f} \alpha_j 1_{E_j}$ is a simple function where $\{\alpha_j\}_{j=1}^{p_f}$ are distinct real numbers, already ordered from the greatest to the smallest, and $\{E_j\}_{j=1}^{p_f}$ are pairwise disjoint non-empty events whose union is Ω, then

$$\int f d\nu = \sum_{i=1}^{p_f} (\alpha_i - \alpha_{i+1}) \nu \left(\bigcup_{j=1}^{i} E_j \right) \tag{5}$$

where we set $\alpha_{p_f+1} = 0$ (see, e.g., [23, p. 257]).

Quantum Expectations and Quantum Mechanics. In the usual formulation of Quantum Mechanics, the primitives are, loosely speaking, not a measurable state space and bounded random variables defined on it, but mathematical objects that carry some similarities as well as striking differences to the measurable setting (see, e.g., [24, 15]). *Pure states* are identified with unit vectors of a separable complex Hilbert space H that, for the sake of simplicity, in the introduction we assume to be finite dimensional. Random variables, which in this context are also called *observables*, are replaced by bounded self-adjoint operators on H, i.e. elements of

[e] An early version is often attributed to Chebyshev (see, e.g., [13, 3] as well as [8, p. 304]). Moreover, in Mathematics often two functions/vectors that are comonotonic are also said to be "similarly ordered" (see, e.g., [17, p. 43]).

$B(H)_{sa}$. Finally, *expectations*, which are also termed *mixed states*, are normalized,[f] positive, and linear maps $\varphi : B(H)_{sa} \to \mathbb{R}$. The celebrated theorem of Gleason [14] shows that if $\dim H \geq 3$, then mixed states are characterized as expectations with respect to a quantum probability (see Sec. 3.2).[g] Intuitively, one reason why bounded self-adjoint operators can be interpreted as random variables is the spectral theorem that guarantees that if $A \in B(H)_{sa}$, then $A = \sum_{j=1}^{p_A} \alpha_j E_j$ where $\{E_j\}_{j=1}^{p_A}$ are pairwise orthogonal non-zero projections which sum up to the identity I and $\{\alpha_j\}_{j=1}^{p_A}$ are distinct real numbers. If φ is a mixed state, by linearity

$$\varphi(A) = \sum_{j=1}^{p_A} \alpha_j \varphi(E_j) \tag{6}$$

where $(\varphi(E_1), \ldots, \varphi(E_{p_A}))$ is a probability vector, that is, all the components are positive and sum up to 1. In other words, the value $\varphi(A)$ can be interpreted as the average of the spectrum of A. The key property here is that the weights $\varphi(E_j)$ depend on A via its spectral form. We call r_A a vector of probability weights that satisfies (6). A remarkable feature is the following one[h]:

$$A \text{ and } B \text{ commute} \Rightarrow r_A \text{ and } r_B \text{ can be chosen to be the same.} \tag{7}$$

In other words, if A and B commute, then their quantum expectations coincide with a standard average of the spectrum, computed using a common probability.

Our Contributions. Starting from the similarities between (3) and (7) and the stylized fact that in the double-slit experiment probabilities are non-additive, this paper tries to bridge the two theories of Choquet integration and quantum expectation. We offer a notion of comonotonicity for bounded self-adjoint operators and we provide a definition of quantum Choquet expectation.

Since the notion of comonotonicity in (1) is intrinsically based on random variables, we need to resort to a characterization to extend this notion to bounded self-adjoint operators. In light of (4), we say that A and B in $B(H)_{sa}$ are comonotonic if and only if

$$\varphi(A \circ B) - \varphi(A)\varphi(B) \geq 0 \quad \text{for all mixed states } \varphi$$

where $A \circ B$ is the Jordan product. Conceptually, we are declaring A and B comonotonic if and only if their covariance is positive for each possible *quantum* expectation.

[f] That is, $\varphi(I) = 1$ where I is the identity operator.
[g] Gleason's theorem could be interpreted as a counterpart in this setting of the Riesz's representation theorem for integrals over bounded and measurable functions (see, e.g., [25, p. 125]).
[h] If A and B commute, then there exists a set $\{H_j\}_{j=1}^{p}$ of pairwise orthogonal non-zero projections which sum up to the identity I such that $A = \sum_{j=1}^{p} \alpha_j H_j$ and $B = \sum_{j=1}^{p} \beta_j H_j$ where $\{\alpha_j\}_{j=1}^{p}$ and $\{\beta_j\}_{j=1}^{p}$ are two collections of real numbers. Compared to the spectral form, the α_js and β_js might not be distinct. In light of this, in (6) we can define the common vector to be $r = (\varphi(H_1), \ldots, \varphi(H_p))$.

In Theorem 2, we prove that comonotonicity implies commutativity. In Theorem 3, we then characterize comonotonicity as a strong form of commutativity. Indeed, for $A, B \in B(H)_{sa}$ the following statements are equivalent:

(i) The operators A and B are comonotonic;
(ii) There exist $C \in B(H)_{sa}$ and two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$A = f(C) \quad \text{and} \quad B = g(C).$$

Since it is well known that A and B commute if and only if (ii) holds without the requirement that f and g are increasing, we can say that, intuitively, commutativity is to comonotonicity as bounded variation functions are to increasing functions. This statement is made formal in Corollary 1.

Given this notion of comonotonicity, we can define comonotonic additivity and characterize it. To do so, we define the notion of quantum capacity. We denote by $P(H)$ the set of projections of H. A function $\nu : P(H) \to [0, 1]$ is a quantum capacity if and only if

(1) $\nu(0) = 0$ and $\nu(I) = 1$;
(2) If $E, F \in P(H)$ and $F \geq E$, then $\nu(F) \geq \nu(E)$.

In this context, the Choquet expectation, denoted by $\mathbb{E}_\nu(A)$, is then defined as follows

$$\mathbb{E}_\nu(A) = \sum_{i=1}^{p_A} (\alpha_i - \alpha_{i+1}) \nu \left(\sum_{j=1}^{i} E_j \right)$$

where $\sum_{i=1}^{p_A} \alpha_i E_i$ is the spectral form of A and, by convention, we set $\alpha_{p_A+1} = 0$.[i] Clearly, this definition is based on the Choquet integral for simple random variables in (5). It is also immediate to show that when ν is a quantum probability, $\mathbb{E}_\nu(A)$ is a standard quantum expectation. Finally, given a functional $\phi : B(H)_{sa} \to \mathbb{R}$ we say that ϕ is:

(a) c-monotone if and only if

$$A \geq B \quad \text{and} \quad A \text{ and } B \text{ commute} \Rightarrow \phi(A) \geq \phi(B);$$

(b) comonotonic additive if and only if

$$A \text{ and } B \text{ are comonotonic} \Rightarrow \phi(A + B) = \phi(A) + \phi(B).$$

In Theorem 4 we provide a quantum counterpart to Theorem 1. Indeed, we show that, given $\phi : B(H)_{sa} \to \mathbb{R}$, the following statements are equivalent:

(i) ϕ is comonotonic additive, c-monotone, and such that $\phi(I) = 1$;

[i]Actually, the collection $\{\alpha_j\}_{j=1}^{p_A}$ is the set of distinct eigenvalues of A, ordered from the greatest to the smallest.

(ii) There exists a quantum capacity $\nu : P(H) \to [0,1]$ such that

$$\phi(A) = \mathbb{E}_\nu(A) \quad \forall\, A \in B(H)_{sa}.$$

Moreover, ν is unique.

A Physical Point of View. Following Bell [4], Varadarajan [25, pp. 124–125] observes that the requirement of linearity for a mixed state φ might be too stringent from a physical point of view. The idea is that the mathematical conditions on $\phi : B(H)_{sa} \to \mathbb{R}$ that have a physical interpretation should be the ones that only involve commuting observables. For example, the following additivity condition

$$\phi(A + B) = \phi(A) + \phi(B) \quad \text{provided } AB = BA \tag{8}$$

abides by this requirement. Similarly, comonotonic additivity is another property in line with this view (see Theorem 3 and the previous discussion). Yet, linearity and monotonicity are not properties which only involve commuting observables. Varadarajan calls *physical states* the real-valued functionals on $B(H)_{sa}$ that satisfy (8) and are also positive and normalized. Physical states correspond exactly to the functionals generated by a quantum probability.[j] On the same vein, in our work we call *physical Choquet states* the functionals that are comonotonic additive, *c*-monotone, and such that $\phi(I) = 1$. Given Gleason's and von Neumann's theorem, if $\dim H \geq 3$ one can then conclude that there are no dispersion-free physical states. In contrast, there are plenty of dispersion-free physical Choquet states (see Remark 3).

Related Literature. To the best of our knowledge, only Vourdas [26] tried to extend the notion of comonotonicity and Choquet integration to the Quantum Mechanics framework. Despite having a similar goal, the final result seems to be very different. Most strikingly, the Choquet integral studied there is an operator-valued map from $B(H)_{sa}$ to $B(H)_{sa}$ and, in our language, an integral with respect to a specific projection-valued capacity. Consequently, the corresponding notion of comonotonicity is different.

Organization of the Paper. In Sec. 2, we introduce the main mathematical preliminaries. In Sec. 3, the core of our paper, we study comonotonicity and Choquet integration for the space of bounded self-adjoint operators, namely, $B(H)_{sa}$.[k] Since proofs are often long, we place them in Appendix A along with some ancillary results.

[j]Recall that Gleason's theorem proves that, if $\dim H \geq 3$, then the notion of physical state coincides with that of mixed state. Nowadays, most of the theory is discussed for functionals defined over the entire space $B(H)$, since the extension from $B(H)_{sa}$ is seamless. In contrast, we operate over $B(H)_{sa}$ since, already in a commutative framework, it is not obvious how to characterize Choquet integration when integrands are allowed to be complex-valued.
[k]A similar mathematical analysis can be carried out for commutative and associative Banach algebras which admit a concrete representation as a space $C(K)$ where K is an Hausdorff and compact space.

2. Preliminaries

Let H be a non-trivial complex Hilbert space with inner product $\langle\ ,\ \rangle$.[l] We denote by $B(H)$ the space of bounded linear operators $A : H \to H$, endowed with the operator norm. This space is a Banach algebra once we consider as multiplication the composition. We denote the product of two elements simply by juxtaposition, so A times B is written AB. By $B(H)_{sa}$ we denote the subspace of self-adjoint operators endowed with the operator norm. Recall that $B(H)_{sa}$ is a *real Jordan* algebra under the Jordan product $A \circ B = \frac{1}{2}(AB + BA)$ (see, e.g., [2, Chap. 1]).[m] An element A in $B(H)$ is positive, written $A \geq 0$, if and only if

$$\langle A(x), x\rangle \geq 0 \quad \forall\, x \in H.$$

In particular, this implies that $A \in B(H)_{sa}$. The binary relation $\geq$ defined by

$$A \geq B \Leftrightarrow A - B \geq 0$$

makes $B(H)_{sa}$ a real ordered vector space. A linear functional $\varphi : B(H)_{sa} \to \mathbb{R}$ is positive (respectively, normalized) if and only if $\varphi(A) \geq 0$ for all $A \geq 0$ (respectively, $\varphi(I) = 1$). In the literature of operator algebras, a normalized, positive, and linear functional $\varphi : B(H)_{sa} \to \mathbb{R}$ is also called (*mixed*) *state*.[n]

As common in this literature, we define the state space and its extreme points (see, e.g., [1, pp. 10–11]):

$$S = \{\varphi \in B(H)^*_{sa} : \varphi \geq 0 \text{ and } \varphi(I) = 1\} \quad \text{and} \quad K = \operatorname{ext} S.$$

Elements of K are called *pure states*. If H is finite dimensional, then self-adjoint operators can be identified with Hermitian matrices and it is well known (see, e.g., [22]) that φ is an extreme point of S if and only if it can be written as

$$\varphi(A) = \langle A(w), w\rangle \quad \forall\, A \in B(H)_{sa},$$

where w is a unit vector in H.

We conclude by recalling the spectral theorem when H is finite dimensional (see, e.g., [16, p. 156]). Given $A \in B(H)_{sa}$, we have that there exist $p_A \in \mathbb{N}$, $\{\alpha_j\}_{j=1}^{p_A} \subseteq \mathbb{R}$, and $\{E_j\}_{j=1}^{p_A} \subseteq B(H)_{sa}$ such that:

(1) $\alpha_j \neq \alpha_k$, provided $j \neq k$;
(2) E_j are pairwise orthogonal projections which are all different from 0[o];
(3) $\sum_{j=1}^{p_A} E_j = I$;
(4) $A = \sum_{j=1}^{p_A} \alpha_j E_j$.

[l]We refer the reader to [5] for most of the definitions and facts regarding complex Hilbert spaces.
[m]We remind readers that Jordan algebras are typically *not* associative. Moreover, if $A, B \in B(H)_{sa}$, then $AB \in B(H)_{sa}$ if and only if A and B commute and that is a reason why we resort to the symmetrized product $\circ$. Indeed, $A \circ B \in B(H)_{sa}$ for all $A, B \in B(H)_{sa}$.
[n]Note that a mixed state, by being normalized, positive, and linear, is also automatically Lipschitz continuous.
[o]An element $E \in B(H)$ is a projection if and only if $E \in B(H)_{sa}$ and $E^2 = E$.

Given $A \in B(H)_{sa}$ with H possibly not finite dimensional, we will refer to either the triple $(p_A, \{\alpha_j\}_{j=1}^{p_A}, \{E_j\}_{j=1}^{p_A})$ or $\sum_{j=1}^{p_A} \alpha_j E_j$ as the (finite) *spectral form* of A if and only if properties 1–4 are satisfied and, without loss of generality, the scalars αs (and the corresponding projections) have already been ordered from the greatest to the smallest, so that $\alpha_1 > \alpha_2 > \cdots > \alpha_{p_A}$.[p] Finally, given $f : \mathbb{R} \to \mathbb{R}$ and $A \in B(H)_{sa}$ with finite spectral form $(p_A, \{\alpha_j\}_{j=1}^{p_A}, \{E_j\}_{j=1}^{p_A})$, as usual we denote by $f(A)$ the element of $B(H)_{sa}$ such that

$$f(A) = \sum_{j=1}^{p_A} f(\alpha_j) E_j.$$

Since in what follows we are interested in the values that f takes on the finite set $\{\alpha_1, \ldots, \alpha_{p_A}\}$, we can assume that f is a polynomial, unless f needs to satisfy some extra property (e.g., monotonicity).

3. Main Results

3.1. *Comonotonicity and commutativity*

We start by defining a notion of comonotonicity for a pair of elements in $B(H)_{sa}$ that builds on some of the ideas presented in the introduction.

Definition 1. Let $A, B \in B(H)_{sa}$. We say that A and B are comonotonic if and only if

$$\varphi(A \circ B) - \varphi(A)\varphi(B) \geq 0 \quad \forall \varphi \in S \tag{9}$$

where $A \circ B = \frac{1}{2}(AB + BA)$ is the Jordan product.

It is well known that in Quantum Mechanics (see, e.g., [24]) the elements of S can be interpreted as expectations, while the elements of $B(H)_{sa}$ are observables. Note that (9) has a simple interpretation: A and B are comonotonic if and only if their covariance is positive for all possible mixed states.[q] In light of this interpretation, for each mixed state $\varphi \in S$ we define $Cov_\varphi(\ , \ ) : B(H)_{sa} \times B(H)_{sa} \to \mathbb{R}$ by

$$Cov_\varphi(A, B) = \varphi(A \circ B) - \varphi(A)\varphi(B) \quad \forall A, B \in B(H)_{sa}.$$

[p] Note that under these requirements the finite spectral form is unique, that is, if $(p'_A, \{\alpha'_j\}_{j=1}^{p'_A}, \{E'_j\}_{j=1}^{p'_A})$ is another finite spectral decomposition, then $p_A = p'_A$ as well as $\alpha_j = \alpha'_j$ and $E_j = E'_j$ for all $j \in \{1, \ldots, p_A\}$.

[q] With the caveat that here the product of A and B is taken with respect to the symmetrized product $\circ$ and not with the respect to the operation of composition. This interpretation is very much in line with the characterization of comonotonicity reported in the Introduction for a pair of bounded and measurable functions $f, g : \Omega \to \mathbb{R}$ (cf. (4)). Finally, observe that $\circ$ is commutative but not associative (except in trivial cases).

It is immediate to verify that:

Lemma 1. *Let $A, B, C \in B(H)_{sa}$ and $\lambda, \mu, \gamma, \delta \in \mathbb{R}$. The following statements are true*:

(1) *For each $\varphi \in S$,*

$$Cov_\varphi(A, B) = Cov_\varphi(B, A). \tag{10}$$

(2) *For each $\varphi \in S$,*

$$Cov_\varphi(\lambda A + \gamma C, B) = \lambda Cov_\varphi(A, B) + \gamma Cov_\varphi(C, B).$$

(3) *For each $\varphi \in S$,*

$$Cov_\varphi(\lambda A + \gamma I, \mu B + \delta I) = \lambda\mu Cov_\varphi(A, B). \tag{11}$$

(4) *If A and B are comonotonic and $\lambda, \mu \geq 0$, then $\lambda A + \gamma I$ and $\mu B + \delta I$ are comonotonic.*

(5) *A and B are comonotonic if and only if $-A$ and $-B$ are comonotonic.*

Our main results show that comonotonicity implies commutativity.

Theorem 2. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. If A and B are comonotonic, then they commute.*

In the finite dimensional case, we fully characterize comonotonicity and the extent to which commutativity and comonotonicity are tied together.

Theorem 3. *Let H be finite dimensional and $A, B \in B(H)_{sa}$. The following statements are equivalent*:

(i) *The operators A and B are comonotonic*;

(ii) *There exist $C \in B(H)_{sa}$ and two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that*

$$A = f(C) \quad \text{and} \quad B = g(C). \tag{12}$$

In a finite dimensional Hilbert space, it is well known that A and B commute if and only if there exist $C \in B(H)_{sa}$ and two (not necessarily increasing) functions $f, g : \mathbb{R} \to \mathbb{R}$ such that (12) holds (see, e.g., [16, p. 171]). In light of this result, comonotonicity is a strong form of commutativity in that f and g are required to be increasing as well. The next result fully characterizes commutativity in terms of comonotonicity.

Corollary 1. *Let H be finite dimensional and $A, B \in B(H)_{sa}$. The following statements are equivalent*:

(i) *The operators A and B commute*;

(ii) *There exist $A_1, A_2, B_1, B_2 \in B(H)_{sa}$ which are pairwise comonotonic and such that*

$$A = A_1 - A_2 \quad \text{and} \quad B = B_1 - B_2. \tag{13}$$

Loosely speaking, we could say that commutativity is to comonotonicity as bounded variation functions are to increasing functions.

One could also explore the following notion of comonotonicity:

Definition 2. Let $A, B \in B(H)_{sa}$. We say that A and B are dually *comonotonic* if and only if

$$[\varphi(A) - \varphi'(A)][\varphi(B) - \varphi'(B)] \geq 0 \quad \forall\, \varphi, \varphi' \in K.$$

Conceptually, we replaced the role of the states in (1) with the *pure states* in K. This is also in line with the notion of dual comonotonicity studied in [7, p. 8524]. The next result shows that dual comonotonicity is an extremely strong condition which yields a very strong form of commutativity.

Proposition 1. *Let H be finite dimensional and $A, B \in B(H)_{sa}$. The following statements are equivalent*:

(i) *The operators A and B are dually comonotonic*;
(ii) *There exist $\lambda \geq 0$ and $\mu \in \mathbb{R}$ such that either $A = \lambda B + \mu I$ or $B = \lambda A + \mu I$.*

The next corollary is an easy consequence of the previous proposition and it shows that dual comonotonicity and comonotonicity are equivalent only when $\dim H \leq 2$. In the opposite case, dual comonotonicity only implies comonotonicity.

Corollary 2. *Let H be finite dimensional and $A, B \in B(H)_{sa}$. If A and B are dually comonotonic, then A and B are comonotonic. Dual comonotonicity and comonotonicity are equivalent only when* $\dim H \leq 2$.

Remark 1. Another possible way to generalize comonotonicity to bounded self-adjoint operators is through the following condition which mimics the condition in (1), where states are replaced by unit vectors and the product by the inner product, that is,

$$\langle A(w) - A(w'), B(w) - B(w')\rangle \geq 0 \quad \forall\, w, w' \in H \text{ s.t. } \|w\| = \|w'\| = 1.$$

This notion does not seem to lead to any fruitful conclusion. It is indeed equivalent to A and B being such that $AB = BA \geq 0$, that is, commuting and having a positive product. We omit the standard proof.

3.2. *Quantum Choquet states*

In this section, we assume that H is *finite dimensional*. We denote by $\mathbb{S}$ the unit sphere $\{w \in H : \|w\| = 1\}$. Recall that a state is a linear, positive, and normalized (that is, $\varphi(I) = 1$) functional $\varphi : B(H)_{sa} \to \mathbb{R}$. It is immediate to observe that a state φ is monotone, that is, $A \geq B$ implies $\varphi(A) \geq \varphi(B)$. We call a functional $\phi : B(H)_{sa} \to \mathbb{R}$ a *Choquet state* if and only if ϕ is comonotonic additive, monotone, and normalized. The only property we need to discuss is comonotonic additivity.

Definition 3. A functional $\phi : B(H)_{sa} \to \mathbb{R}$ is comonotonic additive if and only if $\phi(A + B) = \phi(A) + \phi(B)$ whenever A and B are comonotonic.

Example 1. Define $\phi : B(H)_{sa} \to \mathbb{R}$ to be such that

$$\phi(A) = \min_{w\in\mathbb{S}} \langle A(w), w\rangle \quad \forall\, A \in B(H)_{sa}. \tag{14}$$

Clearly, ϕ is monotone and normalized. Consider now A and B comonotonic. It follows that there exist $C \in B(H)_{sa}$ and two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $A = f(C)$ and $B = g(C)$. If we assume that the spectral form of C is $\{p_C, \{\gamma_j\}_{j=1}^{p_C}, \{H_j\}_{j=1}^{p_C}\}$ where $\gamma_1 > \cdots > \gamma_{p_C}$, then $A = \sum_{j=1}^{p_C} \alpha_j H_j$ and $B = \sum_{j=1}^{p_C} \beta_j H_j$ where $\alpha_j = f(\gamma_j)$ and $\beta_j = g(\gamma_j)$ for all $j \in \{1, \ldots, p_C\}$. Since f and g are increasing, we have that $\alpha_{p_C} \le \alpha_j$ and $\beta_{p_C} \le \beta_j$ for all $j \in \{1, \ldots, p_C\}$. It follows that

$$\min_{w\in\mathbb{S}} \langle (A + B)(w), w\rangle = \alpha_{p_C} + \beta_{p_C} = \min_{w\in\mathbb{S}} \langle A(w), w\rangle + \min_{w\in\mathbb{S}} \langle B(w), w\rangle,$$

proving that ϕ is comonotonic additive and therefore a Choquet state. In Quantum Mechanics, the value $\phi(A)$ is the value of A computed at its *ground state*.

Example 2. Define $\phi : B(H)_{sa} \to \mathbb{R}$ to be such that

$$\phi(A) = \max_{w\in\mathbb{S}} \langle A(w), w\rangle \quad \forall\, A \in B(H)_{sa}. \tag{15}$$

By the same arguments contained in Example 1, we can conclude that ϕ is a Choquet state.

As it is rather customary, we denote the set of all projections of H by $P(H)$. Recall that $\rho : P(H) \to [0, 1]$ is a (finitely additive) *quantum probability* if and only if

(1) $\rho(0) = 0$ and $\rho(I) = 1$;
(2) $\rho(E + F) = \rho(E) + \rho(F)$, provided $EF = 0$.

We say that $\nu : P(H) \to [0, 1]$ is a *quantum capacity* if and only if

(1) $\nu(0) = 0$ and $\nu(I) = 1$;
(2) $\nu(F) \ge \nu(E)$, provided $F \ge E$.

Clearly, a quantum probability is a quantum capacity.

We can define the expectation of an observable A with respect to a quantum probability ρ by

$$\mathbb{E}_\rho(A) = \sum_{i=1}^{p_A} \alpha_i \rho(E_i) \tag{16}$$

where $\sum_{i=1}^{p_A} \alpha_i E_i$ is the spectral form of A.[r] At the same time, if $\dim H \geq 3$, Gleason's theorem yields that $\varphi \in S$ if and only if there exists a unique quantum probability $\rho : P(H) \to [0,1]$ such that $\varphi(A) = \mathbb{E}_\rho(A)$ for all $A \in B(H)_{sa}$.[s]

Note that the expression in (16) admits the following rewriting:

$$\mathbb{E}_\rho(A) = \sum_{i=1}^{p_A} \alpha_i \rho(E_i) = \sum_{i=1}^{p_A} (\alpha_i - \alpha_{i+1}) \rho\left(\sum_{j=1}^{i} E_j\right), \tag{17}$$

where, by convention, we set $\alpha_{p_A+1} = 0$. Given a quantum capacity ν, we can use (17) to define the notion of *quantum Choquet expectation*. More specifically,

$$\mathbb{E}_\nu(A) = \sum_{i=1}^{p_A} (\alpha_i - \alpha_{i+1}) \nu\left(\sum_{j=1}^{i} E_j\right), \tag{18}$$

where $\sum_{i=1}^{p_A} \alpha_i E_i$ is the spectral form of A and $\alpha_{p_A+1} = 0$. Since the spectral form of a bounded self-adjoint operator is unique, (18) is well defined.

Remark 2. (i) If A is not positive, then the last addendum, $(\alpha_{p_A} - \alpha_{p_A+1})\nu(I) = \alpha_{p_A}\nu(I) = \alpha_{p_A}$, in (18) is negative.

(ii) It is easy to check that the definition in (18) is valid also for the decomposition of an element A of the kind $A = \sum_{i=1}^{m} \tilde{\alpha}_i \tilde{E}_i$. Indeed, assume that $\{\tilde{\alpha}_i\}_{i=1}^m$ is a collection of real numbers such that $\tilde{\alpha}_1 \geq \tilde{\alpha}_2 \geq \cdots \geq \tilde{\alpha}_m$ and $\{\tilde{E}_i\}_{i=1}^m$ is a collection of non-zero and pairwise orthogonal projections whose sum is I. Assume also that $A = \sum_{i=1}^{m} \tilde{\alpha}_i \tilde{E}_i$. In other words, compared to the spectral form, we do not necessarily require that $\tilde{\alpha}_i \neq \tilde{\alpha}_j$ whenever $i \neq j$. By setting $\tilde{\alpha}_{m+1} = 0$, it follows that

$$\sum_{i=1}^{m} (\tilde{\alpha}_i - \tilde{\alpha}_{i+1}) \nu\left(\sum_{j=1}^{i} \tilde{E}_j\right) = \mathbb{E}_\nu(A) = \sum_{i=1}^{p_A} (\alpha_i - \alpha_{i+1}) \nu\left(\sum_{j=1}^{i} E_j\right).$$

(iii) We also have that

$$\mathbb{E}_\nu(A) = \sum_{i=1}^{m} (\tilde{\alpha}_i - \tilde{\alpha}_{i+1}) \nu\left(\sum_{j=1}^{i} \tilde{E}_j\right) = \sum_{i=1}^{m} \tilde{\alpha}_i \left[\nu\left(\sum_{j=1}^{i} \tilde{E}_j\right) - \nu\left(\sum_{j=1}^{i-1} \tilde{E}_j\right)\right]$$

where we set $\nu(\sum_{j=1}^{i-1} \tilde{E}_j) = 0$ if $i = 1$.

We next list some of the mathematical properties that quantum Choquet expectations satisfy.

[r] Recall the assumption $\alpha_1 > \cdots > \alpha_{p_A}$.
[s] See [14] and [11, Theorem 3.2.16]. In this section of the paper, recall that we assume that H is finite dimensional. In general, Gleason proved his theorem for the case $\dim H \geq 3$ where H is a complex separable Hilbert space and ρ is countably additive.

Proposition 2. *Let H be finite dimensional, $\nu : P(H) \to [0,1]$ a quantum capacity, and $A, B \in B(H)_{sa}$. The following statements are true*:

(1) (*positive homogeneity*): $\mathbb{E}_\nu(\lambda A) = \lambda \mathbb{E}_\nu(A)$ *for all* $\lambda \geq 0$.
(2) (*translation invariance*): $\mathbb{E}_\nu(A + \lambda I) = \mathbb{E}_\nu(A) + \lambda$ *for all* $\lambda \in \mathbb{R}$.
(3) (*comonotonic additivity*): $\mathbb{E}_\nu(A + B) = \mathbb{E}_\nu(A) + \mathbb{E}_\nu(B)$ *provided* A *and* B *are comonotonic.*
(4) (*c-monotonicity*): $A \geq B$ *implies* $\mathbb{E}_\nu(A) \geq \mathbb{E}_\nu(B)$ *provided* A *and* B *commute.*
(5) (*positivity/negativity*): $A \geq 0$ (*respectively*, ≤ 0) *implies* $\mathbb{E}_\nu(A) \geq 0$ (*respectively*, ≤ 0).

We proceed by characterizing quantum Choquet expectations as physical Choquet states.

Definition 4. Let H be finite dimensional and $\phi : B(H)_{sa} \to \mathbb{R}$. We say that ϕ is a physical Choquet state if and only if ϕ is comonotonic additive, c-monotone, and such that $\phi(I) = 1$.

Theorem 4. *Let H be finite dimensional and $\phi : B(H)_{sa} \to \mathbb{R}$. The following statements are equivalent*:

(i) *The functional ϕ is a physical Choquet state*;
(ii) *There exists a quantum capacity $\nu : P(H) \to [0,1]$ such that*

$$\phi(A) = \mathbb{E}_\nu(A) \quad \forall\, A \in B(H)_{sa}. \tag{19}$$

Moreover, ν is unique and such that $\nu(E) = \phi(E)$ for all $E \in P(H)$.

In light of the previous result we characterize the representing quantum capacities for the Choquet states presented in Examples 1 and 2.

Example 3. On the one hand, if ϕ is the functional in (14), the associated quantum capacity is $\nu(E) = 0$ for every $E \neq I$ and $\nu(I) = 1$. On the other hand, to the quantum capacity $\nu(E) = 1$ for $E \neq 0$ and $\nu(0) = 0$ there corresponds the functional ϕ in (15).

Clearly, Choquet states are physical Choquet states. Therefore, Choquet states admit a representation as quantum Choquet expectations (see, e.g., Example 3). The converse is not true as Example 4 shows. Namely, there are functionals induced by Choquet expectations that are not Choquet states, that is, they fail to be *fully* monotone. Indeed, the only difference between physical Choquet states and Choquet states is that the former ones are monotone only when the observables considered commute too, while the latter are always monotone.

Example 4. Consider $\bar{E} \in P(H) \backslash \{0\}$. Define

$$\mathbb{F}_{\bar{E}} = \{E \in P(H) : E \geq \bar{E}\}.$$

On the side, note that $\mathbb{F}_{\bar{E}}$ has the following properties:

(1) $0 \notin \mathbb{F}_{\bar{E}}$;
(2) $I \in \mathbb{F}_{\bar{E}}$;
(3) If $E \in \mathbb{F}_{\bar{E}}$ and $P(H) \ni F \geq E$, then $F \in \mathbb{F}_{\bar{E}}$;
(4) If $E \in \mathbb{F}_{\bar{E}}$ and $F \in \mathbb{F}_{\bar{E}}$, then $E \wedge F \in \mathbb{F}_{\bar{E}}$.[t]

Define $\nu : P(H) \to [0,1]$ by

$$\nu(E) = \begin{cases} 1 & E \in \mathbb{F}_{\bar{E}} \\ 0 & E \notin \mathbb{F}_{\bar{E}} \end{cases} \quad \forall\, E \in P(H). \tag{20}$$

In light of properties 1–3, we have that ν is a quantum capacity. Note that[u]:

$$\mathbb{E}_\nu(A) = \max_{\{i \in \{1,\ldots,p_A\} \,:\, \sum_{j=1}^{i} E_j \geq \bar{E}\}} \alpha_i \quad \forall\, A \in B(H)_{sa}.$$

Assume now that $\dim H = 3$ as well as that $\{e_1, e_2, e_3\}$ is an orthonormal basis. Define an orthonormal basis $\{f_1, f_2, f_3\}$ where $f_2 = e_2$, $f_1 = \sqrt{\gamma} e_1 + \sqrt{(1-\gamma)} e_3$ with $\gamma = (\frac{99}{100})^2$ and f_3 is a unit vector orthogonal to f_1 and f_2. Let E_i be the projections associated to the spaces $\mathrm{span}\{e_i\}$ for all $i \in \{1,\ldots,3\}$. Similarly, let F_i be the projections associated to the spaces $\mathrm{span}\{f_i\}$ for all $i \in \{1,\ldots,3\}$. Consider now the scalars $\alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 2$ and $\beta_1 = 3, \beta_2 = 2, \beta_3 = 1$. Define

$$A = \sum_{j=1}^{3} \alpha_j E_j \quad \text{and} \quad B = \sum_{j=1}^{3} \beta_j F_j.$$

Some tedious computations yield that $A \geq B$. Now let $\bar{E} = F_1$. We have that

$$\mathbb{E}_\nu(A) = (\alpha_1 - \alpha_2)\nu(E_1) + (\alpha_2 - \alpha_3)\nu(E_1 + E_2) + \alpha_3. \tag{21}$$

By contradiction, assume that $E_1 + E_2 \geq F_1$. This would imply that

$$\mathrm{span}\{f_1\} \subseteq \mathrm{span}\{e_1\} \oplus \mathrm{span}\{e_2\} = \mathrm{span}\{e_1, e_2\}$$

and, in particular, $f_1 \in \mathrm{span}\{e_1, e_2\}$. But, by construction we have that $f_1 = \sqrt{\gamma} e_1 + \sqrt{1-\gamma} e_3$, yielding that $e_3 \in \mathrm{span}\{e_1, e_2\}$, a contradiction. We conclude that $E_1 + E_2 \notin \mathbb{F}_{\bar{E}}$ and $\nu(E_1) = \nu(E_1 + E_2) = 0$. Thus, given (21) we obtain that $\mathbb{E}_\nu(A) = \alpha_3$, while

$$\mathbb{E}_\nu(B) = (\beta_1 - \beta_2)\nu(F_1) + (\beta_2 - \beta_3)\nu(F_1 + F_2) + \beta_3 = \beta_1.$$

This proves that

$$\mathbb{E}_\nu(A) = \alpha_3 < \beta_1 = \mathbb{E}_\nu(B),$$

thus violating monotonicity.

[t]Recall that $E \wedge F$ is the projection associated to the closed vector subspace $\mathrm{Range}\, E \cap \mathrm{Range}\, F$.
[u]Recall that $\alpha_1 > \cdots > \alpha_{p_A}$.

Remark 3. Let $\infty > \dim H \geq 3$. In Quantum Mechanics, a state $\varphi \in S$ is dispersion-free (see, e.g., [19]) if and only if

$$\sigma(\varphi) = \sup_{E \in P(H)} [\varphi(E) - \varphi(E)^2] = 0. \tag{22}$$

It is well known that, under the current assumptions, there are no dispersion-free states. In other words, physical states cannot be dispersion-free. Conversely, if we consider physical Choquet states, then there are several which are dispersion-free: for example, all the ones generated by the quantum capacities defined as in (20). Another interesting example of dispersion-free physical Choquet state is the quantum median as defined in Example 6.

We close with some examples of physical Choquet states, quantum capacities, and comonotonicity.

Example 5 (Courant–Fisher). Examples 1 and 2 can be generalized by considering Courant–Fisher's minimax functionals. Let $\dim H = n$. For each $k \in \{1, \ldots, n\}$ define $\phi_k : B(H)_{sa} \to \mathbb{R}$ by

$$\phi_k(A) = \lambda_k \quad \forall\, A \in B(H)_{sa}$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ are the not necessarily distinct eigenvalues of A.[v] It can be shown that (see [16, p. 181])

$$\phi_k(A) = \inf_{V : \dim V = n-k+1} \max_{w \in \mathbb{S} \cap V} \langle A(w), w \rangle \quad \forall\, A \in B(H)_{sa}.$$

Given Theorem 3, it is routine to check that ϕ_k is a Choquet state and, in particular, a physical Choquet state. By Theorem 4, the associated quantum capacities are

$$\nu_k(E) = \phi_k(E) = \begin{cases} 1 & \text{if } \dim \operatorname{Range} E > k-1 \\ 0 & \text{if } \dim \operatorname{Range} E \leq k-1 \end{cases} \quad \forall\, E \in P(H). \tag{23}$$

Example 6 (Quantum Median and Quantum Quantiles). Let $\rho : P(H) \to [0,1]$ be a quantum probability. Define $\phi : B(H)_{sa} \to \mathbb{R}$ by

$$\phi(A) = \alpha_r \quad \forall\, A \in B(H)_{sa}$$

where given the spectral form of $\sum_{i=1}^{p_A} \alpha_i E_i$ of A the value $r \in \{1, \ldots, p_A\}$ is the minimum value such that[w]

$$\sum_{j=1}^{r-1} \rho(E_j) < \frac{1}{2} \quad \text{and} \quad \sum_{j=1}^{r} \rho(E_j) \geq \frac{1}{2}.$$

In words, α_r is the maximum value of A such that the (quantum) probability of observing a value greater than or equal to α_r is at least 0.5, i.e. the (quantum)

[v] Thus, accounting for multiplicity.
[w] With the convention that $\sum_{j=1}^{r-1} \rho(E_j) = 0$ if $r = 1$.

median. It is not difficult to show that ϕ is a physical Choquet state with representing quantum capacity

$$\nu(E) = \begin{cases} 1 & \rho(E) \geq \dfrac{1}{2} \\ 0 & \rho(E) < \dfrac{1}{2} \end{cases} \quad \forall E \in P(H).$$

Note that the quantum median is another dispersion-free physical Choquet state. Generalizing the previous discussion to any quantile is rather straightforward. Indeed, all is needed to do is to replace everywhere $1/2$ with $1 - q$ with $q \in (0, 1)$.

Example 7. Building on some examples coming from Economics, one could think of a quantum capacity defined as $\nu = f \circ \rho$ where ρ is a quantum probability and $f : [0, 1] \to [0, 1]$ is a strictly increasing and continuous function such that $f(0) = 0$ and $f(1) = 1$. These quantum capacities are very different from the ones described above since they are typically not $\{0, 1\}$-valued.[x]

Example 8. Assume that $E, F \in P(H)$ commute. It is well known that $E + F - EF, EF \in P(H)$. Moreover, we have that

$$E + F - EF = (E - F)^2 + EF \geq EF.$$

What is less obvious to see is that $E + F - EF$ and EF are comonotonic. Indeed, since $E + F - EF \in P(H)$, we have that, for each $\varphi \in S$, $\varphi(E + F - EF) \in [0, 1]$ and

$$\begin{aligned} Cov_\varphi(E + F - EF, EF) &= \varphi(EF) - \varphi(E + F - EF)\varphi(EF) \\ &= \varphi(EF)(1 - \varphi(E + F - EF)) \geq 0, \end{aligned}$$

yielding comonotonicity. By Theorem 4, if ν is a quantum capacity, then

$$\begin{aligned} \mathbb{E}_\nu(E + F) = \mathbb{E}_\nu(E + F - EF + EF) &= \mathbb{E}_\nu(E + F - EF) + \mathbb{E}_\nu(EF) \\ &= \nu(E + F - EF) + \nu(EF). \end{aligned}$$

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[x]In Economics, Choquet expectations with respect to distortions of additive probabilities have been originally used to explain choice patterns not consistent with the linear expected utility model of von Neumann and Morgenstern.

Appendix A

A.1. *Comonotonicity and commutativity*

The main goal of this appendix is to show that comonotonicity implies commutativity. The rest will be rather standard. We begin by showing that comonotonicity implies a form of comonotonicity which is similar to dual comonotonicity, yet less stringent. We term such a property DP comonotonicity. Then, we prove that, given $A, B \in B(H)_{sa}$ with finite spectral forms $\sum_{j=1}^{p_A} \alpha_j E_j$ and $\sum_{j=1}^{p_B} \beta_j F_j$, if they are DP comonotonic, then the projections E_1 and F_1 commute.[y] We will then proceed by induction (Lemmas 6 and 7 as well as Theorem 5) and show that each projection E_i commutes with each projection F_l. This in turn yields that A and B commute.

In order to do so, we need the next two simple, yet crucial, results and a well known fact. We start by the well known fact.[z]

Lemma 2. *Let E, F, and G be three projections such that $G \leq F$. The following statements are equivalent*:

(i) $EF = FE$;
(ii) $F(\text{Range}\, E) \subseteq \text{Range}\, E$;
(iii) $E(\text{Range}\, F) \subseteq \text{Range}\, F$.

In particular, if E and F commute, then the following statements are equivalent:

(a) $EG = GE$;
(b) $G(\text{Range}\, E \cap \text{Range}\, F) \subseteq \text{Range}\, E \cap \text{Range}\, F$.

Lemma 3. *Let $A, B \in B(H)_{sa}$. If $\varphi, \varphi' \in S$ are such that $\varphi(A \circ B) = \varphi(A)\varphi(B)$ and $\varphi'(A \circ B) = \varphi'(A)\varphi'(B)$, then*

$$Cov_{\bar{\varphi}}(A, B) = \frac{1}{4}[\varphi(A) - \varphi'(A)][\varphi(B) - \varphi'(B)]$$

where $\bar{\varphi} = \frac{1}{2}\varphi + \frac{1}{2}\varphi'$.

Proof. Consider the state $\bar{\varphi} = \frac{1}{2}\varphi + \frac{1}{2}\varphi' \in S$. Since $\varphi(A \circ B) = \varphi(A)\varphi(B)$ and $\varphi'(A \circ B) = \varphi'(A)\varphi'(B)$, it follows that

$$\begin{aligned} Cov_{\bar{\varphi}}(A, B) &= \bar{\varphi}(A \circ B) - \bar{\varphi}(A)\bar{\varphi}(B) \\ &= \frac{1}{2}\varphi(A \circ B) + \frac{1}{2}\varphi'(A \circ B) \\ &\quad - \frac{1}{4}\varphi(A)\varphi(B) - \frac{1}{4}\varphi(A)\varphi'(B) - \frac{1}{4}\varphi'(A)\varphi(B) - \frac{1}{4}\varphi'(A)\varphi'(B) \end{aligned}$$

[y] Throughout the appendix, the notation $\sum_{j=1}^{p_A} \alpha_j E_j$ and $\sum_{j=1}^{p_B} \beta_j F_j$ will indicate the spectral forms of A and B respectively, where $\alpha_1 > \cdots > \alpha_{p_A}$ as well as $\beta_1 > \cdots > \beta_{p_B}$.
[z] The reader can find a version of this result in [10, Lemma 9.2]. As the reader can verify, the result and its proof hold also when H is a complex Hilbert space. Alternatively, one could apply [5, Theorem 4, p. 158].

$$= \frac{1}{2}\varphi(A)\varphi(B) + \frac{1}{2}\varphi'(A)\varphi'(B)$$

$$- \frac{1}{4}\varphi(A)\varphi(B) - \frac{1}{4}\varphi(A)\varphi'(B) - \frac{1}{4}\varphi'(A)\varphi(B) - \frac{1}{4}\varphi'(A)\varphi'(B)$$

$$= \frac{1}{4}\varphi(A)\varphi(B) + \frac{1}{4}\varphi'(A)\varphi'(B) - \frac{1}{4}\varphi(A)\varphi'(B) - \frac{1}{4}\varphi'(A)\varphi(B)$$

$$= \frac{1}{4}\varphi(A)[\varphi(B) - \varphi'(B)] + \frac{1}{4}\varphi'(A)[\varphi'(B) - \varphi(B)]$$

$$= \frac{1}{4}[\varphi(A) - \varphi'(A)][\varphi(B) - \varphi'(B)],$$

proving the statement. □

Given a unit vector $w \in H$, we denote by $\varphi_w : B(H)_{sa} \to \mathbb{R}$ the state such that

$$\varphi_w(A) = \langle A(w), w\rangle \quad \forall\, A \in B(H)_{sa}.$$

Vice versa, by φ_w, we will always mean a state induced by a unit vector w, as above.

Lemma 4. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. If A and B are comonotonic, then for each $i \in \{1, \ldots, p_A\}$ and $l \in \{1, \ldots, p_B\}$ and for each pair of unit vectors $w \in \operatorname{Range} E_i$ and $w' \in \operatorname{Range} F_l$*

$$[\varphi_w(A) - \varphi_{w'}(A)][\varphi_w(B) - \varphi_{w'}(B)] \geq 0.$$

Proof. Let $i \in \{1, \ldots, p_A\}$ and $l \in \{1, \ldots, p_B\}$. Define $\varphi = \varphi_w$ with $w \in \operatorname{Range} E_i$. It follows that

$$\langle AB(w), w\rangle = \langle B(w), A^*(w)\rangle = \langle B(w), A(w)\rangle = \alpha_i\langle B(w), w\rangle$$

and

$$\langle BA(w), w\rangle = \langle A(w), B^*(w)\rangle = \langle A(w), B(w)\rangle = \alpha_i\langle w, B(w)\rangle = \alpha_i\langle B(w), w\rangle.$$

We can conclude that $\varphi(A \circ B) = \alpha_i\langle B(w), w\rangle = \varphi(A)\varphi(B)$. Define $\varphi' = \varphi_{w'}$ with $w' \in \operatorname{Range} F_l$. Similar computations yield that $\varphi'(A \circ B) = \beta_l\langle A(w'), w'\rangle = \varphi'(A)\varphi'(B)$. By Lemma 3 and since A and B are comonotonic, we have that

$$[\varphi(A) - \varphi'(A)][\varphi(B) - \varphi'(B)] = 4Cov_{\bar{\varphi}}(A, B) \geq 0,$$

proving the statement. □

In light of Definition 2 and Lemma 4, we make the following definition.

Definition 5. Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. We say that A and B are DP comonotonic if and only if for each

$i \in \{1, \ldots, p_A\}$ and $l \in \{1, \ldots, p_B\}$ and for each pair of unit vectors $w \in \operatorname{Range} E_i$ and $w' \in \operatorname{Range} F_l$

$$[\varphi_w(A) - \varphi_{w'}(A)][\varphi_w(B) - \varphi_{w'}(B)] \geq 0. \tag{24}$$

We term this type of comonotonicity DP comonotonicity, since it is very close to *dual* comonotonicity (see Definition 2), but it is also tightly connected to the *projections* representing A and B. We are ready to prove that if A and B are DP comonotonic, then E_1 and F_1 commute.

Lemma 5. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. If A and B are DP comonotonic, then E_1 and F_1 commute, that is, $E_1F_1 = F_1E_1$.*

Proof. Clearly, we can assume that both p_A and p_B are strictly greater than 1. Otherwise, either $E_1 = I$ or $F_1 = I$ and the statement trivially follows. Let $w \in \operatorname{Range} E_1$ and $w' \in \operatorname{Range} F_1$. Since $E_1, F_1 \neq 0$, we can choose them to be such that $\|w\|^2 = 1 = \|w'\|^2$. Note that

$$\varphi_w(A) = \alpha_1 \quad \text{and} \quad \varphi_{w'}(A) = \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2,$$

where $w'_j = E_j(w')$ for all $j \in \{1, \ldots, p_A\}$ as well as

$$\varphi_w(B) = \sum_{j=1}^{p_B} \beta_j \|w_j\|^2 \quad \text{and} \quad \varphi_{w'}(B) = \beta_1,$$

where $w_j = F_j(w)$ for all $j \in \{1, \ldots, p_B\}$. Since A and B are DP comonotonic, we have that

$$\left(\alpha_1 - \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2\right)\left(\sum_{j=1}^{p_B} \beta_j \|w_j\|^2 - \beta_1\right) \geq 0. \tag{25}$$

Since w and w' are unit vectors and the elements $\{E_j\}_{j=1}^{p_A}$ (respectively, $\{F_j\}_{j=1}^{p_B}$) are pairwise orthogonal, it follows that $\sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2$ and $\sum_{j=1}^{p_B} \beta_j \|w_j\|^2$ are weighted averages.

By contradiction, assume now that E_1 and F_1 do not commute. By Lemma 2, there exist $w \in \operatorname{Range} E_1$ and $w' \in \operatorname{Range} F_1$ such that $F_1(w) \notin \operatorname{Range} E_1$ and $E_1(w') \notin \operatorname{Range} F_1$. Clearly, we have that $w \neq 0 \neq w'$. Without loss of generality, we can assume that w and w' are unit vectors. Since $\sum_{j=1}^{p_B} \|w_j\|^2 = \|w\|^2 = 1$, it follows that $1 > \|w_1\|^2 \geq 0$.[aa] In particular, this implies that $1 \geq \|w_j\|^2 > 0$ for some $j \in \{2, \ldots, p_B\}$. Similarly, since $\sum_{j=1}^{p_A} \|w'_j\|^2 = \|w'\|^2 = 1$, it follows that $1 > \|w'_1\|^2 \geq 0$ and $1 \geq \|w'_j\|^2 > 0$ for some $j \in \{2, \ldots, p_A\}$. Since $\beta_1 > \beta_j$ for all

[aa]Otherwise, we would have that $\|w_1\|^2 = 1$. This would imply that $\|w_j\|^2 = 0$ for all $j \in \{2, \ldots, p_B\}$, that is, $w_j = 0$ for all $j \in \{2, \ldots, p_B\}$. In turn, this would yield that $\operatorname{Range} E_1 \ni w = w_1 = F_1(w) \notin \operatorname{Range} E_1$, a contradiction.

$j \in \{2, \ldots, p_B\}$ and $\alpha_1 > \alpha_j$ for all $j \in \{2, \ldots, p_A\}$, this implies that

$$\alpha_1 - \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2 > 0 \quad \text{and} \quad \sum_{j=1}^{p_B} \beta_j \|w_j\|^2 - \beta_1 < 0,$$

a contradiction with (25). □

We now extend the previous statement by showing that if A and B are DP comonotonic, then E_1 commutes with all the elements of $\{F_j\}_{j=1}^{p_B}$ and F_1 commutes with all the elements of $\{E_j\}_{j=1}^{p_A}$.

Lemma 6. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. If A and B are DP comonotonic, then E_1 and F_j commute for all $j \in \{1, \ldots, p_B\}$. Similarly, F_1 and E_j commute for all $j \in \{1, \ldots, p_A\}$.*

Proof. As in the proof of Lemma 5, we can assume that both p_A and p_B are strictly greater than 1. Otherwise, either $E_1 = I$ or $F_1 = I$ and the statement trivially follows. We prove by induction that E_1 commutes with all the elements of $\{F_j\}_{j=1}^{k}$ for all $k \in \{1, \ldots, p_B\}$.

Initial Step. $k = 1$. It follows from Lemma 5.

Inductive Step. Assume the statement is true for $1 \leq k < p_B$. We next show it holds for $k+1$. We only need to show that $E_1 F_{k+1} = F_{k+1} E_1$. Define $F = I - \sum_{j=1}^{k} F_j$. Since E_1 commutes with each F_j with $j \in \{1, \ldots, k\}$, we have that E_1 and F commute

$$\begin{aligned} E_1 F = E_1 \left(I - \sum_{j=1}^{k} F_j \right) &= E_1 - \sum_{j=1}^{k} E_1 F_j \\ &= E_1 - \sum_{j=1}^{k} F_j E_1 = \left(I - \sum_{j=1}^{k} F_j \right) E_1 = F E_1. \end{aligned} \tag{26}$$

We now have two cases:

(1) $k+1 = p_B$. It follows that $F_{k+1} = I - \sum_{j=1}^{p_B - 1} F_j = I - \sum_{j=1}^{k} F_j = F$. By (26), this implies that E_1 and F_{k+1} commute.

(2) $k+1 < p_B$. Since $\sum_{j=k+1}^{p_B} F_j = F$, observe that $F_{k+1} \leq F$. By contradiction, assume that E_1 and F_{k+1} do not commute. By Lemma 2, this implies that there exists $w \in \operatorname{Range} E_1 \cap \operatorname{Range} F$ such that $F_{k+1}(w) \notin \operatorname{Range} E_1 \cap \operatorname{Range} F$. Without loss of generality, we can assume that w is a unit vector. Given w, define $w_j = F_j(w)$ for all $j \in \{1, \ldots, p_B\}$. Since $w \in \operatorname{Range} F$, it follows that $w_j = 0$ for all $j \in \{1, \ldots, k\}$. At the same time, since $\|w\|^2 = 1$, we have that

$$1 = \|w\|^2 = \left\| \sum_{j=k+1}^{p_B} w_j \right\|^2 = \sum_{j=k+1}^{p_B} \|w_j\|^2$$

and $1 > \|w_{k+1}\|^2 \geq 0$.[ab] In particular, we have that $1 \geq \|w_j\|^2 > 0$ for some $j \in \{k+2, \ldots, p_B\}$. By Lemma 2 and since E_1 and F_{k+1} do not commute, there exists $w' \in \operatorname{Range} F_{k+1}$ such that $E_1(w') \notin \operatorname{Range} F_{k+1}$. Without loss of generality, we can assume that w' is a unit vector. Given w', define $w'_j = E_j(w')$ for all $j \in \{1, \ldots, p_A\}$. Since $\|w'\|^2 = 1$, we have that

$$1 = \|w'\|^2 = \left\| \sum_{j=1}^{p_A} w'_j \right\|^2 = \sum_{j=1}^{p_A} \|w'_j\|^2$$

and $1 > \|w'_1\|^2 \geq 0$.[ac] In particular, we have that $1 \geq \|w'_j\|^2 > 0$ for some $j \in \{2, \ldots, p_A\}$. It follows that

$$\alpha_1 - \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2 > 0 \quad \text{and} \quad \sum_{j=1}^{p_B} \beta_j \|w_j\|^2 - \beta_{k+1} = \sum_{j=k+1}^{p_B} \beta_j \|w_j\|^2 - \beta_{k+1} < 0.$$

We can conclude that there exist two unit vectors $w \in \operatorname{Range} E_1$ and $w' \in \operatorname{Range} F_{k+1}$ such that

$$0 > \left(\alpha_1 - \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2 \right) \left(\sum_{j=1}^{p_B} \beta_j \|w_j\|^2 - \beta_{k+1} \right)$$
$$= [\varphi_w(A) - \varphi_{w'}(A)][\varphi_w(B) - \varphi_{w'}(B)] \geq 0,$$

a contradiction with A and B being DP comonotonic, proving the inductive step.

The first part of the statement follows by induction. The rest of the statement follows given the symmetric role of A and B in the definition of DP comonotonicity. □

Next, we extend the previous result.

Lemma 7. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form where $p_A > 1$. If A and B are DP comonotonic and if all the elements of $\{E_j\}_{j=1}^k$*

[ab]Otherwise, if $\|w_{k+1}\|^2 = 1$, we would have that $\|w_j\|^2 = 0$ for all $j \in \{k+2, \ldots, p_B\}$, yielding that $w_j = 0$ for all $j \in \{1, \ldots, p_B\} \backslash \{k+1\}$. This would imply that

$$\operatorname{Range} E_1 \cap \operatorname{Range} F \ni w = w_{k+1} = F_{k+1}(w) \notin \operatorname{Range} E_1 \cap \operatorname{Range} F,$$

a contradiction.

[ac]Otherwise, if $\|w'_1\|^2 = 1$, we would have that $\|w'_j\|^2 = 0$ for all $j \in \{2, \ldots, p_A\}$, that is, $w'_j = 0$ for all $j \in \{2, \ldots, p_A\}$. This would imply that

$$\operatorname{Range} F_{k+1} \ni w' = w'_1 = E_1(w') \notin \operatorname{Range} F_{k+1},$$

a contradiction.

pairwise commute with all the elements of $\{F_j\}_{j=1}^{p_B}$ *for some* $1 \leq k < p_A$, *then* E_{k+1} *commutes with* F_l *for all* $l \in \{1, \ldots, p_B\}$.

Proof. Assume that A and B are DP comonotonic. Assume also that all the elements of $\{E_j\}_{j=1}^{k}$ pairwise commute with all the elements of $\{F_j\}_{j=1}^{p_B}$ for some $1 \leq k < p_A$. We want to show that $E_{k+1}F_l = F_l E_{k+1}$ for all $l \in \{1, \ldots, p_B\}$. If $p_B = 1$, then the statement is trivial since $F_1 = I$. Otherwise, $p_B > 1$ and we proceed by induction on l.

Initial Step. $l = 1$. It follows from Lemma 6.

Inductive Step. Consider $l \in \{1, \ldots, p_B - 1\}$ such that $E_{k+1}F_{l'} = F_{l'}E_{k+1}$ for all $l' \in \{1, \ldots, l\}$. We only need to show that $E_{k+1}F_{l+1} = F_{l+1}E_{k+1}$. We have two cases:

(1) $l + 1 = p_B$. In this case, we have that $l = p_B - 1$. It follows that

$$\begin{aligned} E_{k+1} - E_{k+1}F_{p_B} &= E_{k+1}(I - F_{p_B}) = E_{k+1}\left(\sum_{l'=1}^{p_B-1} F_{l'}\right) = \sum_{l'=1}^{p_B-1} E_{k+1}F_{l'} \\ &= \sum_{l'=1}^{p_B-1} F_{l'}E_{k+1} = \left(\sum_{l'=1}^{p_B-1} F_{l'}\right)E_{k+1} = (I - F_{p_B})E_{k+1} \\ &= E_{k+1} - F_{p_B}E_{k+1}, \end{aligned}$$

yielding that $F_{p_B}E_{k+1} = E_{k+1}F_{p_B}$.

(2) $l + 1 < p_B$. By *hypothesis*, F_{l+1} and E_j commute for all $j \in \{1, \ldots, k\}$. This implies that F_{l+1} commutes with $E = I - \sum_{j=1}^{k} E_j = \sum_{j=k+1}^{p_A} E_j$. We have two subcases:

(a) $k + 1 = p_A$. In this case, we have that $E_{k+1} = E$, yielding that E_{k+1} and F_{l+1} commute.

(b) $k + 1 < p_A$. By contradiction, assume that F_{l+1} does not commute with E_{k+1}. By Lemma 2 and since $E_{k+1} \leq E$, it follows that there exists $w' \in \operatorname{Range} F_{l+1} \cap \operatorname{Range} E$ such that $E_{k+1}(w') \notin \operatorname{Range} F_{l+1} \cap \operatorname{Range} E$. Without loss of generality, we can assume that w' is a unit vector. Given w', define $w'_j = E_j(w')$ for all $j \in \{1, \ldots, p_A\}$. Since $w' \in \operatorname{Range} E$, it follows that $w'_j = 0$ for all $j \in \{1, \ldots, k\}$. At the same time, since $\|w'\|^2 = 1$, we have that

$$1 = \|w'\|^2 = \left\|\sum_{j=k+1}^{p_A} w'_j\right\|^2 = \sum_{j=k+1}^{p_A} \|w'_j\|^2$$

and $1 > \|w'_{k+1}\|^2 \geq 0$.[ad] In particular, we have that $1 \geq \|w'_j\|^2 > 0$ for some $j \in \{k+2, \ldots, p_A\}$. It follows that

$$\varphi_{w'}(B) = \beta_{l+1} \quad \text{and} \quad \varphi_{w'}(A) = \sum_{j=1}^{p_A} \alpha_j \|w'_j\|^2 = \sum_{j=k+1}^{p_A} \alpha_j \|w'_j\|^2 < \alpha_{k+1}. \tag{27}$$

By *inductive hypothesis* all the elements in $\{F_{l'}\}_{l'=1}^{l}$ pairwise commute with E_{k+1}. Define $F = \sum_{l'=l+1}^{p_B} F_{l'}$. It follows that

$$\begin{aligned} E_{k+1}F &= E_{k+1}\left(I - \sum_{l'=1}^{l} F_{l'}\right) = E_{k+1} - E_{k+1}\sum_{l'=1}^{l} F_{l'} \\ &= E_{k+1} - \sum_{l'=1}^{l} E_{k+1}F_{l'} = E_{k+1} - \sum_{l'=1}^{l} F_{l'}E_{k+1} \\ &= E_{k+1} - \left(\sum_{l'=1}^{l} F_{l'}\right)E_{k+1} = \left(I - \sum_{l'=1}^{l} F_{l'}\right)E_{k+1} \\ &= FE_{k+1}. \end{aligned}$$

This proves that E_{k+1} and F commute. By Lemma 2 and since $F_{l+1} \leq F$ and F_{l+1} does not commute with E_{k+1}, it follows that there exists $w \in \operatorname{Range} E_{k+1} \cap \operatorname{Range} F$ such that $F_{l+1}(w) \notin \operatorname{Range} E_{k+1} \cap \operatorname{Range} F$. Without loss of generality, we can assume that w is a unit vector. Given w, define $w_j = F_j(w)$ for all $j \in \{1, \ldots, p_B\}$. Since $w \in \operatorname{Range} F$, it follows that $w_j = 0$ for all $j \in \{1, \ldots, l\}$. At the same time, since $\|w\|^2 = 1$, we have that

$$1 = \|w\|^2 = \left\|\sum_{j=l+1}^{p_B} w_j\right\|^2 = \sum_{j=l+1}^{p_B} \|w_j\|^2$$

and $1 > \|w_{l+1}\|^2 \geq 0$.[ae] In particular, we have that $1 \geq \|w_j\|^2 > 0$ for some $j \in \{l+2, \ldots, p_B\}$. It follows that

$$\varphi_w(B) = \sum_{j=1}^{p_B} \beta_j \|w_j\|^2 = \sum_{j=l+1}^{p_B} \beta_j \|w_j\|^2 < \beta_{l+1} \quad \text{and} \quad \varphi_w(A) = \alpha_{k+1}. \tag{28}$$

[ad]Otherwise, if $\|w'_{k+1}\|^2 = 1$, we would have that $\|w'_j\|^2 = 0$ for all $j \in \{k+2, \ldots, p_A\}$. Moreover, we would have that $w'_j = 0$ for all $j \in \{1, \ldots, p_A\} \backslash \{k+1\}$. This would imply that

$$\operatorname{Range} F_{l+1} \cap \operatorname{Range} E \ni w' = w'_{k+1} = E_{k+1}(w') \notin \operatorname{Range} F_{l+1} \cap \operatorname{Range} E,$$

a contradiction.

[ae]Otherwise, if $\|w_{l+1}\|^2 = 1$, we would have that $\|w_j\|^2 = 0$ for all $j \in \{l+2, \ldots, p_B\}$. Moreover, we would have that $w_j = 0$ for all $j \in \{1, \ldots, p_B\} \backslash \{l+1\}$. This would imply that

$$\operatorname{Range} E_{k+1} \cap \operatorname{Range} F \ni w = w_{l+1} = F_{l+1}(w) \notin \operatorname{Range} E_{k+1} \cap \operatorname{Range} F,$$

a contradiction.

Equations (27) and (28) yield that there exist $w \in \text{Range}\, E_{k+1}$ and $w' \in \text{Range}\, F_{l+1}$ such that

$$0 \leq [\varphi_w(A) - \varphi_{w'}(A)][\varphi_w(B) - \varphi_{w'}(B)] < 0,$$

a contradiction with A and B being DP comonotonic, proving the inductive step.

The statement follows by induction. □

We are ready to prove that DP comonotonicity implies commutativity.

Theorem 5. *Let $A, B \in B(H)_{sa}$ be such that they both admit a finite spectral form. If A and B are DP comonotonic, then they commute.*

Proof. As in the proof of Lemma 5, we can assume that both p_A and p_B are strictly greater than 1. Otherwise, either $E_1 = I$ or $F_1 = I$ and the statement trivially follows. We next prove by induction that for each $k \in \{1, \ldots, p_A\}$ all the elements of $\{E_j\}_{j=1}^{k}$ pairwise commute with all the elements of $\{F_j\}_{j=1}^{p_B}$.

Initial Step. $k = 1$. It follows from Lemma 6.

Inductive Step. Assume the statement is true for $1 \leq k < p_A$. We next show it holds for $k+1$. Let $i \in \{1, \ldots, k+1\}$ and $l \in \{1, \ldots, p_B\}$ and consider E_i and F_l. We have two cases:

(1) $i \leq k$. By inductive hypothesis, if $i \leq k$, then E_i and F_l commute.
(2) $i = k+1$. By Lemma 7, $E_i = E_{k+1}$ and F_l commute.

Points 1 and 2 prove the inductive step.

The statement follows by induction. In particular, by setting $k = p_A$, we have that all the elements of $\{E_j\}_{j=1}^{p_A}$ and $\{F_j\}_{j=1}^{p_B}$ pairwise commute. This yields that A and B commute. □

Proof of Theorem 2. By Lemma 4 and since A and B are comonotonic, A and B are DP comonotonic. By Theorem 5, this implies that A and B commute. □

Before discussing the finite dimensional case, we need a useful fact about comonotonic vectors.

Example 9. Let $p > 1$ and consider $\mathbb{R}^p$. Consider also the set

$$\Delta_{p-1} = \left\{ r \in \mathbb{R}_+^p : \sum_{j=1}^{p} r_j = 1 \right\}.$$

By [9, Proposition 4.5], one can prove that the following three conditions are equivalent:

(1) x and y in $\mathbb{R}^p$ are such that

$$\sum_{j=1}^{p}(x_j y_j)r_j - \left(\sum_{j=1}^{p} x_j r_j\right)\left(\sum_{j=1}^{p} y_j r_j\right) \geq 0 \quad \forall\, r \in \Delta_{p-1};$$

(2) x and y in $\mathbb{R}^p$ are such that

$$(x_i - x_j)(y_i - y_j) \geq 0 \quad \forall\, i, j \in \{1, \ldots, p\}; \tag{29}$$

(3) There exist a vector $z \in \mathbb{R}^p$ such that $z_i \neq z_j$ whenever $i \neq j$ and two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$x_i = f(z_i) \quad \text{and} \quad y_i = g(z_i) \quad \forall\, i \in \{1, \ldots, p\}.$$

Theorem 6. *Let H be finite dimensional and $A, B \in B(H)_{sa}$. The following statements are equivalent*:

(i) *The operators A and B are comonotonic*;
(ii) *The operators A and B are DP comonotonic*;
(iii) *The operators A and B are DP comonotonic and commute*;
(iv) *There exist $C \in B(H)_{sa}$ and two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that*

$$A = f(C) \quad \text{and} \quad B = g(C).$$

Proof. Before starting, recall that given $C \in B(H)_{sa}$, the notation $f(C)$ means that

$$f(C) = \sum_{j=1}^{p_C} f(\gamma_j) H_j$$

where $\sum_{j=1}^{p_C} \gamma_j H_j$ is the spectral form of C. Consider now a matrix $A = \sum_{j=1}^{p} \alpha_j H_j$ where $\{H_j\}_{j=1}^{p}$ is a collection of non-zero pairwise orthogonal projections such that $\sum_{j=1}^{p} H_j = I$.[af] Observe that trivially

$$\varphi(A) = \sum_{j=1}^{p} \alpha_j \varphi(H_j) \quad \forall\, \varphi \in S, \tag{30}$$

where $(\varphi(H_1), \ldots, \varphi(H_p))$ is a vector that belongs to Δ_{p-1}, that is, $\varphi(H_j) \geq 0$ for all $j \in \{1, \ldots, p\}$ and $\sum_{j=1}^{p} \varphi(H_j) = \varphi(\sum_{j=1}^{p} H_j) = \varphi(I) = 1$. Finally, since H is finite dimensional, we have that A and B both admit a finite spectral form.

(i) implies (ii). By Lemma 4 and since A and B are comonotonic and both admit a finite spectral form, A and B are DP comonotonic.

[af] Note that this might not be the spectral form of A, since we did not require the elements of $\{\alpha_j\}_{j=1}^{p}$ to be distinct.

(ii) implies (iii). By Theorem 5 and since A and B both admit a finite spectral form and are DP comonotonic, it follows that A and B commute too.

(iii) implies (iv). Since A and B commute (see, e.g., [16, p. 171]), there exist $C \in B(H)_{sa}$ and $f, g : \mathbb{R} \to \mathbb{R}$ such that $A = f(C)$ and $B = g(C)$. Assume that the spectral form of C is $C = \sum_{j=1}^{p_C} \gamma_j H_j$. If $p_C = 1$, the statement trivially follows, since f and g can be taken to be constant. If $p_C > 1$, then define $\alpha, \beta \in \mathbb{R}^{p_C}$ as the vectors such that $\alpha_j = f(\gamma_j)$ and $\beta_j = g(\gamma_j)$ for all $j \in \{1, \ldots, p_C\}$. Let $(p_A, \{\alpha'_k\}_{k=1}^{p_A}, \{E_k\}_{k=1}^{p_A})$ (respectively, $(p_B, \{\beta'_k\}_{k=1}^{p_B}, \{F_k\}_{k=1}^{p_B})$) be the spectral form of A (respectively, B). It follows that $p_A, p_B \leq p_C$ and for each $i, j \in \{1, \ldots, p_C\}$ we have that $H_i \leq E_k$ and $H_j \leq F_{k'}$ for some $k \in \{1, \ldots, p_A\}$ and $k' \in \{1, \ldots, p_B\}$. Let w and w' be unit vectors such that $w \in \operatorname{Range} H_i \subseteq \operatorname{Range} E_k$ and $w' \in \operatorname{Range} H_j \subseteq \operatorname{Range} F_{k'}$. Since A and B are DP comonotonic, we have that

$$(\alpha_i - \alpha_j)(\beta_i - \beta_j) = [\varphi_w(A) - \varphi_{w'}(A)][\varphi_w(B) - \varphi_{w'}(B)] \geq 0.$$

By Example 9, we can conclude that α and β are comonotonic vectors as in (29), therefore, there exist a vector $\hat{\gamma} \in \mathbb{R}^{p_C}$ and $\hat{f}, \hat{g} : \mathbb{R} \to \mathbb{R}$ increasing such that $\hat{\gamma}_i \neq \hat{\gamma}_j$ for all $i \neq j$ and $\alpha_j = \hat{f}(\hat{\gamma}_j)$ and $\beta_j = \hat{g}(\hat{\gamma}_j)$ for all $j \in \{1, \ldots, p_C\}$. Define

$$\hat{C} = \sum_{j=1}^{p_C} \hat{\gamma}_j H_j.$$

It is immediate to see that $A = \hat{f}(\hat{C})$ and $B = \hat{g}(\hat{C})$, proving the implication.

(iv) implies (i). Consider C and its spectral form $\sum_{j=1}^{p_C} \gamma_j H_j$. By assumption, it follows that $A = \sum_{j=1}^{p_C} f(\gamma_j) H_j$ and $B = \sum_{j=1}^{p_C} g(\gamma_j) H_j$. Define $\alpha, \beta \in \mathbb{R}^{p_C}$ as the vectors such that $\alpha_j = f(\gamma_j)$ and $\beta_j = g(\gamma_j)$ for all $j \in \{1, \ldots, p_C\}$. Let $\varphi \in S$. Define by $r \in \mathbb{R}^{p_C}$ the probability vector $r_j = \varphi(H_j)$ for all $j \in \{1, \ldots, p_C\}$. By (30), we can conclude that

$$\varphi(A) = \sum_{j=1}^{p_C} \alpha_j \varphi(H_j) = \sum_{j=1}^{p_C} \alpha_j r_j \quad \text{and} \quad \varphi(B) = \sum_{j=1}^{p_C} \beta_j \varphi(H_j) = \sum_{j=1}^{p_C} \beta_j r_j.$$

Since A and B commute, we have that $A \circ B = AB = \sum_{j=1}^{p_C} (\alpha_j \beta_j) H_j$. It follows that

$$\varphi(A \circ B) = \sum_{j=1}^{p_C} (\alpha_j \beta_j) \varphi(H_j) = \sum_{j=1}^{p_C} (\alpha_j \beta_j) r_j.$$

By construction and Example 9, we have that α and β are comonotonic as in (29). This implies that

$$\varphi(A \circ B) - \varphi(A)\varphi(B) = \sum_{j=1}^{p_C} (\alpha_j \beta_j) r_j - \left(\sum_{j=1}^{p_C} \alpha_j r_j \right) \left(\sum_{j=1}^{p_C} \beta_j r_j \right) \geq 0. \qquad (31)$$

Since φ was arbitrarily chosen, it follows that (31) holds for all $\varphi \in S$, proving that A and B are comonotonic. ◻

Proof of Theorem 3. It follows from the equivalence of (i) and (iv) of Theorem 6. □

Proof of Corollary 1. (i) implies (ii). Since A and B commute, there exist $C \in B(H)_{sa}$ and $f, g : \mathbb{R} \to \mathbb{R}$ such that $A = f(C)$ and $B = g(C)$. Assume that the spectral form of C is $C = \sum_{j=1}^{p_C} \gamma_j H_j$. Since we are only interested in the values that f and g take on the finite set $\{\gamma_j\}_{j=1}^{p_C} \subseteq \mathbb{R}$, we can consider f and g being of bounded variation and write them as difference of two increasing functions: namely, $f = f_1 - f_2$ and $g = g_1 - g_2$ where f_1, f_2, g_1, g_2 are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Define $A_1 = f_1(C), A_2 = f_2(C), B_1 = g_1(C)$, and $B_2 = g_2(C)$. Clearly, we have that $A = A_1 - A_2$ and $B = B_1 - B_2$. By construction and Theorem 3, we have that A_1, A_2, B_1, and B_2 are pairwise comonotonic.

(ii) implies (i). By Theorem 3 and since A_1, A_2, B_1, and B_2 are pairwise comonotonic, we have that they pairwise commute. By (13), this implies that

$$\begin{aligned} AB &= (A_1 - A_2)(B_1 - B_2) = A_1 B_1 - A_1 B_2 - A_2 B_1 + A_2 B_2 \\ &= B_1 A_1 - B_2 A_1 - B_1 A_2 + B_2 A_2 \\ &= (B_1 - B_2)(A_1 - A_2) = BA, \end{aligned}$$

proving that A and B commute. □

Proof of Proposition 1. (i) implies (ii). Clearly, dual comonotonicity implies DP comonotonicity. By Theorem 6, DP comonotonicity implies that A and B commute. Since A and B commute, there exist $C \in B(H)_{sa}$ and $f, g : \mathbb{R} \to \mathbb{R}$ such that $A = f(C)$ and $B = g(C)$. Assume that the spectral form of C is $C = \sum_{j=1}^{p_C} \gamma_j H_j$. Define $\alpha, \beta \in \mathbb{R}^{p_C}$ to be the vectors such that $\alpha_j = f(\gamma_j)$ and $\beta_j = g(\gamma_j)$ for all $j \in \{1, \ldots, p_C\}$. Hence, $A = \sum_{j=1}^{p_C} \alpha_j H_j$ and $B = \sum_{j=1}^{p_C} \beta_j H_j$. For each $j \in \{1, \ldots, p_C\}$ fix a unit vector $w_j \in \operatorname{Range} H_j$. Next, consider the space of affine functions over the set $\Delta_{p_C - 1}$, that is, $\operatorname{Aff}(\Delta_{p_C-1})$. Define also for each $r \in \Delta_{p_C-1}$ the vector $w_r \in H$ by $w_r = \sum_{j=1}^{p_C} \sqrt{r_j} w_j$. Since the elements of $\{H_j\}_{j=1}^{p_C}$ are pairwise orthogonal, we have that the vectors in $\{w_j\}_{j=1}^{p_C}$ are pairwise orthogonal. This implies that $\|w_r\|^2 = \|\sum_{j=1}^{p_C} \sqrt{r_j} w_j\|^2 = \sum_{j=1}^{p_C} \|\sqrt{r_j} w_j\|^2 = \sum_{j=1}^{p_C} r_j = 1$, that is, w_r is a unit vector. Define $\tilde{A} \in \operatorname{Aff}(\Delta_{p_C-1})$ to be such that $\tilde{A}(r) = \sum_{j=1}^{p_C} \alpha_j r_j = \sum_{j=1}^{p_C} \alpha_j r_j \|w_j\|^2$ for all $r \in \Delta_{p_C-1}$. Note also that for each $r \in \Delta_{p_C-1}$

$$\tilde{A}(r) = \sum_{j=1}^{p_C} \alpha_j r_j \|w_j\|^2 = \sum_{j=1}^{p_C} \alpha_j \|\sqrt{r_j} w_j\|^2 = \langle A(w_r), w_r \rangle = \varphi_{w_r}(A), \tag{32}$$

where φ_{w_r} is the pure state induced by the unit vector w_r. Define also $\tilde{B} \in \operatorname{Aff}(\Delta_{p_C-1})$ to be such that $\tilde{B}(r) = \sum_{j=1}^{p_C} \beta_j r_j = \sum_{j=1}^{p_C} \beta_j r_j \|w_j\|^2$ for all $r \in \Delta_{p_C-1}$. It follows that (32) holds also for $\tilde{B}$ by replacing A with B and α_j with β_j. Since A and B are dually comonotonic, this implies that for each $r, r' \in \Delta_{p_C-1}$

$$[\tilde{A}(r) - \tilde{A}(r')][\tilde{B}(r) - \tilde{B}(r')] = [\varphi_{w_r}(A) - \varphi_{w_{r'}}(A)][\varphi_{w_r}(B) - \varphi_{w_{r'}}(B)] \geq 0,$$

proving that $\tilde{A}$ and $\tilde{B}$ are comonotonic. By [9, Proposition 4.5], it follows that $\tilde{A} = \varphi(\tilde{A} + \tilde{B})$ and $\tilde{B} = \psi(\tilde{A} + \tilde{B})$ where $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing functions. Since $\tilde{A}$, $\tilde{B}$, and $\tilde{A} + \tilde{B}$ are affine and φ is increasing, it follows that φ can be chosen to be also affine,[ag] that is, $\varphi(t) = \bar{\lambda} t + \bar{\mu}$ where $\bar{\lambda} \geq 0$ and $\bar{\mu} \in \mathbb{R}$. This implies that $\tilde{A} = \bar{\lambda}\tilde{A} + \bar{\lambda}\tilde{B} + \bar{\mu}$, that is, $(1 - \bar{\lambda})\tilde{A} = \bar{\lambda}\tilde{B} + \bar{\mu}$. We have three cases:

(1) $\bar{\lambda} > 1$. In this case, we have that $\tilde{A} = \hat{\lambda}\tilde{B} + \hat{\mu}$ where $\hat{\lambda} = \frac{\bar{\lambda}}{1-\bar{\lambda}}$ and $\hat{\mu} = \frac{\bar{\mu}}{1-\bar{\lambda}}$. Note that $\hat{\lambda} < 0$. This implies that

$$\begin{aligned} 0 &\leq [\tilde{A}(r) - \tilde{A}(r')][\tilde{B}(r) - \tilde{B}(r')] \\ &= [\hat{\lambda}\tilde{B}(r) + \hat{\mu} - \hat{\lambda}\tilde{B}(r') - \hat{\mu}][\tilde{B}(r) - \tilde{B}(r')] \\ &= \hat{\lambda}[\tilde{B}(r) - \tilde{B}(r')]^2 \leq 0 \quad \forall\, r, r' \in \Delta_{p_C - 1}. \end{aligned}$$

This implies that $\tilde{B} = k$ for some $k \in \mathbb{R}$. Consider $r \in \Delta_{p_C-1}$ such that $r_j = 1$ and $r_i = 0$ for $i \neq j$. It follows that

$$\beta_j = \tilde{B}(r) = k.$$

Since $j \in \{1, \ldots, p_C\}$ was arbitrarily chosen, we can conclude that $B = kI$. If we define $\lambda = 0$ and $\mu = k$, we have that $B = \lambda A + \mu I$.

(2) $\bar{\lambda} = 1$. In this case, we have that $\tilde{B} = -\bar{\mu}$. Consider $r \in \Delta_{p_C-1}$ such that $r_j = 1$ and $r_i = 0$ for $i \neq j$. It follows that

$$\beta_j = \tilde{B}(r) = -\bar{\mu}.$$

Since $j \in \{1, \ldots, p_C\}$ was arbitrarily chosen, we can conclude that $B = \lambda A + \mu I$ where $\lambda = 0$ and $\mu = -\bar{\mu}$.

(3) $\bar{\lambda} < 1$. In this case, we have that $\tilde{A} = \lambda\tilde{B} + \mu$ where $\lambda = \frac{\bar{\lambda}}{1-\bar{\lambda}}$ and $\mu = \frac{\bar{\mu}}{1-\bar{\lambda}}$. Since $\bar{\lambda} \geq 0$, note that $\lambda \geq 0$. Consider $r \in \Delta_{p_C-1}$ such that $r_j = 1$ and $r_i = 0$ for $i \neq j$. It follows that

$$\alpha_j = \tilde{A}(r) = \lambda\tilde{B}(r) + \mu = \lambda\beta_j + \mu.$$

Since $j \in \{1, \ldots, p_C\}$ was arbitrarily chosen, we can conclude that $A = \lambda B + \mu I$.

(ii) implies (i). It is trivial. □

A.2. *Choquet integration*

Proof of Proposition 2. Properties 1 and 2 follow from the definition of quantum Choquet expectation and the properties of the spectral form.

[ag] Indeed, given the assumptions, φ turns out to be affine on the range of $\tilde{A} + \tilde{B}$.

(3) Let A and B be comonotonic. By Theorem 3, $A = f(C)$ and $B = g(C)$ where $C \in B(H)_{sa}$ and f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Let C have the following spectral form $\sum_{i=1}^{p_C} \gamma_i H_i$. Since f and g are increasing, we have that

$$A + B = \sum_{j=1}^{p_C} (f(\gamma_j) + g(\gamma_j))H_j$$

where $f(\gamma_1) \geq \cdots \geq f(\gamma_{p_C}), g(\gamma_1) \geq \cdots \geq g(\gamma_{p_C})$, and $f(\gamma_1) + g(\gamma_1) \geq \cdots \geq f(\gamma_{p_C}) + g(\gamma_{p_C})$. By (iii) of Remark 2, we conclude that

$$\begin{aligned}\mathbb{E}_\nu(A + B) &= \sum_{i=1}^{p_C} (f(\gamma_i) + g(\gamma_i)) \left[\nu\left(\sum_{j=1}^{i} H_j\right) - \nu\left(\sum_{j=1}^{i-1} H_j\right)\right] \\ &= \sum_{i=1}^{p_C} f(\gamma_i) \left[\nu\left(\sum_{j=1}^{i} H_j\right) - \nu\left(\sum_{j=1}^{i-1} H_j\right)\right] \\ &\quad + \sum_{i=1}^{p_C} g(\gamma_i) \left[\nu\left(\sum_{j=1}^{i} H_j\right) - \nu\left(\sum_{j=1}^{i-1} H_j\right)\right] \\ &= \mathbb{E}_\nu(A) + \mathbb{E}_\nu(B),\end{aligned}$$

as desired.

(4) Assume that A and B commute with $A \geq B \geq 0$. It is well known that $A = f(C)$ and $B = g(C)$ where $C \in B(H)_{sa}$ and f and g are two functions from $\mathbb{R}$ to $\mathbb{R}$. Let C have the following spectral form $\sum_{i=1}^{p_C} \gamma_i H_i$. Define $\alpha_j = f(\gamma_j)$ and $\beta_j = g(\gamma_j)$ for all $j \in \{1, \ldots, p_C\}$. Consider A. Note that there exists a bijection $\pi : \{1, \ldots, p_C\} \to \{1, \ldots, p_C\}$ such that $i \leq j$ implies $\alpha_{\pi(i)} \geq \alpha_{\pi(j)}$. Define $\tilde{\alpha}_i = \alpha_{\pi(i)}$ and $\tilde{E}_i = H_{\pi(i)}$ for all $i \in \{1, \ldots, p_C\}$. Let also $\tilde{\alpha}_{p_C+1} = 0$. Clearly, we have that $A = \sum_{i=1}^{p_C} \alpha_i H_i = \sum_{i=1}^{p_C} \alpha_{\pi(i)} H_{\pi(i)} = \sum_{i=1}^{p_C} \tilde{\alpha}_i \tilde{E}_i$. Define the following two mathematical objects:

(a) $I_{A,t} = \{j \in \{1, \ldots, p_C\} : \alpha_j \geq t\}$ for all $t \in [0, \infty)$;
(b) $\nu_A : [0, \infty) \to [0, 1]$ to be such that $\nu_A(t) = \nu(\sum_{j \in I_{A,t}} H_j)$ for all $t \in [0, \infty)$ with the convention that if $I_{A,t} = \emptyset$, then $\sum_{j \in I_{A,t}} H_j = 0$ and $\nu_A(t) = 0$.

On the one hand, by (ii) of Remark 2 we have that

$$\mathbb{E}_\nu(A) = \sum_{i=1}^{p_C} (\tilde{\alpha}_i - \tilde{\alpha}_{i+1})\nu\left(\sum_{j=1}^{i} \tilde{E}_j\right).$$

On the other hand, since $A \geq 0$ we have that for each $t \in [0, \infty)$,[ah]

$$\nu_A(t) = \begin{cases} 0 & t > \tilde{\alpha}_1 \\ \nu\left(\sum_{j=1}^{i} \tilde{E}_j\right) & \tilde{\alpha}_i \geq t > \tilde{\alpha}_{i+1} \quad \text{and} \quad i \in \{1, \ldots, p_C\} \\ 1 & t = 0 \end{cases}.$$

Note that ν_A is a decreasing function which eventually vanishes. So, it is Riemann integrable and

$$\int_0^\infty \nu_A(t)dt = \int_0^{\tilde{\alpha}_1} \nu_A(t)dt = \sum_{i=1}^{p_C} (\tilde{\alpha}_i - \tilde{\alpha}_{i+1})\nu\left(\sum_{j=1}^{i} \tilde{E}_j\right).$$

We can conclude that

$$\mathbb{E}_\nu(A) = \int_0^\infty \nu_A(t)dt. \tag{33}$$

If we define $I_{B,t}$ for all $t \in [0, \infty)$ and ν_B similarly, then the same arguments yield that

$$\mathbb{E}_\nu(B) = \int_0^\infty \nu_B(t)dt. \tag{34}$$

Since $A \geq B$, we have that $I_{B,t} \subseteq I_{A,t}$ for all $t \geq 0$, proving that $\nu_B(t) \leq \nu_A(t)$ for all $t \geq 0$. By (33) and (34), we conclude that $\mathbb{E}_\nu(A) \geq \mathbb{E}_\nu(B)$. Finally, assume that A and B commute and $A \geq B$. It follows that there exists $\lambda \geq 0$ such that $\tilde{A} \geq \tilde{B} \geq 0$ where $\tilde{A} = A + \lambda I$ and $\tilde{B} = B + \lambda I$. Clearly, $\tilde{A}$ and $\tilde{B}$ commute. By the previous part of the proof and point 2, we have that

$$\mathbb{E}_\nu(A) + \lambda = \mathbb{E}_\nu(A + \lambda I) = \mathbb{E}_\nu(\tilde{A}) \geq \mathbb{E}_\nu(\tilde{B}) = \mathbb{E}_\nu(B + \lambda I) = \mathbb{E}_\nu(B) + \lambda,$$

proving point 4.

(5) If we define $B = 0$, then clearly A and B commute. By point 4 and since $\mathbb{E}_\nu(B) = 0$, it follows that $\mathbb{E}_\nu(A) \geq \mathbb{E}_\nu(B) = 0$ (respectively, $0 = \mathbb{E}_\nu(B) \geq \mathbb{E}_\nu(A)$). □

Lemma 8. *Let H be finite dimensional and $\phi\colon B(H)_{sa} \to \mathbb{R}$. The following statements are true*:

(1) *If ϕ is a Choquet state, then ϕ is a physical Choquet state.*

[ah]Observe that if $\tilde{\alpha}_i = \tilde{\alpha}_{i+1}$, then there does not exist any $t \geq 0$ such that $\tilde{\alpha}_i \geq t > \tilde{\alpha}_{i+1}$. Hence, the equality $\nu_A(t) = \nu(\sum_{j=1}^{i} \tilde{E}_j)$ is vacuously true.

(2) *If ϕ is a physical Choquet state, then*

$$\phi(\lambda A) = \lambda\phi(A) \quad \forall\, \lambda \geq 0, \quad \forall\, A \in B(H)_{sa} \ s.t.\ A \geq 0 \tag{35}$$

and

$$\phi(A + \gamma I) = \phi(A) + \gamma \quad \forall\, A \in B(H)_{sa}, \quad \forall\, \gamma \in \mathbb{R}. \tag{36}$$

(3) *If ϕ is a Choquet state, then ϕ is Lipschitz continuous.*

Proof. (1) It is trivial. (2) Clearly, 0 is comonotonic with itself. This implies that $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$, proving that $\phi(0) = 0$. Consider $A \in B(H)_{sa}$. By Lemma 1 and since $Cov_\varphi(A, A) \geq 0$ for all $\varphi \in S$, observe that kA is always comonotonic with $k'A$, provided $k, k' \in [0, \infty)$. In light of this, we are going to prove that a physical Choquet state is positively homogeneous when the argument is a positive element. Indeed, if $k \in \mathbb{N}$, we have that $kA = (k - 1)A + A$. Since $(k - 1)A$ and A are comonotonic and ϕ is comonotonic additive, it follows that

$$\phi(kA) = \phi((k - 1)A) + \phi(A) \quad \forall\, k \in \mathbb{N}.$$

By induction and since A was arbitrarily chosen, it follows that

$$\phi(kA) = k\phi(A) \quad \forall\, A \in B(H)_{sa}, \quad \forall\, k \in \mathbb{N}. \tag{37}$$

Next, consider $B \in B(H)_{sa}$ and $n \in \mathbb{N}$. Define $A = \frac{1}{n}B$. It follows that $B = nA$. By (37), we can conclude that $\phi(B) = \phi(nA) = n\phi(A) = n\phi(\frac{1}{n}B)$. Since B and n were arbitrarily chosen, it follows that

$$\phi\left(\frac{1}{n}B\right) = \frac{1}{n}\phi(B) \quad \forall\, B \in B(H)_{sa}, \quad \forall\, n \in \mathbb{N}. \tag{38}$$

Consider now $C \in B(H)_{sa}$ and $q \in \mathbb{Q} \cap (0, \infty)$. It follows that $q = \frac{k}{n}$ for some $k, n \in \mathbb{N}$. By combining (37) and (38), we have that $\phi(qC) = \phi(\frac{k}{n}C) = k\phi(\frac{1}{n}C) = \frac{k}{n}\phi(C) = q\phi(C)$. Since C and q were arbitrarily chosen, it follows that

$$\phi(qC) = q\phi(C) \quad \forall\, C \in B(H)_{sa}, \quad \forall\, q \in \mathbb{Q} \cap (0, \infty). \tag{39}$$

Next, consider $\lambda \geq 0$ and $C \in B(H)_{sa}$ such that $C \geq 0$. We have two cases:

(1) $\lambda = 0$. Since $\phi(0) = 0$, we have that $\phi(\lambda C) = \phi(0) = 0 = \lambda\phi(C)$.
(2) $\lambda > 0$. It follows that $qC \leq \lambda C \leq rC$ for all $r, q \in \mathbb{Q} \cap (0, \infty)$ such that $q \leq \lambda \leq r$. By (39) and since ϕ is c-monotone, this implies that

$$q\phi(C) = \phi(qC) \leq \phi(\lambda C) \leq \phi(rC) = r\phi(C).$$

By taking two sequences $\{q_n\}_{n\in\mathbb{N}}, \{r_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q} \cap (0, \infty)$ such that $q_n \leq \lambda \leq r_n$ for all $n \in \mathbb{N}, q_n \to \lambda$, and $r_n \to \lambda$, we can conclude that $\phi(\lambda C) = \lambda\phi(C)$.

Since C and λ were arbitrarily chosen, points 1 and 2 show that

$$\phi(\lambda C) = \lambda\phi(C) \quad \forall\, \lambda \in [0, \infty), \quad \forall\, C \in B(H)_{sa} \quad \text{s.t. } C \geq 0, \tag{40}$$

proving (35). We next prove (36). By definition of comonotonicity, we have that A and λI are comonotonic for all $A \in B(H)_{sa}$ and for all $\lambda \geq 0$. By (40) and since ϕ is comonotonic additive and normalized, this implies that

$$\phi(A+\lambda I) = \phi(A) + \phi(\lambda I) = \phi(A) + \lambda\phi(I) = \phi(A) + \lambda \quad \forall\, A \in B(H)_{sa}, \quad \forall\, \lambda \geq 0. \tag{41}$$

Given (41), we only need to prove that (41) also holds for $\lambda < 0$. Consider $\lambda < 0$, hence $-\lambda > 0$ and $A + \lambda I \in B(H)_{sa}$. By (41), this implies that

$$\phi(A) = \phi((A+\lambda I) + (-\lambda I)) = \phi(A+\lambda I) - \lambda,$$

proving (36).

(3) Let ϕ be a Choquet state. By point 1, also the results in point 2 hold. We next prove Lipschitz continuity. Let $A, B \in B(H)_{sa}$. Recall that for each $C \in B(H)_{sa}$ we have that $\|C\|I \geq C \geq -\|C\|I$. This implies that $A + \|B-A\|I \geq B$. By (36) and since ϕ is monotone, this yields that

$$\phi(A) + \|B-A\| = \phi(A + \|B-A\|I) \geq \phi(B),$$

that is, $\|B-A\| \geq \phi(B) - \phi(A)$. Given the symmetric role of A and B, we can conclude that $\|B-A\| = \|A-B\| \geq \phi(A) - \phi(B)$, that is, $\|B-A\| \geq |\phi(A) - \phi(B)|$, proving Lipschitz continuity. □

Proof of Theorem 4. (i) implies (ii). Define $\nu : P(H) \to \mathbb{R}$ by $\nu(E) = \phi(E)$ for all $E \in P(H)$. By point 2 of Lemma 8, we have that $\phi(0) = 0$. Since $\phi(I) = 1$, this implies that $\nu(0) = \phi(0) = 0 = \phi(I) - 1 = \nu(I) - 1$. Note that if $E, F \in P(H)$ and $F \geq E$, then E and F commute. Thus, $\nu(F) = \phi(F) \geq \phi(E) = \nu(E)$. Since ϕ is c-monotone, this implies that ν maps the elements of $P(H)$ in $[0, 1]$ and is a quantum capacity. By point 2 of Lemma 8 and since ϕ is a physical Choquet state, observe also that

$$\phi(\lambda I) = \lambda\phi(I) \quad \forall\, \lambda \in \mathbb{R}.$$

Consider now $A \in B(H)_{sa}$ with spectral form $A = \sum_{i=1}^{p_A} \alpha_i E_i$. We show by induction on p_A that (19) holds.

Initial Step. $p_A = 1$. In this case, we have that $A = \alpha_1 E_1$ with $E_1 = I$. It follows that

$$\phi(A) = \phi(\alpha_1 E_1) = \alpha_1\phi(E_1) = \alpha_1\nu(E_1) = \sum_{i=1}^{p_A}(\alpha_i - \alpha_{i+1})\nu\left(\sum_{j=1}^{i} E_j\right).$$

Inductive Step. We assume that (19) holds for each $A \in B(H)_{sa}$ such that $p_A \in \{1, \ldots, k\}$. We next show that (19) holds for all $A \in B(H)_{sa}$ such that $p_A \in \{1, \ldots, k+1\}$. Since $p_A \leq \dim H$ for all $A \in B(H)_{sa}$, if $\dim H \leq k$, then there is nothing to prove. Otherwise, we have that $k \leq \dim H - 1$. Let $A \in B(H)_{sa}$ with spectral form $\sum_{i=1}^{p_A} \alpha_i E_i$ such that $p_A \leq k+1$. By the inductive hypothesis, if

$p_A \leq k$, then (19) holds. Otherwise, since $p_A = k+1 > 1$, we have that $\alpha_1 - \alpha_2 > 0$. Consider two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha_i) = \begin{cases} \alpha_i & i \in \{2, \ldots, k+1\} \\ \alpha_2 & i = 1 \end{cases}$$

and

$$g(\alpha_i) = \begin{cases} 0 & i \in \{2, \ldots, k+1\} \\ \alpha_1 - \alpha_2 & i = 1 \end{cases}.$$ [ai]

Define $B = f(A)$ and $C = g(A)$. By Theorem 3, we have that B and C are comonotonic as well as $A = C + B$. We have two cases:

(1) $k = 1$. In this case, we have that $p_A = 2$ and $B = \alpha_2 I$ as well as $C = (\alpha_1 - \alpha_2)E_1 + 0(I - E_1)$. By the initial step as well as point 2 of Lemma 8 and since ϕ is comonotonic additive, we have that

$$\begin{aligned} \phi(A) &= \phi(C + B) = \phi(C) + \phi(B) = \phi((\alpha_1 - \alpha_2)E_1) + \phi(\alpha_2 I) \\ &= (\alpha_1 - \alpha_2)\phi(E_1) + \alpha_2\phi(E_1 + E_2) = \mathbb{E}_\nu(A). \end{aligned}$$

(2) $k > 1$. In this case, we have that

$$\begin{aligned} B &= \alpha_2 E_1 + \sum_{i=2}^{k+1} \alpha_i E_i = \alpha_2(E_1 + E_2) + \sum_{i=3}^{k+1} \alpha_i E_i \\ C &= (\alpha_1 - \alpha_2)E_1 + 0(I - E_1). \end{aligned}$$

We can conclude that:

(a) The spectral form of B is $\sum_{i=1}^{p_B} \beta_i F_i$ where $p_B = p_A - 1 \leq k$, $\beta_i = \alpha_{i+1}$ for all $i \in \{1, \ldots, p_B\}$, $F_1 = E_1 + E_2$, and $F_i = E_{i+1}$ for all $i \in \{2, \ldots, p_B\}$. Thus, by inductive hypothesis, we have that

$$\phi(B) = \sum_{i=1}^{p_B} (\beta_i - \beta_{i+1})\nu\left(\sum_{j=1}^{i} F_j\right) = \sum_{i=1}^{p_A - 1} (\alpha_{i+1} - \alpha_{i+2})\nu\left(\sum_{j=1}^{i+1} E_j\right).$$

[ai]For example, define f and g from $\mathbb{R}$ to $\mathbb{R}$ to be such that

$$f(t) = \begin{cases} t & t \leq \alpha_2 \\ \alpha_2 & t \geq \alpha_2 \end{cases} \quad \text{and}$$

$$g(t) = \begin{cases} 0 & t \leq \alpha_2 \\ t - \alpha_2 & t \geq \alpha_2 \end{cases}.$$

(b) The spectral form of C is $\sum_{i=1}^{p_C} \gamma_i H_i$ where $p_C = 2$, $\gamma_1 = \alpha_1 - \alpha_2$, $\gamma_2 = 0$, $H_1 = E_1$, and $H_2 = I - E_1$. Thus, by inductive hypothesis, we have that

$$\phi(C) = \sum_{i=1}^{p_C} (\gamma_i - \gamma_{i+1}) \nu \left(\sum_{j=1}^{i} H_j \right) = (\gamma_1 - \gamma_2)\nu(E_1) = (\alpha_1 - \alpha_2)\nu(E_1).$$

Since ϕ is comonotonic additive, this implies that

$$\begin{aligned}
\phi(A) &= \phi(C + B) = \phi(C) + \phi(B) \\
&= (\alpha_1 - \alpha_2)\nu(E_1) + \sum_{i=1}^{p_A - 1} (\alpha_{i+1} - \alpha_{i+2}) \nu \left(\sum_{j=1}^{i+1} E_j \right) \\
&= (\alpha_1 - \alpha_2)\nu(E_1) + \sum_{i=2}^{p_A} (\alpha_i - \alpha_{i+1}) \nu \left(\sum_{j=1}^{i} E_j \right) \\
&= \sum_{i=1}^{p_A} (\alpha_i - \alpha_{i+1}) \nu \left(\sum_{j=1}^{i} E_j \right) = \mathbb{E}_\nu(A).
\end{aligned}$$

Case 1 and 2 prove the inductive step.

The implication follows by induction.

(ii) implies (i). It follows from Proposition 2.

As for uniqueness, by the previous part of the statement, if either (i) or (ii) holds, then (19) holds. If ν_1 and ν_2 are two quantum capacities that satisfy (19), it follows that

$$\nu_2(E) = \mathbb{E}_{\nu_2}(E) = \phi(E) = \mathbb{E}_{\nu_1}(E) = \nu_1(E) \quad \forall\, E \in P(H),$$

proving that $\nu_1 = \nu_2$ and that they both coincide with ϕ restricted to $P(H)$. □

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