Exercise: Income taxation[1]

Consider the following setup. Income q is produced with effort e according to a production function $q = \theta e$, where θ is individual's productivity parameter. θ can take two values: θ_L and θ_H, with $\theta_L < \theta_H$. We will assume that a proportion β of individuals is of low productivity θ_L and $1-\beta$ is of high θ_H. All the individuals have the same utility function

$$u\left(q - t - \frac{e^2}{2}\right),$$

where t is the tax (a subsidy if t is negative) the individual has to pay to the government, and u a strictly concave utility function. The government has to balance the budget. It can not give out more money than it gets, therefore

$$\beta t_L + (1-\beta)\, t_H \geq 0,$$

where t_L is the tax levied on low productivity individuals and t_H on high productivity individuals.

The question we will try to answer is: What is the optimal taxation under the perfect information?

The maximization problem is

$$\max_{t_H, t_L, q_H, q_L} \beta u\left(q_L - t_L - \frac{1}{2}\left(\frac{q_L}{\theta_L}\right)^2\right) + (1-\beta)\, u\left(q_H - t_H - \frac{1}{2}\left(\frac{q_H}{\theta_H}\right)^2\right)$$

$$s.t.$$

$$0 \leq \beta t_L + (1-\beta)\, t_H.$$

Notice that we do not impose any IR constraints for the consumers here; government can impose whatever taxes it wants. Consumers can not decide not to pay. We did impose a budget constraint on the government, though. Government can not create the money from thin air.

At the optimum the budget constraint will be binding: if it was not, the government would be withholding money, which could be effectively used to increase the welfare. We will use a Lagrangian to solve for the optimal contract:

$$L = \beta u\left(q_L - t_L - \frac{1}{2}\left(\frac{q_L}{\theta_L}\right)^2\right) + (1-\beta)\, u\left(q_H - t_H - \frac{1}{2}\left(\frac{q_H}{\theta_H}\right)^2\right) + \lambda\left(\beta t_L + (1-\beta)\, t_H\right),$$

where λ is the Lagrange multiplier. The solution is obtained by taking derivatives of L with

[1] This is an Exercise prepared by Nenad Kos, nenad.kos@unibocconi.it, for the students of the course Theory of Incentives and Contracts. The solutions are written in *italics*. Comments and suggestions are welcome.

respect to each variable and equating it with zero, which gives a system of four equations:

$$\begin{aligned}
\beta u'\left(q_L^* - t_L^* - \frac{1}{2}\left(\frac{q_L^*}{\theta_L}\right)^2\right) * \left(1 - \frac{q_L^*}{\theta_L^2}\right) &= 0, \\
(1-\beta)\, u'\left(q_H^* - t_H^* - \frac{1}{2}\left(\frac{q_H^*}{\theta_H}\right)^2\right) * \left(1 - \frac{q_H^*}{\theta_H^2}\right) &= 0, \\
-\beta u'\left(q_L^* - t_L^* - \frac{1}{2}\left(\frac{q_L^*}{\theta_L}\right)^2\right) + \lambda\beta &= 0, \\
-(1-\beta)\, u'\left(q_H^* - t_H^* - \frac{1}{2}\left(\frac{q_H^*}{\theta_H}\right)^2\right) + (1-\beta)\,\lambda &= 0.
\end{aligned}$$

First two are zero only if $1 = \frac{q_L^*}{\theta_L^2}$ *and* $1 = \frac{q_H^*}{\theta_H^2}$*; combining the last two yields*

$$u'\left(q_L^* - t_L^* - \frac{1}{2}\left(\frac{q_L^*}{\theta_L}\right)^2\right) = u'\left(q_H^* - t_H^* - \frac{1}{2}\left(\frac{q_H^*}{\theta_H}\right)^2\right).$$

This equation has an important interpretation. Namely, marginal utilites of both types are equal at the first best. Furthermore, since we assumed u *is strictly concave the last equation implies*

$$q_L^* - t_L^* - \frac{1}{2}\left(\frac{q_L^*}{\theta_L}\right)^2 = q_H^* - t_H^* - \frac{1}{2}\left(\frac{q_H^*}{\theta_H}\right)^2$$

If we add to those equations the budget constraint holding with equality we have the following system of equations

$$\begin{aligned}
1 &= \frac{q_L^*}{\theta_L^2}, \\
1 &= \frac{q_H^*}{\theta_H^2}, \\
q_L^* - t_L^* - \frac{1}{2}\left(\frac{q_L^*}{\theta_L}\right)^2 &= q_H^* - t_H^* - \frac{1}{2}\left(\frac{q_H^*}{\theta_H}\right)^2, \\
\beta t_L^* + (1-\beta)\, t_H^* &= 0,
\end{aligned}$$

which should be easy to solve. The solution is

$$\begin{aligned}
q_L^* &= \theta_L^2, \\
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t_H^* &= \frac{\beta}{2}\left(\theta_H^2 - \theta_L^2\right), \\
t_L^* &= \frac{(1-\beta)}{2}\left(\theta_L^2 - \theta_H^2\right).
\end{aligned}$$

Notice that $t_H^* > 0$ *and* $t_L^* < 0$*, which means that the high types pay money to the government (taxes) and the government gives the money to the low types (subsidies).*