Appendix

To derive the affine bond pricing formulas and yield curve equations, consider the case with prices of risk $\lambda_t = \begin{bmatrix} \lambda_t^1 & \lambda_t^2 \end{bmatrix}^\top$. (Note that equation (9) can be obtained from (10) by setting the prices of risk to zero.) There are two ways to derive thes formulas. First, we can construct a risk-neutral probability measure under which the risk-neutral pricing formula (7) holds. Second, we can start from the Euler equation $E\left[d\left(m_t F_t\right)\right] = 0$.

Risk-neutral probability

Under the risk-neutral probability measure, the process B^* which solves $dB_t^* = dB_t + \lambda_t dt$ is a Brownian motion. By solving for dB_t and inserting this expression into the AR(1) dynamics of the factors (6), we get

$$\begin{aligned}
dx_t^i &= \kappa_i\left(\theta_i - x_t^i\right)dt + \sigma_i(dB_t^{*i} - \lambda_t^i dt) && (11)\\
&= \left(\kappa_i\theta_i - \kappa_i x_t^i - \sigma_i\lambda_0^i - \sigma_i\lambda_1^i x_t^i\right)dt + \sigma_i dB_t^{*i} && (12)\\
&= \left(\kappa_i\theta_i - \sigma_i\lambda_0^i - (\kappa_i + \sigma_i\lambda_1^i)x_t^i\right)dt + \sigma_i dB_t^{*i} && (13)\\
&= (\kappa_i + \sigma_i\lambda_1^i)\left(\frac{\kappa_i\theta_i - \sigma_i\lambda_0^i}{(\kappa_i + \sigma_i\lambda_1^i)} - x_t^i\right)dt + \sigma_i dB_t^{*i} && (14)\\
&= \kappa_i^*\left(\theta_i^* - x_t^i\right)dt + \sigma_i dB_t^{*i}, && (15)
\end{aligned}$$

where

$$\begin{aligned}
\kappa_i^* &= \kappa_i + \sigma_i\lambda_1^i\\
\theta_i^* &= \frac{\kappa_i\theta_i - \sigma_i\lambda_0^i}{\kappa_i + \sigma_i\lambda_1^i}
\end{aligned}$$

The price of the τ-period bond is equal to

$$P_t^{(\tau)} = E_t^*\left(\exp\left(-\int_t^{t+\tau} r_s ds\right)\right),$$

where the expectation operator E^* uses the risk-neutral probability measure. Since the vector $x = (x_1, x_2)^\intercal$ is Markov, this expectation is a function of the state today x_t. Thus, the bond price is a function

$$P_t^{(\tau)} = F(x_t, \tau)$$

of the state vector x_t and time-to-maturity τ. The Feynman-Kac formula says that F solves the partial differential equation

$$F_t r_t = -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2} \left[\frac{\partial F}{\partial x^i} \kappa_i^* \left(\theta_i^* - x_t^i \right) + \frac{1}{2} \frac{\partial^2 F}{\partial x^{i2}} \sigma_i^2 \right]$$

with terminal condition $F(x, 0) = 1$.

We guess the solution

$$F(x_t, \tau) = \exp\left(A(\tau) + B(\tau) \cdot x_t \right) \tag{16}$$

which means that

$$\begin{aligned}
\frac{\partial F}{\partial x^i} &= B_i(\tau) F \\
\frac{\partial^2 F}{\partial x^{i2}} &= B_i(\tau)^2 F \\
\frac{\partial F}{\partial \tau} &= \left(A'(\tau) + B'(\tau) \cdot x_t \right) F.
\end{aligned}$$

Insert these expressions into the partial differential equation and get

$$\begin{aligned}
x_t^1 + x_t^2 &= -A'(\tau) - B_1'(\tau) x_t^1 - B_2'(\tau) x_t^2 \\
&\quad + \sum_{i=1}^{2} \left[B_i(\tau) \kappa_i^* \left(\theta_i^* - x_t^i \right) + \frac{1}{2} B_i(\tau)^2 \sigma_i^2 \right].
\end{aligned}$$

Matching coefficients results in

$$\begin{aligned} A'(\tau) &= \sum_{i=1}^{2} B_i(\tau)\,\kappa_i^*\theta_i^* + \frac{1}{2} B_i(\tau)^2 \sigma_i^2 \\ 1 &= -B_1'(\tau) - B_1(\tau)\,\kappa_1^* \\ 1 &= -B_2'(\tau) - B_2(\tau)\,\kappa_2^*. \end{aligned}$$

The boundary conditions are

$$\begin{aligned} A(0) &= 0 \\ B(0) &= 0_{2\times 1}. \end{aligned}$$

The solution to these ODE's are

$$\begin{aligned} B_1(\tau) &= \frac{(\exp(-\kappa_1^*\tau) - 1)}{\kappa_1^*} \qquad (17a) \\ B_2(\tau) &= \frac{(\exp(-\kappa_2^*\tau) - 1)}{\kappa_2^*}. \end{aligned}$$

We can plug these solutions into the yield equation

$$\begin{aligned} y_t^{(\tau)} &= -\frac{A(\tau)}{\tau} - \frac{B_1(\tau)}{\tau} x_t^1 - \frac{B_2(\tau)}{\tau} x_t^2 \qquad (18) \\ &= a^{NA}(\tau) + b_1^{NA}(\tau)\, x_t^1 + b_2^{NA}(\tau)\, x_t^2 \end{aligned}$$

and get equations (9).

Euler equation approach

The Euler equation is

$$P_t^{(\tau)} = E_t\left[\frac{m_{t+\tau}}{m_t}\right]$$

and the instantaneous equation is

$$E\left[d\left(m_t F_t\right)\right] = 0. \qquad (19)$$

The bond price is a function $F(x,\tau)$ and we can apply Ito's Lemma

$$dF = \mu_F dt + \sigma_F dB_t,$$

where the drift and volatility of F are given by

$$\begin{aligned}
\mu_F &= -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2}\left[\frac{\partial F}{\partial x_i}\kappa_i\left(\theta_i - x^i\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^{i2}}\sigma_i^2\right] \\
\sigma_F &= \sum_{i=1}^{2}\frac{\partial F}{\partial x^i}\sigma_i
\end{aligned}$$

Both m_t and F_t are Ito processes, so their product solves

$$\begin{aligned}
d\left(m_t F_t\right) &= -r_t m_t F_t dt + m_t \mu_t^F dt - m_t \lambda_t \sigma_t^F dt \\
&\quad -F_t m_t \lambda_t dB_t + m_t \sigma_t^F dB_t
\end{aligned}$$

We use the Euler equation (19) and get

$$\begin{aligned}
0 &= -r_t m_t F_t + m_t \mu_t^F - m_t \lambda_t \sigma_t^F \qquad (20) \\
F_t r_t &= \left(-\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2}\left[\frac{\partial F}{\partial x^i}\kappa_i\left(\theta_i - x_t^i\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^{i2}}\sigma_i^2\right]\right) - \sum_{i=1}^{2}\frac{\partial F}{\partial x^i}\sigma_i\lambda_t^i
\end{aligned}$$

Again, guess the exponential-affine solution (16) and insert the expressions into (20), we get

$$\begin{aligned}
x_t^1 + x_t^2 &= -A'(\tau) - B_1'(\tau)\, x_t^1 - B_2'(\tau)\, x_t^2 \\
&\quad + \sum_{i=1}^{2}\left[B_i(\tau)\,\kappa_i\left(\theta_i - x_t^i\right) + \frac{1}{2}B_i(\tau)^2\sigma_i^2\right] \\
&\quad - \sum_{i=1}^{2} B_i(\tau)\,\sigma_i\left(\lambda_0^i + \lambda_1^i x_t^i\right).
\end{aligned}$$

Matching coefficients, we get the ordinary differential equations:

$$\begin{aligned}
A'(\tau) &= \sum_{i=1}^{2} B_i(\tau)\left(\kappa_i\theta_i - \sigma_i\lambda_0^i\right) + \frac{1}{2}B_i(\tau)^2\sigma_i^2 \\
1 &= -B_1'(\tau) - B_1(\tau)\left(\kappa_1 + \sigma_1\lambda_1^1\right) \\
1 &= -B_2'(\tau) - B_2(\tau)\left(\kappa_2 + \sigma_2\lambda_1^2\right).
\end{aligned}$$

From this expression, we can see that we get the coefficients (17a) with risk neutral parameters

$$\begin{aligned}
\kappa_i^* &= \kappa_i + \sigma_i \lambda_1^i \\
\kappa_i^* \theta_i^* &= \kappa_i \theta_i - \sigma_i \lambda_0^i \Longrightarrow \theta_i^* = \frac{\kappa_i \theta_i - \sigma_i \lambda_0^i}{\kappa_i + \sigma_i \lambda_1^i}.
\end{aligned}$$